ELLIPTIC OPERATORS (note: Always assume M is closed!)

$D:\Gamma(E)\to\Gamma(F)$ diff. op. of order $k\geq 1$ btwn vec. bndls $E,F\to M$.
In local coords./trivs. $M \underset{open}{\supseteq} \mathcal{U} \xrightarrow[\cong]{chart} \Omega\subseteq\mathbb{R}^n$: $D\eta(x)=\sum_{|\alpha|\leq k} c_\alpha(x)\partial^\alpha\eta(x)$ for some fns. $c_\alpha:\Omega\to\mathbb{C}^{m\times\ell}$.

For $x\in\Omega$, $D \rightsquigarrow$ homogeneous kth-degree $\mathbb{C}^{m\times\ell}$-valued polynomial fn. of $p\in\mathbb{R}^n$: $\sigma_k^D(x)(p):=\sum_{|\alpha|=k} p^\alpha c_\alpha(x)$ w/ for $p=(p_1,\dots,p_n)\in\mathbb{R}^n$ & $\alpha=(\alpha_1,\dots,\alpha_n)$, $p^\alpha:=p_1^{\alpha_1}\cdots p_n^{\alpha_n}$. "principal symbol of D at x" (in coords)

Coord-inv. version: Given $x_0\in\Omega$, choose any $f\in C^\infty(\Omega,\mathbb{R})$ st. $f(x_0)=0$ & $\nabla f(x_0)=p\in\mathbb{R}^n$. Then $\partial^\alpha f^k(x_0)=\partial^\alpha(f\cdots f)(x_0)=\begin{cases}0 & \text{if } |\alpha|<k\\ k!\,p^\alpha & \text{if } |\alpha|=k\end{cases}$

$\Rightarrow \forall\,\eta\in C^\infty(\Omega,\mathbb{C}^\ell)$, $D(f^k\eta)(x_0)=\sum_{|\alpha|\leq k} c_\alpha(x_0)\partial^\alpha(f^k\eta)(x_0)$

$=k!\sum_{|\alpha|=k} c_\alpha(x_0)p^\alpha\eta(x_0) = k!\,\sigma_k^D(x_0)(p)\,\eta(x_0)$.

defn: The principal symbol of $D:\Gamma(E)\to\Gamma(F)$ is the unique fiber-preserving map $\sigma_k^D: T^*M\oplus E\to F: (p,v)\mapsto\sigma_k^D(p)v$ st. $\forall\, x\in M$ & $p\in T_x^*M$, $\sigma_k^D(p):E_x\to F_x$ is linear & any $\eta\in\Gamma(E)$ & $f\in C^\infty(M,\mathbb{R})$ w/ $f(x)=0$ & $df(x)=p$ satisfies $\sigma_k^D(p)\eta(x)=\frac{1}{k!}D(f^k\eta)(x)$.

ex: $\nabla:\Gamma(E)\to\Gamma(T^*M\otimes E)$ is a connection on $E \Leftrightarrow \nabla(f\eta)=df\otimes\eta+f\nabla\eta$
$\Rightarrow$ if $f(x)=0$, $df(x)=p\in T_x^*M$, then $\sigma_1^\nabla(p):E_x\to T_x^*M\otimes E_x: v\mapsto p\otimes v$

prop: (1) If $D:\Gamma(E)\to\Gamma(F)$ has order k & $D':\Gamma(F)\to\Gamma(G)$ has order k', then $\sigma_{k+k'}^{D'D}(p)=\sigma_{k'}^{D'}(p)\circ\sigma_k^D(p)$.

(2) If E & F are equipped w/ bndl metrics & M has a volume form defining the formal adjoint $D^*:\Gamma(F)\to\Gamma(E)$ by $\langle\xi,D\eta\rangle_{L^2(F)}=\langle D^*\xi,\eta\rangle_{L^2(E)}$ $\forall\,\eta\in\Gamma(E)$ & $\xi\in\Gamma(F)$, then $\sigma_k^{D^*}(p):F_x\to E_x$ for $p\in T_x^*M$ is the adjoint of $\sigma_k^D(p):E_x\to F_x$ w.r.t. the bndl metrics. $\boxtimes$

ex: For $d:\Gamma(\Lambda^m T^*M)\to\Gamma(\Lambda^{m+1}T^*M)$, $d(f\alpha)=df\wedge\alpha+f\,d\alpha$
$\Rightarrow \sigma_1^d(p)\alpha=p\wedge\alpha$. Choosing a Riem. metric g on M, this has adjoint $\sigma_1^{d^*}(p)\beta=-\iota_{p^\#}\beta$, w/ $T^*M\to TM: p\mapsto p^\#$ is the inverse of $TM\to T^*M: X\mapsto g(X,\cdot)$.

$\therefore$ For $\Delta=dd^*+d^*d$, $\sigma_2^\Delta(p)\alpha=-\iota_{p^\#}(p\wedge\alpha)-p\wedge\iota_{p^\#}\alpha \underset{\text{(computation)}}{=} -|p|^2\alpha$.

defn: D (of order k) is elliptic if $\sigma_k^D(p):E_x\to F_x$ is invertible $\forall\,x\in M$, $p\neq 0\in T_x^*M$.

rks: D elliptic $\Leftrightarrow D^*$ elliptic. Impossible unless $\mathrm{rk}\,E=\mathrm{rk}\,F$.
We will use the fact that D elliptic $\Rightarrow\sigma_k^D(p)$ & $\sigma_k^{D^*}(p)$ both injective $\forall\,p\neq 0$

main thm: Spse $D: \Gamma(E) \to \Gamma(F)$ is elliptic of order k, $D^*: \Gamma(F) \to \Gamma(E)$ is its formal adjoint w.r.t. some bdl metrics & volume form, & we consider the extensions D & D^* to bdd linear ops. $H^{m+k}(E) \to H^m(F)$ or $H^{m+k}(F) \to H^m(E)$ for some $m \geq 0$.

(1) $\ker D$ is a fin.-dim. subspace of $\Gamma(E)$ (the smooth sections) ($\Rightarrow$ same true for D^*)

(2) $\operatorname{im} D \subseteq H^m(F)$ is a closed subspace & $H^m(F) = \operatorname{im} D \oplus \ker D^*$.

In particular, $D: H^{m+k}(E) \to H^m(F)$ is Fredholm & $\operatorname{ind}(D) = \dim \ker D - \dim \ker D^*$ is indep. of m.

cor: If $D: \Gamma(E) \to \Gamma(E)$ is elliptic & formally self-adjoint (i.e. $D^* = D$), then $L^2(E) = \bigoplus_{\lambda \in \operatorname{spec}(D)} E_\lambda$ where $\operatorname{spec}(D) \subseteq \mathbb{R}$ is a discrete set of eigenvals. of D that accumulate only at ∞ ($\pm$), & $\forall \lambda \in \operatorname{spec}(D)$, $E_\lambda := \{\eta \in \Gamma(E) \mid D\eta = \lambda \eta\}$ is fin.-dim., with $E_\lambda \perp E_\mu$ for $\lambda \neq \mu$ w.r.t. the L^2-product.

pf of cor: $\operatorname{ind}(D) = -\operatorname{ind}(D^*) = -\operatorname{ind}(D) \Rightarrow \operatorname{ind}(D) = 0$. Then $\forall \lambda \in \mathbb{C}$, $D - \lambda$ is a cpt perturbation of D $\Rightarrow$ also Fredholm w/ $\operatorname{ind}(D-\lambda) = 0$, so inj $\Leftrightarrow$ surj.,
$\therefore$ {eigenvals. of D} $= \{\lambda \in \mathbb{C} \mid D - \lambda : H^k \to L^2$ not an iso.$\}$. Pick $\mu \in \mathbb{C} \setminus \mathbb{R}$, so $D = D^* \Rightarrow \mu$ not an e-val. of $D \Rightarrow \exists$ bdd inverse $K_\mu := (D-\mu)^{-1} : L^2 \to H^k$, which becomes a compact op. $K_\mu : L^2 \to L^2$ when composed w/ the cpt inclusion $H^k \hookrightarrow L^2$.
Now λ = e-val of $D \Leftrightarrow \frac{1}{\lambda - \mu}$ = e-val. of K_μ, w/ same eigenvectors.
Spectral thm for cpt ops. $\Rightarrow$ {e-vals of K_μ} $\subseteq \mathbb{C}$ is a bdd discrete set w/ accumulation only at 0, $\therefore$ same for D w/ accumulation at $\pm\infty$.
Now pick $\mu \in \mathbb{R} \setminus$ {e-vals. of D}, so K_μ is self-adjoint $\Rightarrow$ applying spectral thm again, $L^2(E) = \bigoplus$ eigenspaces, each fin.-dim since $\dim \ker(D - \lambda) < \infty$. ☐

elliptic estimates: Consider $D = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha : C^\infty(\mathbb{R}^n, \mathbb{C}^\ell) \circlearrowleft$, elliptic w/ const. coeffs. Then for $f \in C_0^\infty(\mathbb{R}^n)$, $Df = g \Leftrightarrow \sigma^D(p) \hat{f}(p) = \hat{g}(p)$ for the matrix-valued polynomial $\sigma^D(p) := \sum_{|\alpha| \leq k} (2\pi i p)^\alpha c_\alpha$ whose top degree term is $(2\pi i)^k \sigma_k^D(p)$.

elementary lemma: For $P: \mathbb{R}^n \to \mathbb{C}^{\ell \times \ell'}$ a polynomial of deg. k w/ $P_k :=$ its degree k part, the following are equivalent:

(1) $P_k(x) \in \mathbb{C}^{\ell \times \ell'}$ is injective $\forall x \neq 0$.

(2) $\exists R \geq 0, c > 0$ s.t. $|P(x)v|^2 \geq c(1+|x|^2)^k |v|^2 \quad \forall v \in \mathbb{C}^{\ell'}, x \in \mathbb{R}^n$ w/ $|x| \geq R$ ☐

$$\therefore \|f\|^2_{H^{m+k}} = \int_{\mathbb{R}^n} (1+|p|^2)^{m+k} |\hat{f}(p)|^2 \, d^n p = \int_{\mathbb{D}^n_R} (\cdots) + \int_{\mathbb{R}^n \setminus \mathbb{D}^n_R} (\cdots)$$
$$\leq (1+R^2)^{m+k} \|\hat{f}\|^2_{L^2} + \frac{1}{c} \int_{\mathbb{R}^n \setminus \mathbb{D}^n_R} (1+|p|^2)^m |\underbrace{\sigma^D(p) \hat{f}(p)}_{= \widehat{Df}(p)}|^2 \, d^n p$$
$$\leq c' \|f\|^2_{L^2} + c' \|Df\|^2_{H^m}.$$

$\Rightarrow$ prop: $\exists c > 0$ s.t. $\forall f \in C_0^\infty(\mathring{\mathbb{D}}^n)$, $\|f\|_{H^{m+k}} \leq c\|f\|_{L^2} + c\|Df\|_{H^m}$. ☐

Globally: $D : \Gamma(E) \to \Gamma(F)$ elliptic, $M = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$ finite cover w/ local trivs. s.t. $\forall\, \alpha \in I$, $\exists\, D_\alpha$ w/ const. coeffs. s.t. $\|D - D_\alpha\|$ small on $\mathcal{U}_\alpha$. Then choose partition of unity $\{\rho_\alpha : \mathcal{U}_\alpha \to [0,1]\}_{\alpha \in I}$, so $\Gamma(E) \ni \eta = \sum_\alpha \rho_\alpha \eta$, apply local estimate above to each $\rho_\alpha \eta$ $\Rightarrow$

prop: $\exists\, c > 0$ s.t. $\forall\, \eta \in \Gamma(E)$, $\|\eta\|_{H^{m+k}} \le c\|D\eta\|_{H^m} + c\|\eta\|_{H^{m+k-1}}$. ☐

rks: (1) also valid $\forall\, \eta \in H^{m+k}(E)$ by density. (2) $H^{m+k}(E) \hookrightarrow H^{m+k-1}(E)$ is compact.

lemma: Spse X, Y, Z = Banach spaces, $A : X \to Y$ bdd lin. op. & $K : X \to Z$ a cpct lin. op. s.t. for some $c > 0$, $\|x\|_X \le c\|Ax\|_Y + c\|Kx\|_Z \quad \forall\, x \in X$.
Then $\dim \ker A < \infty$ & $\operatorname{im} A \subseteq Y$ is closed.

pf that $\dim \ker A < \infty$: Suff to show unit ball in $\ker A$ is cpct. If $x_n \in \ker A$ is a bdd seq., $Kx_n \in Z$ converges (after reducing to a subseq.) $\Rightarrow$ is Cauchy, then $\|x_n - x_m\|_X \le c\|Kx_n - Kx_m\|_Z \Rightarrow x_n$ also Cauchy $\Rightarrow$ converges.

pf that $\operatorname{im} A$ closed: $\dim \ker A < \infty \Rightarrow X = \ker A \oplus V$ for a closed subspace $V \subseteq X$, so just consider $\operatorname{im}(A|_V)$ & use the estimate — easy exercise. ☐

rk: Can also use this lemma to prove Fredholm + cpct = Fredholm.

regularity: $D\eta = \xi$ weakly $\Leftrightarrow \forall\, \varphi \in \Gamma(F)$, $\langle \varphi, \xi \rangle_{L^2} = \langle D^*\varphi, \eta \rangle_{L^2}$.

prop: Spse D elliptic, $\eta \in L^2(E)$ & $D\eta = \xi \in H^m(F)$ weakly. Then $\eta \in H^{m+k}(E)$.

cor: $D\eta = 0$ weakly for $\eta \in L^2 \Rightarrow \eta \in \bigcap_{k \ge 0} H^k = C^\infty$. ☐

sketch of pf: Use a mollifier to approximate $\eta \in L^2$ & $\xi \in H^m$ by $\eta_\varepsilon, \xi_\varepsilon \in C^\infty$ s.t. $\eta_\varepsilon \xrightarrow{L^2} \eta$ & $\xi_\varepsilon \xrightarrow{H^m} \xi$ as $\varepsilon \to 0$. Can arrange s.t. $D\eta = \xi \Rightarrow$ $D\eta_\varepsilon$ & ξ_ε related by some bdd op. w/ bound indep. of $\varepsilon > 0$.
Then elliptic estimates $\Rightarrow \|\eta_\varepsilon\|_{H^{m+k}}$ bdd as $\varepsilon \to 0$, so by the Banach–Alaoglu thm, $\exists$ seq. $\varepsilon_n \to 0$ s.t. η_{ε_n} converges weakly to some $\eta_0 \in H^{m+k}$. Since we already know $\eta_{\varepsilon_n} \xrightarrow{L^2} \eta$, conclude $\eta = \eta_0 \in H^{m+k}$. ☐

conclusion of pf of main thm: Claim $H^m(F) = D(H^{m+k}(E)) \oplus \ker D^*$.

trivial intersection: If $\xi \in \operatorname{im} D \cap \ker D^*$, $\xi = D\eta$ for some $\eta \in H^{m+k}$, then
$D^*\xi = 0 \Rightarrow \xi \in C^\infty \Rightarrow \eta \in C^\infty \Rightarrow \|\xi\|^2_{L^2} = \langle \xi, D\eta \rangle_{L^2} = \langle D^*\xi, \eta \rangle_{L^2} = 0 \Rightarrow \xi = 0$.
Still need to show $H^m = D(H^{m+k}) + \ker D^*$.

case $m = 0$: If not, then since $\operatorname{im} D$ closed & $\dim \ker D^* < \infty$, RHS = proper closed subspace $\Rightarrow \exists\, \xi \ne 0 \in L^2$ s.t. $\xi \perp \operatorname{im} D$ & $\xi \perp \ker D^*$. So $\langle D\eta, \xi \rangle_{L^2} = 0$ $\forall\, \eta \in H^k$, in particular $\forall\, \eta \in C^\infty \Rightarrow D^*\xi = 0$ weakly $\Rightarrow \xi \in \ker D^*$, contradiction.

case $m > 0$: Given $\alpha \in H^m \subseteq L^2$, we've shown $\alpha = D\eta + \xi$ for some $\eta \in H^k$, $\xi \in \ker D^* \subseteq C^\infty$, so $D\eta = \alpha - \xi \in H^m \Rightarrow$ by regularity, $\eta \in H^{m+k}$, $\therefore H^m = D(H^{m+k}) + \ker D^*$. ☐