# Connections, Curvature and Characteristic Classes

## 1. Principal bundles and their connections

Let $\mathcal{G}$ be a Lie group, $\rho: \mathcal{G} \to GL(V)$ a representation on a $\mathbb{K}$-vs $V$. Let $M$ be a smooth mfd.

**Def:** A principle $\mathcal{G}$-Bundle is a fiber Bndl $P \xrightarrow{\pi} M$ with fibre $\mathcal{G}$, considered as a right $\mathcal{G}$-space. i.e., there is a right $\mathcal{G}$-action on $P$ which is fibre preserving, and acts transitively and freely on each fiber.

The vector bundle associated to $P \to M$ and $\rho: \mathcal{G} \to GL(V)$ is the bundle $E := P \times_\rho V \to M$ with

$$P \times_\rho V := (P \times V)/_\sim \,, \qquad (p, v) \sim (pg, \rho(g^{-1})v) \quad \forall g \in \mathcal{G},\ p \in P,\ v \in V.$$

**Examples:** Let $E \to M$ be a vec. Bndl of rank $k$. For $x \in M$, let $(GL(E))_x$ be the set of bases of the fibre $E_x$. Then

$$GL(E) = \bigcup_{x \in M} (GL(E))_x$$

is the frame bndle of $E$. It is naturally a principal $GL(k, \mathbb{K})$-bundle.

Similarly, if we have a metric on $E$ or choose an orientation, we can define

- $O(E)$ - principal $O(k, \mathbb{R})$-Bundle,
- $U(E)$ - principal $U(k, \mathbb{C})$-bundle,
- $SO(E)$ - principal $SO(k, \mathbb{R})$-Bundle,
- $SU(E)$ - principal $SU(k, \mathbb{C})$-bundle.

Given the natural repr $\rho: GL(k, \mathbb{K}) \to GL(\mathbb{K}^k)$, we obtain $E$ back as

$$E = GL(E) \times_\rho \mathbb{K}^k.$$

Similarly in the other cases above.

### The vertical subbundle

Let $P \xrightarrow{\pi} M$ be a princ. $\mathcal{G}$-Bndl. Then

$$VP := \ker T\pi \subset TP$$


is the vertical subbundle of $TP$. At each $p \in P$, $VP_p$ is spanned by the vectors $\tilde{X}_p := \frac{d}{dt}\big|_{t=0} p \cdot e^{tX}$, where $X \in \mathfrak{g} = T_e \mathcal{G}$ (Lie algebra of $\mathcal{G}$). They extend to fundamental vector fields $\tilde{X}$ on $P$, i.e. sections of $VP$.

### $\rho$-equivariant functions and forms

**Def:** A function $f \in C^\infty(P, V)$ is $\rho$-equivariant if $f(pg) = \rho(g^{-1}) f(p)$ $\forall p \in P,\ g \in \mathcal{G}$. A diff form $\alpha \in \Omega^m(P, V)$ is $\rho$-equivariant if $R_g^* \alpha = \rho(g^{-1}) \circ \alpha$, where $R_g: P \to P$ is multiplication by $g \in \mathcal{G}$, $\forall g \in \mathcal{G}$. Furthermore, $\alpha$ is called horizontal if $\alpha(X_1, \dots, X_m) = 0$ whenever $X_i \in VP$ for some $i$.

Lemma: There are 1:1-correspondences

$\{\rho\text{-equivariant functions } f: P \to V\} \xleftrightarrow{1:1} \{\text{sections of } E = P \times_\rho V\}$

$\{\rho\text{-equivariant horizontal forms } \alpha \in \Omega^m(P, V)\} \longleftrightarrow \{\text{diff. forms } \alpha \in \Omega^m(M, E)\}$

## Connections

Def: A connection on $P$ is a $G$-invariant complement to $VP$ in $TP$, i.e. a subbundle $HP \subset TP$ s.t. $HP_p \oplus VP_p = TP_p$ and $TR_g(HP_p) = HP_{pg}$ for each $p \in P$, $g \in G$.

Note that $HP$ can be equivalently described by the projection $TP \to VP$. Since $VP_p \cong \mathfrak{g}$ for each $p \in P$, we obtain:

Def: A connection form on $P$ is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ s.t.

i) $\omega$ is Ad-equivariant, i.e. $R_g^* \omega = Ad(g^{-1}) \circ \omega \quad \forall g \in G$.

Remark: This comes from the fact that, for fund v.f. $\tilde{X}$ we have

$$TR_g \tilde{X}_p = \widetilde{Ad(g^{-1})X}_{pg}.$$

ii) $\omega(\tilde{X}) = X$ for each $X \in \mathfrak{g}$.

Specifying $\omega$ or $HP$ is equivalent. In both cases, we get a projection $P_\omega$ of diff. forms on $P$ to horizontal diff. forms on $P$ by

$$P_\omega \alpha_p(X_1, \dots, X_m) := \alpha_p(pr_{HP} X_1, \dots, pr_{HP} X_m)$$

## Associated connection on the vector bundle

We describe briefly how a connection on $P$ defines a v.bundl connection on $E = P \times_\rho V$. Given a path $\gamma: [0,1] \to M$ on $M$, we can lift $\gamma$ to a horizontal path $\gamma^*: [0,1] \to P$ on $P$, i.e. $\dot{\gamma}^*(t) \in HP$ for all $t$ and $\pi \circ \gamma^* = \gamma$. Now given $X_0 \in E_x$ for $\overset{\gamma(0)}{x} \in M$, we can translate $X$ parallely along $\gamma$: Let $X_0 = [(p, v)] \in E$, i.e. $\overset{\gamma^*(0)}{p} \in P$ with $\pi(p) = x$ and $v \in V$. Define the translate as $X_1 = [(\gamma^*(1), v)] \in E_{\gamma(1)}$.
This translation only depends on $X_0$ and $\gamma$ and defines an iso $E_{\gamma(0)} \cong E_{\gamma(1)}$.
Now let $X \in TM_x$ and $s \in \Gamma(E)$. Then let $\Theta_t: E_{\gamma(t)} \xrightarrow{\sim} E_{\gamma(0)}$ with $\gamma(0) = 0$, $\dot{\gamma}(0) = X$, be the above iso.
We define

$$\nabla_X s := \frac{d}{dt}\left(\Theta_t s(\gamma(t))\right)\Big|_{t=0}.$$

Remark: Identifying $s$ with a function $\bar{s}: P \to V$ as above, $\nabla_X s$ corresponds to $d\bar{s}(X^*)$, where $X^*$ is a horizontal lift of $X$.

## Curvature

**Def:** The curvature of a connection $\omega$ is

$$\Omega := P_\omega d\omega \in \Omega^2(P, \mathfrak{g}).$$

$\Omega$ is a horizontal 2-form with values in $\mathfrak{g}$. Some easy properties:

1) $R_g^* \Omega = Ad(g^{-1}) \circ \Omega \qquad \forall g \in G \qquad$ (i.e. $\Omega$ is Ad-equivariant)

2) $\Omega(X_1, X_2) = d\omega(X_1, X_2) + [\omega(X_1), \omega(X_2)]$.

3) $P_\omega d\Omega = 0$.

**Remark:** For $X, Y \in HP_p$, one sees easily that $\widetilde{\Omega_p(X,Y)} = pr_{VP}([X,Y])$, hence $\Omega$ measures how much $HP$ deviates from being integrable. In particular, $\Omega = 0$ iff $HP$ is integrable iff $(P, \omega)$ is locally trivial.

Let $\rho_* = T\rho_e : \mathfrak{g} \to \mathfrak{gl}(V) = End(V)$. Then $\rho_* \omega$ and $\rho_* \Omega$ are $End(V)$ valued. We get the following important computations:

**Prop:** Let $\alpha$ be a horizontal, $\rho$-equivariant $n$-form on $P$. Then:

i) $P_\omega d\alpha = d\alpha + \rho_* \omega \wedge \alpha$

ii) $P_\omega d P_\omega d\alpha = \rho_* \Omega \wedge \alpha$

Here "$\wedge$" means with respect to the evaluation $End(V) \otimes V \to V$.

**Proof:** For (i), separate between vertical and horizontal vectors, (ii) follows easily from (i). $\square$

**Remark:** $\rho_* \Omega$ is horizontal and $Ad_{End(V)}$-equivariant and hence defines a $End(E)$ valued 2-form $R$ on $M$. Using the prop. (ii) it is not hard to see that

$$R(X,Y)v = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})v.$$

This is the curvature of $\nabla$ on $E$.

## 2. Chern and Pontrjagin classes

**Goal:** Given vec. bndl $E \to M$, construct a class $c(E) \in H^{\cdot}(M)$ that "characterizes" $E$.

**Example:** Let $E \to M$ be complex and of rank 1. Then $E$ is associated to some principal $\mathbb{C}^* = GL(1,\mathbb{C})$-bundle $P \to M$. Choose a connection on $P$ with curvature $\Omega \in \Omega^2_{hor}(P,\mathbb{C})$. Since $R_g^*\Omega = Ad(g^{-1})\circ\Omega$ and $Ad_{\mathbb{C}^*} = id$, $\Omega$ is invariant under $G$ and hence defines some $\Omega \in \Omega^2(M,\mathbb{C})$. We denote $c_1(E) := -\frac{1}{2\pi i}[\Omega] \in H^2(M,\mathbb{C})$.

**Def:** Let, for simplicity, $G = GL(k,\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(k,\mathbb{C}) = Mat(k,k,\mathbb{C})$. An invariant polynomial on $\mathfrak{gl}_k(\mathbb{C})$ is a polynomial function $P: \mathfrak{gl}_k(\mathbb{C}) \to \mathbb{C}$ s.t.

$$P(Ad_Y X) = P(YXY^{-1}) \overset{!}{=} P(X) \qquad \forall X, Y \in \mathfrak{gl}_k(\mathbb{C}).$$

Let $\Omega$ be the curvature of a connection on a principal $G$-bundle. Then $P\circ\Omega$ is $G$ invariant and hence def. an element $P(\Omega) \in \Omega^{\cdot}(M,\mathbb{C})$.

**Theorem:** i) $P(\Omega) \in \Omega^{\cdot}(M,\mathbb{C})$ is closed.

ii) Given the curvature $\Omega'$ of another connection on $P$, $\Omega - \Omega'$ is exact.

**Proof:** i) We have $P\circ\Omega = \pi^*(P(\Omega))$, hence $d(P\circ\Omega) = d(\pi^* P(\Omega)) = \pi^* dP(\Omega)$ is horizontal, therefore $d(P\circ\Omega) = P_\omega d(P\circ\Omega) = 0$ since $P_\omega d\Omega = 0$.

ii) Let $\omega_t = (1-t)\omega + t\omega'$. Then $\omega_t$ is a path of connections from $\omega$ to $\omega'$. For the curvatures $\Omega_t$ we get

$$\frac{d}{dt}\Omega_t = P_{\omega_t} d(\omega - \omega').$$

W.l.o.g. assume $P$ is homogeneous of degree $m$; then it can be considered as a multilinear map $f: \mathfrak{gl}_k(\mathbb{C})^m \to \mathbb{C}$ and

$$P\circ\Omega = f\circ(\Omega\wedge\ldots\wedge\Omega).$$

Define $\varphi = m\cdot\int_0^1 f\circ((\omega-\omega')\wedge\Omega_t\wedge\ldots\wedge\Omega_t)\,dt$. Then $d\varphi = f\circ(\Omega\wedge\ldots\wedge\Omega) - f\circ(\Omega'\wedge\ldots\wedge\Omega')$. □

**Def:** Let

$$c_k(X) = (-2\pi i)^{-k}\, tr(\Lambda^k X): \mathfrak{gl}_m(\mathbb{C}) \to \mathbb{C},$$

where $\Lambda^k X$ is the action of $X$ on $\Lambda^k \mathbb{C}^m$. [1)]

Given a $\mathbb{C}$-vec. bndl of rank $m$, $E \to M$, let $P = GL(E)$ the frame bndl of $E$ and $\Omega$ a connection on $P$. By the thm, we get an element

$$c_k(E) \in H^{2k}(M,\mathbb{C}), \quad c_k(E) := c_k(\Omega), \qquad \text{and } c(E) = c_0(E) + c_1(E) + \ldots + c_m(E) \in H^{\cdot}(M,\mathbb{C}).$$

which is independent of the choosen connection on $P$. It is called the $k$-th Chern class of $E$.

**Lemma:** The ring of invariant polynomials is a polynomial ring generated by the $c_k$.

**Proof:** Note that every invariant polynomial is a symmetric polynomial on the eigenvalues (using that diagonalizable matrices are dense in $\mathbb{C}^{m\times m}$). The $c_k$ correspond to the elementary symmetric polynomials. □

**Remark:** i) For $\mathbb{C}$-v.bndl $E, E' \to M$ we have $c(E\oplus E') = c(E)\cdot c(E')$

ii) For smooth $\varphi: N \to M$ we have $c(\varphi^* E) = \varphi^*(c(E))$.

iii) Actually, $c(E) \in H^{\cdot}(M,\mathbb{R})$: Choose a metric on $E$ and a compatible connection. This corresponds to a connection on $U(E)$ with values in $\mathfrak{u}(m)$. But $c_k|_{\mathfrak{u}(m)}$ has values in $\mathbb{R}$.

Now let $E \to M$ be a real v.bndl and $E_{\mathbb{C}} := E\otimes\mathbb{C}$. Then $c_k(E_{\mathbb{C}}) = 0$ for odd $k$, because for a metric on $E$ we see that the curvature can be taken to have values in $\mathfrak{o}(m)$, so that $tr(\Lambda^k\Omega) = (-1)^k tr(\Lambda^k\Omega)$.

**Def:** For a real v.bdl $E \to M$, the $k$-th Pontrjagin class is

$$p_k(E) := (-1)^k c_{2k}(E\otimes\mathbb{C}).$$

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1) In particular, $c_1(X) = -\frac{1}{2\pi i} tr(X)$, $c_m(X) = \left(-\frac{1}{2\pi i}\right)^m \det(X)$.

# 3. Genera

Since high powers of $\Omega$ vanish, we can extend the definition of $P(\Omega)$ to invariant formal power series $P: \mathfrak{gl}_m(\mathbb{C}) \to \mathbb{C}$.
If $f(z)$ is a function holomorphic near $z=0$, define the invariant power series $\Pi_f$ by

$$\Pi_f(X) = \det\left(f\left(-\tfrac{1}{2\pi i}X\right)\right),$$

called the Chern $f$-genus. It is uniquely determined by the properties

i) For a $\mathbb{C}$-line bundl $L \to M$, $\Pi_f(L) = f(c_1(L))$.

ii) $\Pi_f(E \oplus E') = \Pi_f(E) \cdot \Pi_f(E')$.

If $x_1, \dots, x_m$ are the eigenvalues of $X$, then $\Pi_f(X) = \prod_{j=1}^{m} f(x_j)$.

<u>Example:</u> $c(E)$ is the genus associated to $f(z) = 1+z$.

<u>Example 2:</u> The genus associated to $f(z) = \frac{1}{1+z}$ is computed by expanding $\prod \frac{1}{1+x_j}$ and using the fact that $c_k$ is the $k$-th elementary symm. poly. in $(x_j)$:

$$(1 - x_1 + x_1^2 - \dots)(1 - x_2 + x_2^2 - \dots) \dots = 1 - c_1 + (c_1^2 - c_2) + \dots$$

<u>Def:</u> The Chern character is the characteristic class assoc. to the invariant power series given by $X \mapsto \operatorname{tr}\exp\left(-\frac{1}{2\pi i}X\right)$. In eigenvalues we get $ch(X) = \sum e^{x_j}$, i.e. $ch(V) = (\dim V) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \dots$

$ch$ is not a genus, but

$$ch(E \oplus E') = ch(E) + ch(E'), \qquad ch(E \otimes E') = ch(E) \cdot ch(E').$$

There is a similar construction for $\mathbb{R}$-v. bundl: Let $g$ be holomorphic near $0$, $g(0) = 1$. Let $f$ be the branch of $z \mapsto (g(z^2))^{1/2}$ with $f(0) = 1$. The Pontrjagin $g$-genus is the Chern $f$-genus of the complexification.

<u>Lemma:</u> The $g$-genus is equal to $\prod_j g(y_j)$ with $y_j$ formal variables whose elementary symmetric polynomials are the Pontrjagin classes.

<u>Examples:</u> The $\hat{A}$-genus is the genus assoc. to $g: z \mapsto \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$. The Hirzebruch $\mathcal{L}$-genus is the genus assoc. to $g: z \mapsto \frac{\sqrt{z}}{\tanh\sqrt{z}}$.