Lecture 5 (Emilio Lauret) 16.11.2017

Clifford algebras and Dirac operators
(Chapter 3 in Roe's book)

<u>Motivation:</u> Laplacian $\Delta : \Gamma(\Lambda^* T^* M \otimes_{\mathbb{R}} \mathbb{C}) \circlearrowleft$

$$\Delta = d^* d + d d^* = (d + d^*)^2 = D^2$$

$D := d + d^*$ Dirac operator, Γ

The main goal is to generalize this situation.

§1. Clifford bundles and Dirac operators.

<u>Def:</u> Let V be a vect. space with $(\,,\,)$ a symmetric bilinear form. A Clifford alg for $(V, (\,,\,))$ is

- A an algebra with unity 1.
- $\varphi : V \to A$ with $\varphi(v)^2 = -(v,v)\,1$
- (universality) if $\varphi' : V \to A'$ satisfies $\varphi'(v) = -(v,v)\,1$, then $\exists!\ \Phi : A \to A'$ s.t. $\Phi \circ \varphi = \varphi'$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & A \arrow[d, dashed, "\Phi"] \\ & A' \end{tikzcd}$$

<u>Exercise:</u> $(\,,\,) \equiv 0 \rightsquigarrow A = \Lambda^* V$ exterior alg.

~~Fact:~~

<u>Prop:</u> For $(V, (\,,\,))$ $\exists!$ Clifford alg up to isomorphism.

<u>Proof:</u> Uniqueness follows by universality.
Existence (idea): $\{e_1, \dots, e_n\}$ a basis of V, $A := \operatorname{span}_{\mathbb{R}}\left\{ e_{i_1} \cdots e_{i_r} : \begin{smallmatrix} i_1 < \dots < i_r \\ 0 \le r \le n \end{smallmatrix} \right\}$

The product $e_{i_1} \cdots \underbrace{e_{i_r} e_{j_1}}_{\shortparallel} \cdots e_{j_{r'}}$

$$- e_{j_1} e_{i_r} - 2(e_{i_r}, e_{j_1})$$

$$\varphi(e+f)^2 = -(e+f, e+f)$$

$$\cancel{\varphi(e)^2} + \cancel{\varphi(f)^2} + \varphi(e)\varphi(f) + \varphi(f)\varphi(e) = -\cancel{(e,e)} - \cancel{(f,f)} - 2(e,f)$$

$$\varphi(e)\varphi(f) + \varphi(f)\varphi(e) = -2(e,f)$$

$\dim \mathcal{C}\ell(V) = 2^n$

<u>Notation</u>: $Cl(V) :=$ the Clifford alg of $(V, \langle,\rangle)$.

<u>Convention</u>: We will ~~consider~~ assume that every $Cl(V)$-modules is a complex vect. space and the left action is by $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$.

<u>Recall</u>: Let V be a vect. bundle on a smooth mfd M.

- A connection on V is $\nabla : \Gamma(TM) \otimes \Gamma(V) \longrightarrow \Gamma(V)$ s.t.
  (i) $\nabla_{fX} Y = f \nabla_X Y \quad \forall X \in \Gamma(TM), Y \in \Gamma(V), f \in C^\infty(M)$
  (ii) $\nabla_X (fY) = f \nabla_X Y + (X.f) Y$ ''

- Fact: $\nabla_X Y$ at a point $m \in M$ depends only on X_p.
Thus

$$\nabla \leadsto \overline{\nabla} : \Gamma(V) \longrightarrow \Gamma(T^*M \otimes V) =: \Omega^1(V)$$

$$(\overline{\nabla} Y)_m (X_m) = (\nabla_{X_p} Y)_m$$

space of V-valued 1-forms

$$T_m^* M \otimes V_m \simeq Hom(T_m M, V_m)$$

- $V = TM \leadsto \nabla :=$ Levi-Civita connection $\nabla_X Y - \nabla_Y X = [X, Y]$

compatibility $(\nabla_X Y_1, Y_2) + (Y_1, \nabla_X Y_2) = X \cdot (Y_1, Y_2)$

- The curvature operator K of V on M is the $End(V)$-valued 2-form

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \forall \begin{matrix} X, Y \in \Gamma(TM) \\ Z \in \Gamma(V) \end{matrix}$$

- $V = TM$, $R =$ Riemann curvature op

$$R(X,Y)Z = \quad \forall X, Y, Z \in \Gamma(TM).$$

$\nabla =$ L-C connection

(M,g) Riem mfd $\leadsto$ bundle of Clifford algs

$$Cl(TM) := \bigcup_{m\in M} Cl(T_mM, g_m).$$

<u>Def</u>: Let S be a bundle of Clifford modules (i.e. S_m at m is a $\mathbb{C}$-vect space with a left $Cl(T_mM)\otimes_{\mathbb{R}}\mathbb{C}$-action) S is a "Clifford bundle" if it is equipped with a Hermitian metric and compatible connection ∇ s.t.

(i) $\overset{\text{At } m,}{}$ $(v\cdot s_1, s_2) = -(s_1, v\cdot s_2)$ $\quad \forall \begin{array}{l} v\in T_mM \\ s_1,s_2\in S_m\end{array}$

(ii) $\nabla_X(Y\cdot s) = (\nabla_X Y)\cdot s + Y\cdot\nabla_X s \quad \forall \begin{array}{l} X,Y\in\Gamma(TM) \\ s\in\Gamma(S)\end{array}$

Clifford mult

<u>Def</u>: the "Dirac operator" D of S is the first order diff op on $\Gamma(S)$ given by the following composition:

$$\Gamma(S) \xrightarrow{\nabla \text{ connection}} \Gamma(T^*M\otimes S) \xrightarrow{\text{identif } TM\cong T^*M} \Gamma(TM\otimes S) \xrightarrow{\text{Cliff prod}} \Gamma(S)$$

Locally ~~form~~: $\{e_i\}$ a local o.n.b. of $\Gamma(TM)$ with dual basis $\{\hat{e}_i\}$ in $\Gamma(T^*M)$

$$s\in\Gamma(S) \longmapsto \nabla s = \sum_i \hat{e}_i\otimes\nabla_{e_i}s \longmapsto \sum_i e_i\otimes\nabla_{e_i}s \longmapsto \sum_i e_i\nabla_{e_i}s$$

$$\boxed{Ds = \sum_i e_i(\nabla_{e_i}s).}$$

At $m \in M$

$$D^2 s = \sum_i e_i \nabla_{e_i}\Big(\sum_j e_j \nabla_{e_j} s\Big) = \sum_{i,j} e_i \overbrace{\nabla_{e_i}(e_j \nabla_{e_j} s)}^{(\nabla_{e_i} e_j)^{=0} \nabla_{e_j} s + e_j \nabla_{e_j}\nabla_{e_i} s}$$

$$= \sum_{i,j} e_i e_j \nabla_{e_i} \nabla_{e_j} s$$

$$= -\sum_i \nabla_{e_i}\nabla_{e_i} s + \sum_{i<j} e_i e_j \underbrace{(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})}_{K(e_i,e_j) + \nabla_{[e_i,e_j]^{=0}}} s$$

$$= \nabla^* \nabla s + \underbrace{\sum_{i<j} e_i e_j K(e_i,e_j) s}_{\bar{K} s}$$

"Weitzembock formula"

$\bar{K} s$ = Clifford contraction of the curvature.

$$D^2 s = \nabla^*\nabla s + \bar{K} s$$

Explanation of $\underbrace{\nabla^*\nabla}$:

connection Laplacian
Rough Laplacian
Bochner Laplacian

$$\Gamma(S) \underset{\nabla^*}{\overset{\nabla}{\rightleftarrows}} \Gamma(T^*M \otimes S)$$

Lemma: $\{e_i\}$ a ~~synchronous~~ orthonormal frame

$$\nabla^*\Big(\sum_i \hat{e}_i \otimes s_i\Big) = -\sum_i \nabla_{e_i} s_i$$

Note that $\langle \nabla^*\nabla s, s\rangle = \langle \nabla s, \nabla s\rangle = |\nabla s|^2 \geq 0$.

Theorem (Bochner): If the least eigenvalue of $\bar{K}$ at each point of a compact M is strictly positive, then

$$D^2 s \equiv 0 \Rightarrow s \equiv 0.$$

Proof: $(\bar{K}_m s_m, s_m)_m \geq \overset{>0}{C_m} \|s_m\|_m^2 \overset{M \text{ cpct}}{\Longrightarrow} \langle \bar{K} s, s\rangle \geq C|s|^2,\ C>0.$

$\Rightarrow 0 = \langle D^2 s, s\rangle = \|\nabla s\|^2 + \langle \bar{K}s, s\rangle \geq 0 + C|s|^2 \Rightarrow |s|^2 \equiv 0 \Rightarrow s \equiv 0$ ▨

Prop: D is self-adjoint, i.e. $\langle Ds_1, s_2\rangle = \langle s_1, Ds_2\rangle$ $\forall s_1, s_2 \in \Gamma(S)$ with one of them compactly supported.

Proof: At $m \in M$,

$$(Ds_1, s_2) - (s_1, Ds_2) = \sum_i (e_i \underbrace{\nabla_{e_i}}_{\nabla_{e_i} e_i = 0} s_1, s_2) - (s_1, e_i \nabla_{e_i} s_2)$$

$$= \sum_i (\nabla_{e_i}(e_i s_1), s_2) + (e_i s_1, \nabla_{e_i} s_2)$$

compatibility

$$= \sum_i \nabla_{e_i}(e_i s_1, s_2) = d^* \omega, \quad \omega(X) = -(X s_1, s_2)$$

$$\Rightarrow 0 \underset{\text{divergence}}{=} \int_M d^*\omega \, dvol = \int_M \big((Ds_1, s_2) - (s_1, Ds_2)\big)\, dvol =$$

$$= \langle Ds_1, s_2\rangle - \langle s_1, Ds_2\rangle. \quad \blacksquare$$

§2. Clifford bundles and curvature. ("Refine Weitzenböck formula")

Notation: $c: TM \to End(S)$, $c(X)\cdot s = X \cdot s$. (Cliff prod.)

Lemma: In $End(S)$

$$[K(X,Y), c(Z)] = c(R(X,Y)\cdot s)$$

(Cliff prod)

Proof: $K(X,Y)(Z.s) = \nabla_X\nabla_Y(Zs) - \nabla_Y\nabla_X(Zs) - \nabla_{[X,Y]}(Z.s)$

$$= \nabla_X(\nabla_Y Z.s + Z\nabla_Y s) - \nabla_Y(\nabla_X Z.s + Z\nabla_X s) - (\nabla_{[X,Y]}Z)s - Z\nabla_{[X,Y]}s$$

$$= (\nabla_X\nabla_Y Z)s + \cancel{\nabla_Y Z \nabla_X s} + \cancel{\nabla_X Z \nabla_Y s} + Z\nabla_X\nabla_Y s$$

$$-(\nabla_Y\nabla_X Z)s - \cancel{\nabla_X Z \nabla_Y s} - \cancel{\nabla_Y Z \nabla_X s} - Z\nabla_Y\nabla_X s$$

$$-(\nabla_{[X,Y]}Z)s - Z\nabla_{[X,Y]}s = (R(X,Y)Z).s + Z.K(X,Y)s \quad \blacksquare$$

[$K(X,Y)$ is not a $Cl(TM)\otimes\mathbb{C}$-morphism of Cliff bundles since it doesn't commute with the Cliff multiplication]

Def: The Riemann endomorphism R^S of S is the $End(S)$-valued 2-form

$$R^S(X,Y) = \frac{1}{4}\sum_{k,l} c(e_k)\,c(e_l)\,\underbrace{(R(X,Y)e_k, e_l)}_{\in\mathbb{R}}$$

Lemma: $[R^S(X,Y), c(Z)] = c(R(X,Y)Z)$

$F^S := K - R^S \Rightarrow [F^S(X,Y), c(Z)] = 0$

$\Rightarrow F^S$ commutes with the action of the Cliff alg.

= twisting curvature of S.

Prop: $D^2 = \nabla^*\nabla + \overline{F^S} + \frac{scal}{4}$.

where $\overline{F^S} = \sum_{i<j} c(e_i)\,c(e_j)\,F^S(e_i,e_j)$ the Clifford contraction of F^S

and scal is the scalar curvature. $\overline{R^S} \overset{?}{=} \frac{scal}{4}$

The proof uses $R(X,Y) = -R(Y,X)$, Bianchi identity + ...

§3. Example of Clifford bundles

$S := \Lambda^* T^* M \otimes \mathbb{C} \simeq Cl(TM) \otimes \mathbb{C}$

$\wr\wr$ isomorphism of vect spaces.

$\Lambda^* TM \otimes \mathbb{C}$

We make $\Lambda^* T^* M \otimes \mathbb{C}$ a bundle of left $Cl(TM) \otimes \mathbb{C}$-modules

<u>Lemma:</u> $c(e) \cdot \omega = e^* \wedge \omega + e^* \lrcorner\, \omega \quad \forall \, e \in TM, \ \omega \in \Lambda^* T^* M.$

where

$e^* \lrcorner\, \omega = (-1)^{mk+m+1} * (e \wedge * \omega)$ interior product

$e^* \equiv e$ under the identification $TM \equiv T^*M$.

<u>Lemma:</u> S is a Clifford bundle.

<u>Proof (idea):</u> Check $(\omega_1, e^* \lrcorner\, \omega_2) = -(e^* \wedge \omega_1, \omega_2)$

$\Rightarrow c(e)$ is skew-adjoint. ∎

$$D\omega = \sum_i c(e_i) \nabla_{e_i} \omega$$

$$= \sum_i e_i \wedge \nabla_{e_i} \omega + \sum_i e_i \lrcorner\, \nabla_{e_i} \omega$$

$$= d\omega + d^* \omega$$

$$\Rightarrow D^2 = dd^* + d^* d = \Delta$$