The heat and wave equations

goal: define $\exp(-tD^2)$ for Dirac operators D (generalization of the matrix exponential)

1. Functional calculus (→ chapter 5)

Let M be closed, oriented, Riemannian; let $S \to M$ be a Clifford bundle with Dirac op. D.

⇝ D is elliptic, self-adjoint of order 1.

Recall (→ talk #3)

Prop: $L^2(S) = \bigoplus_{\lambda \in \mathrm{Spec}(D)} S_\lambda$ as an orthogonal (i.e. $E_\mu \perp E_\lambda$) direct sum

where S_λ denotes the λ-Eigenspace of D

- $\mathrm{Spec}(D) \subseteq \mathbb{R}$ is discrete
- $S_\lambda \subseteq \Gamma(S)$ (smooth! sections) is finite dimensional

Elliptic Estimate: $\forall s \in \Gamma(S)$: $\|s\|_{H^k} \leq C\left(\|s\|_{H^{k-1}} + \|Ds\|_{H^{k-1}}\right)$

Prop: Each $s \in L^2(S)$ can be written as $s = \sum_\lambda s_\lambda$ (decomposition into Eigenvectors)

Then: s is smooth $\iff \forall k \in \mathbb{N}: \|s_\lambda\|_{L^2} \in \mathcal{O}(|\lambda|^{-k})$ ⊛

Proof: Sobolev Embedding theorem: s is smooth $\iff s \in \bigcap_{k \in \mathbb{N}} H^k(S)$

Elliptic Estimate for s_λ, $\lambda \neq 0$:

$\|s_\lambda\|_{H^k} \leq C(|\lambda|+1)\|s_\lambda\|_{H^{k-1}} \leq C'|\lambda| \cdot \|s_\lambda\|_{H^{k-1}}$ (Spec(D) is discrete)

$\Rightarrow \|s_\lambda\|_{H^k} \leq C_k |\lambda|^k \|s_\lambda\|_{L^2}$ with C_k independent of λ

$\Rightarrow$ Condition ⊛ $\iff s = \sum_\lambda s_\lambda$ converges in each Sobolev Space H^k

$\iff s \in \bigcap_k H^k$ □

Idea for $\exp(-tD^2)$

Let $L \in \mathbb{R}^{m \times m}$ be diagonalizable, write $v = \sum_\lambda v_\lambda$ (decomp. into Eigenvectors).

Then $\exp(-tL^2) \cdot v = \sum_\lambda \exp(-t\lambda^2) v_\lambda$ ⇝ can be generalized for D.

Def: Let $f: \mathrm{Spec}(D) \to \mathbb{C}$ be bounded, write $s = \sum_\lambda s_\lambda \in L^2(S)$.

$f(D)s := \sum_\lambda f(\lambda) s_\lambda$ ⇝ $f(D): L^2(S) \to L^2(S)$

- f bounded ⇝ series converges

Prop: Properties of $f(D)$

- $f \mapsto f(D)$ is a unital ring homomorphism (from the bounded functions on $\mathrm{Spec}(D)$ to the bounded linear functions $L^2(S) \to L^2(S)$)
- $\|f(D)\| \leq \|f\|_\infty$
- $AD = DA \Rightarrow Af(D) = f(D)A \quad \forall A: L^2(S) \to L^2(S)$
- $f(D)$ restricts to $f(D)|_{\Gamma(S)}: \Gamma(S) \to \Gamma(S)$ (follows from smoothness condition ⊛)

Def: $S \boxtimes S^* \to M \times M$ is the vector bundle with $(S \boxtimes S^*)_{p,q} = \mathrm{Hom}(S_q, S_p)$

A bounded operator $A: L^2(S) \to L^2(S)$ is called smoothing operator

$:\Longleftrightarrow \exists k \in \Gamma(S \boxtimes S^*): \quad \forall s \in L^2(S) \quad As(p) = \int_M k(p,q) s(q) \, \mathrm{vol}\, q$

- Interchange Integration & Differentiation $\Rightarrow$ As is smooth (hence "smoothing" op.)
- k is called the smoothing kernel of A

Def: $f: \mathbb{R} \to \mathbb{C}$ is called rapidly decreasing $:\Longleftrightarrow \forall k \in \mathbb{N} \; \exists C > 0: |f(\lambda)| \leq C|\lambda|^{-k} \; \forall \lambda$ (i.e. $|f(\lambda)| \in O(|\lambda|^{-k})$)

$\mathcal{R}(\mathbb{R}) := \{ f \text{ rapidly decreasing} \}$

Prop: $f \in \mathcal{R}(\mathbb{R}) \Rightarrow f(D)$ is a smoothing operator

The map $\mathcal{R}(\mathbb{R}) \to \Gamma(S \boxtimes S^*)$, $f \mapsto$ kernel of $f(D)$ is continuous.

Proof: Write $f(D) = \sum_\lambda f(\lambda) P_\lambda$ with $P_\lambda :=$ projection to S_λ.

1.) $P_\lambda: L^2(S) \to L^2(S)$ is smoothing:

Let $\{s_1, \dots, s_n\}$ be an ONB of S_λ.

Define $k_\lambda(p,q) := \frac{1}{\mathrm{Vol}(M)} \sum_{i=1}^n \langle \cdot, s_i(q) \rangle_q \cdot s_i(p) \in \mathrm{Hom}(S_q, S_p)$

$\Rightarrow \int_M k_\lambda(p,q) s(q) \, \mathrm{vol}\, q = \frac{1}{\mathrm{Vol}(M)} \sum_{i=1}^n \underbrace{\left(\int_M \langle s(q), s_i(q) \rangle_q \, \mathrm{vol}\, q \right)}_{= \begin{cases} 0, & \text{if } s \perp S_\lambda \\ \alpha_i, & \text{if } s = \sum_i \alpha_i s_i \in S_\lambda \end{cases}} \cdot s_i(p)$

$\Rightarrow \int_M k_\lambda(p,q) s(q) \, \mathrm{vol}\, q = \begin{cases} 0, & \text{if } s \perp S_\lambda \\ s, & \text{if } s \in S_\lambda \end{cases}$

2.) The kernel $\sum_\lambda f(\lambda) k_\lambda$ converges in each H^k $\rightsquigarrow$ smooth

Sobolev-estimate (ex): $\forall k \; \exists \ell(k): \|k_\lambda\|_{H^k} \leq C_k |\lambda|^{\ell(k)}$

rapid decrease of f $\Rightarrow$ convergence in each H^k

Rem: Step 2.) implies:

$\exists N$ (depending on $\dim M$): $f(\lambda) \in O(|\lambda|^{-N}) \Rightarrow f(D)$ has a continuous kernel

2. Existence and Uniqueness results

M closed oriented Riemannian ; $S \to M$ Clifford bundle with Dirac operator D

heat equation $\frac{\partial s}{\partial t} + D^2 s = 0$

with $s_t \in \Gamma(S)$ smoothly depending on $t \in \mathbb{R}$

wave equation $\frac{\partial s}{\partial t} - iDs = 0$

Prop (Existence and Uniqueness of the homogeneous IVP)

$\forall s_0 \in \Gamma(S)$ $\exists!$ solution s_t of

- the heat equation $(t \geq 0)$ satisfying $\|s_t\| \leq \|s_0\|$
- the wave equation $(t \in \mathbb{R})$ satisfying $\|s_t\| = \|s_0\|$

Proof: 1.) Inequalities

(heat) $\frac{\partial}{\partial t}\|s_t\|^2 = \frac{\partial}{\partial t}\langle s_t ; s_t\rangle = \langle \frac{\partial s_t}{\partial t} ; s_t\rangle + \langle s_t ; \frac{\partial s_t}{\partial t}\rangle$ $\left(\langle \cdot , \cdot \rangle = \int_M (\cdot , \cdot) \operatorname{vol}\right)$

$= -\langle D^2 s_t ; s_t\rangle - \langle s_t ; D^2 s_t\rangle = -2\|Ds_t\|^2 \leq 0$ (D is self-adjoint)

(wave) $\frac{\partial}{\partial t}\|s_t\|^2 = \langle iDs_t ; s_t\rangle + \langle s_t ; iDs_t\rangle = \langle iDs_t ; s_t\rangle - \langle iDs_t ; s_t\rangle = 0$

2.) Uniqueness

$s_t ; s_t'$ two solutions with $s_0 = s_0'$ $\Rightarrow s_t - s_t'$ solution with $s_0 - s_0' = 0$

$\Rightarrow s_t - s_t' = 0$ (Inequalities)

3.) Existence

(heat) $s_t = e^{-tD^2} s_0$ (wave) $s_t = e^{itD} s_0$ defined by functional calculus □

heat kernel

$\forall t > 0$: e^{-tx^2} is rapidly decreasing $\Rightarrow e^{-tD^2}$ is smoothing

$\Rightarrow \exists k_t \in \Gamma(S \boxtimes S^*)$: $e^{-tD^2} s(p) = \int_M k_t(p,q)\, s(q) \operatorname{vol} q$ $\forall s \in \Gamma(S)$

Prop (Properties of the heat kernel)

(i) $\forall q \in M$: $k_t(\cdot, q) \in \Gamma(S \otimes S_q^*)$ satisfies the heat equation in p:

$$\left[\frac{\partial}{\partial t} + D_p^2\right] k_t(p,q) = 0$$

(ii) k_t tends to a δ-function as $t \to 0$, i.e.

$$\forall s \in \Gamma(S) : \int_M k_t(\cdot, q)\, s(q) \operatorname{vol} q \xrightarrow{t \to 0} s$$

k_t is the unique section of $S \boxtimes S^*$ (C^2 in p,q & C^1 in t) satisfying (i) and (ii)

Proof: (i) interchange $\int_M$ and $\left[\frac{\partial}{\partial t} + D_p^2\right]$

(ii) $e^{-tx^2} \xrightarrow{t \to 0} 1$ as rapidly decreasing functions

$\Rightarrow e^{-tD^2} \xrightarrow{t \to 0} 1_{L^2(S)}$ as bounded operators on $L^2(S)$

Uniqueness

Let k_t satisfy (i) and (ii) with smoothing operator K_t; to show: $K_t = e^{-tD^2}$ $\forall t>0$.

Let $\varepsilon>0$, then $K_t s = K_{(t-\varepsilon)+\varepsilon} s = e^{-(t-\varepsilon)D^2} K_\varepsilon s$ by uniqueness of the heat equ. $\forall t \geq \varepsilon$

$e^{-(t-\varepsilon)D^2} \xrightarrow{\varepsilon \to 0} e^{-tD^2}$ as $L^2(S)$-operators

$\Rightarrow K_t s = e^{-tD^2} s$

$K_\varepsilon s \xrightarrow{\varepsilon \to 0} s$ by (ii) □

Def: A section k_t' of $S \boxtimes S^*$ (C^2 in p,q & C^1 in $t \geq 0$) is an approximative heat kernel of order m if it satisfies (ii) and

(i') $\left[\frac{\partial}{\partial t} + D_p^2\right] k_t'(p,q) = t^m r_t(p,q)$ with r_t C^m in p,q & C^0 in $t \geq 0$

goal: if k_t' is of order m'; m' suff. large; then the error $k_t - k_t'$ is also of order t^m

Prop: (Duhamel's principle; inhomogeneous IVP)

For each C^2-section s_t of S, C^0 in t, there is a unique smooth section $\tilde{s}_t$ of S, C^1 in t

with $\tilde{s}_0 = 0$ and $\left[\frac{\partial}{\partial t} + D^2\right] \tilde{s}_t = s_t$.

It is given by $\tilde{s}_t = \int_0^t e^{-(t-t')D^2} s_{t'} \, dt'$

Sobolev Estimate for $\tilde{s}$: $\|\tilde{s}_t\|_{H^k} \leq C_k t \sup\{\|s_{t'}\|_{H^k} : 0 \leq t' \leq t\}$

Proof: Uniqueness of hom. problem $\Rightarrow$ Uniqueness of inhom. problem

Let $f(t_1,t_2) = \int_0^{t_1} e^{-(t_2-t')D^2} s_{t'} \, dt$, then:

$$\underbrace{\partial_t f(t,t)}_{\tilde{s}_t} = \partial_{t_1} f(t,t) + \partial_{t_2} f(t,t) = e^{-(t-t)D^2} s_t + \int_0^t \partial_t (e^{-(t-t')D^2} s_{t'}) \, dt' = s_t - D^2 \tilde{s}_t$$

$$\Leftrightarrow \left[\frac{\partial}{\partial t} + D^2\right] \tilde{s}_t = s_t$$

Estimate: $\|\tilde{s}_t\|_{H^k} \leq \int_0^t \|e^{-(t-t')D^2} s_{t'}\|_{H^k} dt \leq t \sup\{\|e^{-(t-t')D^2}\| \cdot \|s_{t'}\|_{H^k} : 0 \leq t' \leq t\}$

$\rightsquigarrow$ suffices to show: $\|e^{-tD^2}\|_{\mathcal{L}(H^k)} \leq C_k$

- true for $k=0$ ($e^{-t\lambda^2}$ is uniformly bounded in t)
- for $k>0$: $\|e^{-tD^2} s\|_{H^k} \leq C\left(\|D e^{-tD^2} s\|_{H^{k-1}} + \|e^{-tD^2} s\|_{H^{k-1}}\right)$ (Elliptic Estimate)

$$\leq C C_{k-1} \left(\|Ds\|_{H^{k-1}} + \|s\|_{H^{k-1}}\right) \leq C_k \|s\|_{H^k}$$ □

Prop: $\forall m \; \exists m' > m$: every approx. heat kernel of order m' satisfies

$k_t(p,q) - k_t'(p,q) = t^m e_t(p,q)$ with e_t C^m in p,q & C^0 in $t \geq 0$

Proof: $\left[\frac{\partial}{\partial t} + D_p^2\right]\left(k_t(p,q) - k_t'(p,q)\right) = -t^{m'} r_t(p,q) = \left[\frac{\partial}{\partial t} + D_p^2\right] \tilde{s}_t(p,q)$ (Duhamal)

Uniqueness of the heat kernel: $k_t(p,q) - k_t'(p,q) = \tilde{s}_t(p,q)$

Take $e_t = t^{-m} \tilde{s}_t$, to show: $e_t \in C^m$ cont. in $t \geq 0$, let $m' \geq \frac{1}{2}\dim M + m$

Sobolev-Estimate for $\tilde{s}_t$: $\|\tilde{s}_t\|_{H^{m'}} \leq C t^{m'+1}$, use Sobolev-Embedding □

3. Asymptotic expansion for the heat kernel

Def: Let E be a Banach space, let $f, a_k : \mathbb{R}^+ \longrightarrow E$ $(k \in \mathbb{N})$.

The formal series $\sum_{k=0}^{\infty} a_k(t)$ is an asymptotic expansion of f near 0

write: $f(t) \sim \sum_{k=0}^{\infty} a_k(t)$

$:\Longleftrightarrow \forall n\ \exists \ell_n\ \forall \ell \geq \ell_n\ \exists C_{\ell,n} > 0\ \forall t$, $|t|$ small: $\| f(t) - \sum_{k=0}^{\ell} a_k(t) \| < C_{\ell,n} |t|^n$

i.e. $\forall n$ the error between f and the partial sums $\sum_{k=0}^{\ell} a_k(t)$ is of order $|t|^n$ for almost all ℓ.

Observe: $k_t^J(p,q) := \sum_{j=1}^{J} a_j(t,p,q)$ approximative heat kernel of order m $\forall J \geq J_0$

$\Rightarrow h_t(p,q) \sim \sum_{j=0}^{\infty} a_j$

heat kernel on $\mathbb{R}^n$: $h_t(x,y) = (4\pi t)^{-\frac{n}{2}} \exp\left(\frac{-|x-y|^2}{4t}\right)$

$\rightsquigarrow$ first approximation of k_t: $h_t(p,q) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d(p,q)^2}{4t}\right)$

goal: find asymptotic expansion $k_t \sim h_t \sum_{j=0}^{\infty} \Theta_j t^j$

Choose geodesic coordinates (x^j) at q, let $r^2 := \sum_j (x^j)^2$ $\rightsquigarrow r =$ distance to q

$\Rightarrow h_t(p,q) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{4t}\right)$

Lemma: (Derivatives of h)

a) $\nabla h = -\frac{h}{2t} r \frac{\partial}{\partial r}$

b) $\frac{\partial h}{\partial t} + \Delta h = \frac{rh}{4gt} \frac{\partial g}{\partial r}$ with $g = \det(g_{ij})$

Lemma: Let $s \in \Gamma(S)$ and $f \in C^\infty(M)$.

a) $D(fs) - f Ds = c(\nabla f)s$ with $c =$ Clifford multiplication

b) $D^2(fs) - f D^2 s = (\Delta f)s - 2\nabla_{\nabla f} s$

Proof (Idea): Choose a synchronous orthogonal frame (e_i) and use the formula for D ($\to$ talk 5)

Theorem:

(i) $k_t(p,q) \sim h_t(p,q) \sum_{j=0}^{\infty} \Theta_j(p,q) t^j$ with $\Theta_j \in \Gamma(S \boxtimes S^*)$

(ii) expansion is valid in every $C^k(S \boxtimes S^*)$

(iii) Coefficients at diagonal $(\Theta_j(p,p))$ = algebraic expressions of metric, connection coeff. & derivatives (will write down Θ_0, Θ_1)

Proof: h_t decreases faster than any polynomial outside diagonal in $M \times M$

$\rightsquigarrow$ can work in local geodesic coordinates at q

$$\left[\frac{\partial}{\partial t} + D_p^2\right](hs) = \frac{\partial h}{\partial t} s + h\frac{\partial s}{\partial t} + hD^2 s + \Delta h s - 2\nabla_{\nabla h} s$$
$$= h\frac{r}{4gt}\frac{\partial g}{\partial r} s + h\frac{\partial s}{\partial t} + hD^2 s + \frac{h}{t}\nabla_{r\frac{\partial}{\partial r}} s$$
$$= h\left[\frac{1}{t}\left(\frac{r}{4g}\frac{\partial g}{\partial r} s + \nabla_{r\partial_r} s\right) + \frac{\partial s}{\partial t} + D^2 s\right]$$

- Let $s \sim \sum_{j=0}^{\infty} u_j t^j \Rightarrow \frac{\partial s}{\partial t} \sim \sum_{j=1}^{\infty} j u_j t^{j-1}$ (rest: linear in t)
- $\left[\frac{\partial}{\partial t} + D_p^2\right](hs) \overset{!}{=} 0$, evaluate terms of order t^{j-1}:

$$\frac{r}{4g}\frac{\partial g}{\partial r} u_j + \nabla_{r\partial_r} u_j + j u_j + D^2 u_{j-1} \overset{!}{=} 0 \quad | \cdot r^{j-1} g^{\frac{1}{4}} \quad | - r^{j-1} g^{\frac{1}{4}} D^2 u_{j-1}$$

$$\Longleftrightarrow \nabla_{r\partial_r}(r^j g^{\frac{1}{4}} u_j) \overset{!}{=} -r^{j-1} g^{\frac{1}{4}} D^2 u_{j-1}$$

$$\Longrightarrow \text{solve } \nabla_{r\partial_r}(r^j g^{\frac{1}{4}} u_j) = \begin{cases} 0 & , j=0 \\ -r^{j-1} g^{\frac{1}{4}} D^2 u_{j-1} & , j>0 \end{cases}$$

- $\underline{j=0}$: 1 initial condition: fix $u_0(0) = 1$ $\rightsquigarrow$ uniquely determines u_0
- $\underline{j>0}$ $r^j u_j$ uniquely determined by u_{j-1} up to integration const. C

  u_j smooth at $r=0$ $\rightsquigarrow C \overset{!}{=} 0$ $\rightsquigarrow u_j$ uniquely determined

Coefficients Θ_j: let $\Theta_j(p,q) := u_j(x)$ in local coordinates at q

- construction of u_j & $u_j(0) = \Theta_j(q,q)$ implies (iii) (via induction)
- $\Theta_0(p,p) = 1 \in \mathrm{Hom}(S_p, S_p)$

  $\Rightarrow k_t^J(p,q) := h_t(p,q)\sum_{j=0}^{J}\Theta_j(p,q)t^j$ tends to a δ-function as $t \to 0$

  and the construction of the u_j shows that k_t^J is an approximative heat kernel (for J large), hence $k_t \sim h_t \sum \Theta_j t^j$ □

Prop: $\Theta_0(p,p) = 1$

$\Theta_1(p,p) = \frac{1}{6}\kappa(p) - K(p)$

(κ = scalar curvature ; K = Clifford contracted curvature → (Weitzenböck formula))

4. Finite propagation speed of the wave equation

Prop: $\forall s \in \Gamma_c^\infty(S)$: $\mathrm{supp}(e^{itD}s)$ lies within distance $|t|$ of $\mathrm{supp}(s)$.
(s section with cpt supp)

Proof: Energy estimate : $t \mapsto \int_{B(m,R-t)} |s_t|^2$ is decreasing

where $s_t = e^{itD}s$ and $B(m,R)$ allows geodesic coordinates at m

$\Rightarrow$ For m outside $\mathrm{supp}(s)$ with $\mathrm{dist}(m, \mathrm{supp}(s)) \geq R$: $\int_{B(m,R)} |s|^2 = 0$

$\Rightarrow \int_{B(m,R-t)} |s_t|^2 = 0 \quad \forall\, 0 \leq t \leq R$, i.e. $s_t(m) = e^{itD}s(m) = 0$

Proof of energy estimate:

$$\frac{\partial}{\partial t} \int_{B(m,R-t)} |s_t|^2 = \int_{B(m,R-t)} \partial_t (s_t, s_t) - \int_{S(m,R-t)} |s_t|^2 \, d\underset{\text{surface area}}{\underbrace{\sigma}}$$

wave equ. : $\partial_t (s_t, s_t) = (iDs_t, s_t) + (s_t, iDs_t) = i d^* \omega$ with $\omega(X) = -(Xs_t, s_t)$

divergence thm. $\Rightarrow$ $\int_{B(m,R-t)} \frac{\partial}{\partial t} |s_t|^2 = i \int_{S(m,R-t)} \omega(N)\, d\sigma = -i \int_{S(m,R-t)} (N.s_t, s_t)\, d\sigma$ with N = unit normal to S

Cauchy-Schwartz : $\left| \int_{S(m,R-t)} (N.s_t, s_t)\, d\sigma \right|^2 \leq \int_{S(m,R-t)} \underset{|s_t|^2}{\underbrace{|N.s_t|^2}}\, d\sigma \int_{S(m,R-t)} |s_t|^2 = \left| \int_{S(m,R-t)} |s_t|^2 d\sigma \right|^2$

$\Rightarrow \frac{\partial}{\partial t} \int_{B(m,R-t)} |s_t|^2 \leq 0$ $\square$

Fourier transform

$S(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid \forall \alpha, \beta : x^\alpha \partial^\beta f \text{ bounded} \}$

$\hat{f}(\lambda) = \int f(x) e^{-i\lambda x} dx$ and $f(x) = \frac{1}{2\pi} \int \hat{f}(\lambda) e^{i\lambda x} d\lambda$

$\rightsquigarrow f(D) = \frac{1}{2\pi} \int \hat{f}(\lambda) e^{i\lambda D} d\lambda$ in the weak sense, i.e.

$\langle f(D)x, y \rangle = \frac{1}{2\pi} \int \hat{f}(\lambda) \langle e^{i\lambda D} x, y \rangle d\lambda \quad \forall x, y \in L^2(S)$ (check for eigenvalues x, y of D)

Prop: $f \in S(\mathbb{R})$ with $\mathrm{supp}(\hat{f}) \subseteq [-c; c]$

$\Rightarrow \forall x, y \in \Gamma_c(S)$ with $d(\mathrm{supp}(x), \mathrm{supp}(y)) > c$: $\langle f(D)x, y \rangle = 0$

$\Rightarrow$ smoothing kernel of $f(D)$ supported in c-neighborhood of diagonal in $M \times M$.

Proof: $\langle f(D)x, y \rangle = \frac{1}{2\pi} \int_{-c}^{c} \hat{f}(\lambda) \underset{0 \text{ (finite prop. speed)}}{\underbrace{\langle e^{i\lambda D}x, y \rangle}} d\lambda$

Prop: For $f \in S(\mathbb{R})$: the smoothing kernel of $f(uD) \xrightarrow[u\to 0]{} 0$ outside diagonal of $M \times M$.

Proof (Idea):

Take $U = 2\delta$-neighborhood of the diagonal, find decomposition (via bump function)

$f(ux) = f_1(x) + f_2(x)$ with $\operatorname{supp} f_1 \subseteq [-2\delta; 2\delta]$

and $f_2 \xrightarrow{u\to 0} 0$ in $S(\mathbb{R})$

$\Rightarrow$ outside U: $f(uD) = f_2(D) \xrightarrow{u\to 0} 0$ as smoothing kernels