PROBLEM SET 11
To be discussed: 7.02.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

Prologue: If you have not yet become comfortable with the method of acyclic models, this sheet will either help or will drive you crazy.

1. The goal of this problem is to understand the role played by 0-chains and 0-cochains in the cross and cup product. Let $\{\text{pt}\}$ denote the 1-point space and, for a path-connected space X, let $[\text{pt}] \in H_0(X) = \mathbb{Z}$ denote the canonical generator, i.e. the homology class represented by $\langle x \rangle \in C_0(X)$ for any point $x \in X$ (regarded as a singular 0-simplex). Recall moreover that any coefficient group G has a canonical inclusion $G \hookrightarrow H^0(X; G)$.

   (a) Show that if Y is path-connected, the cross product of any $A \in H_n(X)$ with $[\text{pt}] \in H_0(Y)$ is $A \times [\text{pt}] = i_* A$ for any inclusion map of the form $i : X \hookrightarrow X \times Y : x \mapsto (x, \text{const})$. A similar formula holds for cross products with $[\text{pt}] \in H_0(X)$ if X is path-connected.
   Hint: Remember that $\times$ is induced by a natural chain map $\Phi : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$, so if you have the right formula for $\Phi : C_n(X) \otimes C_0(Y) \to C_n(X \times Y)$, the relation will become obvious. In general, one can make many choices in defining Φ, but there is an obvious choice that one "should" make when one of the chains is 0-dimensional. Review the construction of Φ via acyclic models to show that this choice is always possible.

   (b) Suppose Ψ associates to every space X a chain map $\Psi : C_*(X) \to C_*(X)$. We will say that Ψ is a **natural chain map** $C_*(X) \to C_*(X)$ if it acts as the identity map on 0-chains and for every continuous map $f : X \to Y$, $\Psi \circ f_* = f_* \circ \Psi$. Use the method of acyclic models to show that any two choices of natural chain maps in this sense are chain homotopic for all X.

   (c) Identify the chain complex $C_*(X \times \{\text{pt}\})$ with $C_*(X)$ via the obvious canonical isomorphism between them, and consider the following two maps:

$$C_*(X \times \{\text{pt}\}) \xrightarrow{\theta} C_*(X) \otimes C_*(\{\text{pt}\}) \xrightarrow{\mathbb{1} \otimes \epsilon} C_*(X) \otimes \mathbb{Z} = C_*(X),$$
$$C_*(X \times \{\text{pt}\}) \xrightarrow{(\pi_X)_*} C_*(X),$$

   where $\pi_X : X \times \{\text{pt}\} \to X$ is the canonical projection, θ is any natural chain homotopy inverse for the natural chain map $\Phi : C_*(X) \otimes C_*(\{\text{pt}\}) \to C_*(X \times \{\text{pt}\})$ as used in the construction of the cross product, and $\epsilon : C_*(\{\text{pt}\}) \to \mathbb{Z}$ is the *augmentation* map, which vanishes on $C_n(\{\text{pt}\})$ for $n \neq 0$ and sends the generator $\langle \sigma \rangle \in C_0(\{\text{pt}\})$ to 1. Verify that both of these define natural chain maps, hence by part (b), they are chain homotopic.

   (d) Fix a commutative ring R with unit (denoted by $1 \in R$), and deduce from part (c) that for any space X, the cross product of $\alpha \in H^*(X; R)$ with $1 \in R \subset H^0(\{\text{pt}\}; R)$ satisfies $\alpha \times 1 = \pi_X^* \alpha$.

   (e) Prove the naturality formula $(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta$ for any continuous maps $f : X \to X'$ and $g : Y \to Y'$ with $\alpha \in H^*(X'; R)$ and $\beta \in H^*(Y'; R)$. (This is an easy consequence of the definition of the cohomology cross product.) Now use it to deduce from part (d) that the cross product of any $\alpha \in H^*(X; R)$ with $1 \in R \subset H^0(Y; R)$ satisfies $\alpha \times 1 = \pi_X^* \alpha$, where $\pi_X : X \times Y \to X$ is the projection. (Similarly, $1 \times \beta = \pi_Y^* \beta$ for $1 \in R \subset H^0(X; R)$ and $\beta \in H^*(Y; R)$.)

   (f) Deduce from the above that for any space X, $1 \in R \subset H^0(X; R)$ acts as the unit with respect to the cup product: $\alpha \cup 1 = \alpha = 1 \cup \alpha$.

2. Let's prove that the cross product and cup products are associative.

   (a) Show that for triples of spaces X, Y, Z, all natural chain maps $\Phi : C_*(X) \otimes C_*(Y) \otimes C_*(Z) \to C_*(X \times Y \times Z)$ that act on 0-chains by $\Phi(\langle x \rangle \otimes \langle y \rangle \otimes \langle z \rangle) = \langle (x, y, z) \rangle$ are chain homotopic. Here, "natural" means that for any triple of continuous maps $f : X \to X'$, $g : Y \to Y'$ and $h : Z \to Z'$, $\Phi \circ (f_* \otimes g_* \otimes h_*) = (f \times g \times h)_* \circ \Phi$.
   Remark: The statement implicitly assumes that there is a well-defined notion of the tensor product of three chain complexes, which of course is true since there is a canonical chain isomorphism between $(C_*(X) \otimes C_*(Y)) \otimes C_*(Z)$ and $C_*(X) \otimes (C_*(Y) \otimes C_*(Z))$. Right?

   (b) Given $A \in H_*(X)$, $B \in H_*(Y)$ and $C \in H_*(Z)$, show that the products $(A \times B) \times C$ and $A \times (B \times C) \in H_*(X \times Y \times Z)$ can each be expressed via natural chain maps as in part (a), and conclude that they are identical.

   (c) Convince yourself that the natural chain maps in part (a) also have natural chain homotopy inverses. (You probably won't want to work through the proof in full detail, but once you've started it, it should not be hard to see how the rest goes.)

   (d) Prove that $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma) \in H^*(X \times Y \times Z; R)$ for any $\alpha \in H^*(X; R)$, $\beta \in H^*(Y; R)$ and $\gamma \in H^*(Z; R)$.
   Hint: Try to express each as the composition of $\alpha \otimes \beta \otimes \gamma$ with some natural chain map.

   (e) Prove that for $\alpha, \beta, \gamma \in H^*(X; R)$, $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.

3. We now have enough information to compute the cup product on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ for any (commutative and unital) coefficient ring R. Our main tools for this purpose will be the Künneth formula, the universal coefficient theorem, and the following two relations:

$$\langle \alpha \times \beta, A \times B \rangle = (-1)^{k\ell} \langle \alpha, A \rangle \langle \beta, B \rangle \quad \text{for} \quad \alpha \in H^k(X; R),\ A \in H_k(X),\ \beta \in H^\ell(Y; R),\ B \in H_\ell(Y),$$

   and

$$\pi_X^* \alpha \cup \pi_Y^* \beta = \alpha \times \beta \quad \text{for} \quad \alpha \in H^*(X; R),\ \beta \in H^*(Y; R),$$

   where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the canonical projections.

   Let us identify S^1 with $\mathbb{R}/\mathbb{Z}$ so that we can write $\mathbb{T}^n = S^1 \times \ldots \times S^1$. All the homology groups that appear below will turn out to be free, hence the Ext terms in the universal coefficient theorem vanish and we will always have canonical isomorphisms $H^k(X; R) \to \operatorname{Hom}(H_k(X), R)$. With this in mind, fix a generator $[S^1] \in H_1(S^1)$ and let $\lambda \in H^1(S^1; R)$ denote the unique element such that $\langle \lambda, [S^1] \rangle = 1$.

   Now for each $j = 1, \ldots, n$, the projection $\pi_j : \mathbb{T}^n \to S^1 : (x_1, \ldots, x_n) \mapsto x_j$ defines a cohomology class

$$\lambda_j := \pi_j^* \lambda \in H^1(\mathbb{T}^n; R) \quad \text{for} \quad j = 1, \ldots, n.$$

   (a) Use the Künneth formula to show that for each $k = 1, \ldots, n$, $H_k(\mathbb{T}^n)$ is a free abelian group of rank $\binom{n}{k}$ with a basis consisting of products $A_1 \times \ldots \times A_n$ where exactly k of the A_j are $[S^1] \in H_1(S^1)$ and the rest are $[\mathrm{pt}] \in H_0(S^1)$.
   Hint: I suggest induction on n.

   (b) Deduce from part (a) that $H^k(\mathbb{T}^n; R)$ is similarly a free R-module with a basis consisting of the $\binom{n}{k}$ elements

$$\lambda_{j_1, \ldots, j_k} := \alpha_1 \times \ldots \times \alpha_n \in H^k(\mathbb{T}^n; R)$$

   for $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$, where $\alpha_{j_i} = \lambda \in H^1(S^1; R)$ for each $i = 1, \ldots, k$ and the rest of the α_j are all $1 \in H^0(S^1; R)$.

   (c) Prove that for all k-tuples of integers $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$,

$$\lambda_{j_1} \cup \ldots \cup \lambda_{j_k} = \pm \lambda_{j_1, \ldots, j_k}.$$

   Hint: The statement is trivial for $n = 1$, so try induction on n. It might make your life easier to assume $j_k = n$; this is just a matter of bookkeeping, and if you're not too worried about signs, it is not a loss of generality.

Comment: In light of the relation $\lambda_i \cup \lambda_j = -\lambda_j \cup \lambda_i$, what we've proven here is that the ring $H^*(\mathbb{T}^n; R)$ is isomorphic to the ***exterior algebra*** *$\Lambda_R[\lambda_1, \ldots, \lambda_n]$ on n generators of degree 1.*

4. The **smash product** $X \wedge Y$ of two spaces is defined by choosing base points $x_0 \in X$ and $y_0 \in Y$, and then writing the quotient

$$X \wedge Y := (X \times Y) \big/ ((\{x_0\} \times Y) \cup (X \times \{y_0\})).$$

Let's not worry about the extent to which this depends on the choice of base points, as we are only going to consider examples in which it clearly does not depend. Notice that the subset being quotiented out is homeomorphic to the wedge sum $X \vee Y$, so it is sensible to write $X \wedge Y = X \times Y \big/ X \vee Y$. It is straightforward to check that for any base-point preserving continuous maps $f : (X, x_0) \to (X', x_0')$ and $g : (Y, y_0) \to (Y', y_0')$, the product map $f \times g : X \times Y \to X' \times Y'$ descends to the quotient as a continuous map

$$f \wedge g : X \wedge Y \to X' \wedge Y'.$$

Here is the most important example:

(a) Show that $S^k \wedge S^\ell$ is homeomorphic to $S^{k+\ell}$.
Hint: Think of S^k as $\mathbb{D}^k / \partial\mathbb{D}^k$ with the boundary as the base point.

Now assume X and Y are both CW-complexes, with base points chosen to be 0-cells in their cell decompositions, so the cross product and the Künneth formula are valid for the pairs $(X, \{x_0\})$ and $(Y, \{y_0\})$. Since $(X, \{x_0\}) \times (Y, \{y_0\}) = (X \times Y, X \vee Y)$, the Künneth formula now takes the form

$$0 \to \bigoplus_{k+\ell=n} H_k(X, \{x_0\}) \otimes H_\ell(Y, \{y_0\}) \xrightarrow{\times} H_n(X \times Y, X \vee Y) \longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}(H_k(X, \{x_0\}), H_\ell(Y, \{y_0\})) \to 0,$$

or under the natural isomorphisms $H_*(X, A) = \widetilde{H}_*(X/A)$ for good pairs,

$$0 \to \bigoplus_{k+\ell=n} \widetilde{H}_k(X) \otimes \widetilde{H}_\ell(Y) \xrightarrow{\times} \widetilde{H}_n(X \wedge Y) \longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}(\widetilde{H}_k(X), \widetilde{H}_\ell(Y)) \to 0.$$

(b) Show that for the cross product on reduced homology as described above and the identification of $S^k \wedge S^\ell$ with $S^{k+\ell}$ as indicated in part (a), if $[S^k] \in \widetilde{H}_k(S^k)$ and $[S^\ell] \in \widetilde{H}_\ell(S^\ell)$ are generators, then $[S^k] \times [S^\ell] \in \widetilde{H}_{k+\ell}(S^{k+\ell})$ is also a generator.

(c) Prove that for any two base-point preserving maps $f : S^k \to S^k$ and $g : S^\ell \to S^\ell$, $\deg(f \wedge g) = \deg(f) \cdot \deg(g)$.
Hint: You need the naturality of the Künneth formula for this.

(d) Find an alternative proof of the formula in part (c) using the following fact from differential topology: any continuous map $f : S^k \to S^k$ admits a small perturbation to a smooth map such that for almost every point $x \in S^k$, $f^{-1}(x)$ is a finite set of points at which the local degree of f is ± 1. (The latter is an immediate consequence of Sard's theorem.)

(e) Using the definition of cellular chain maps and the cellular cross product, you are now in a position to justify a claim that was stated but not proved in lecture: if $f : X \to X'$ and $g : Y \to Y'$ are cellular maps, then the diagram

$$\begin{array}{ccc} C_*^{\mathrm{CW}}(X) \otimes C_*^{\mathrm{CW}}(Y) & \xrightarrow{\times} & C_*^{\mathrm{CW}}(X \times Y) \\ \downarrow{\scriptstyle f_* \otimes g_*} & & \downarrow{\scriptstyle (f \times g)_*} \\ C_*^{\mathrm{CW}}(X') \otimes C_*^{\mathrm{CW}}(Y') & \xrightarrow{\times} & C_*^{\mathrm{CW}}(X' \times Y') \end{array}$$

commutes.

5. One can write down convenient chain-level formulas for the cup product using the notion of a **diagonal approximation**. The latter means an assignment to every space X of a chain map $\Psi : C_*(X) \to C_*(X) \otimes C_*(X)$ that satisfies $\Psi\langle x\rangle = \langle (x,x)\rangle$ for all $\langle x\rangle \in C_0(X)$ and is natural in the sense that for every continuous map $f : X \to Y$, $\Psi \circ f_* = (f_* \otimes f_*) \circ \Psi$. An example appears in the definition of the cup product: if $\alpha \in C^k(X;R)$ and $\beta \in C^\ell(X;R)$ are cocycles and $d : X \to X \times X$ denotes the diagonal map, then $[\alpha] \cup [\beta] \in H^{k+\ell}(X;R)$ is represented by the cocycle

$$\alpha \cup \beta := d^*(\alpha \times \beta) = (\alpha \times \beta) \circ d_* = (\alpha \otimes \beta) \circ (\theta \circ d_*),$$

where $d_* : C_*(X) \to C_*(X \times X)$ and $\theta : C_*(X \times X) \to C_*(X) \otimes C_*(X)$ are each natural chain maps, hence $\theta \circ d_*$ is a diagonal approximation.

(a) Show via an acyclic models argument that all diagonal approximations are chain homotopic. Deduce from this that $[\alpha] \cup [\beta]$ can also be represented by a cocycle of the form $\alpha \cup \beta := (\alpha \otimes \beta) \circ \Psi \in C^{k+\ell}(X;R)$ where Ψ is *any* choice of diagonal approximation, and that for any $\alpha \in C^k(X;R)$ and $\beta \in C^\ell(X;R)$, $\alpha \cup \beta$ then satisfies

$$\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta\beta.$$

The most popular choice of Ψ in the literature is known as the **Alexander-Whitney** diagonal approximation, and is defined as follows. Number the vertices of the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ as $0, \dots, n$, and given any integers $0 \leqslant j_0 < j_1 < \dots < j_k \leqslant n$, let

$$[j_0, \dots, j_k] \subset \Delta^n$$

denote the k-simplex spanned by the vertices $j_0, \dots, j_k$, which is identified naturally with the standard k-simplex. For instance, in this notation, the jth boundary face of Δ^n is $\partial_{(j)}\Delta^n = [0, \dots, j-1, j+1, \dots, n]$ for each $j = 0, \dots, n$. Now define $\Psi : C_*(X) \to C_*(X) \otimes C_*(X)$ on each singular n-simplex $\sigma : \Delta^n \to X$ by

$$\Psi\langle\sigma\rangle := \sum_{k+\ell=n} \langle \sigma|_{[0,\dots,k]}\rangle \otimes \langle \sigma|_{[k,\dots,n]}\rangle$$

(b) Verify that Ψ as defined above is a diagonal approximation.

Plugging the Alexander-Whitney approximation into $\alpha \cup \beta = (\alpha \otimes \beta) \circ \Psi$ gives the following formula for the cup product of cochains: for any singular n-simplex $\sigma : \Delta^n \to X$ with $n = k + \ell$,[1]

$$(\alpha \cup \beta)\langle\sigma\rangle = (-1)^{k\ell} \alpha(\langle \sigma|_{[0,\dots,k]}\rangle) \beta(\langle \sigma|_{[k,\dots,n]}\rangle).$$

On its own, this formula is seldom very useful since explicit computations with singular cochains are almost never practical. What is slightly more reasonable, however, is to use the same formula for computing the cup product in the simplicial *cohomology of a simplicial complex, which of course is a special case of cellular cohomology and is therefore isomorphic to its singular cohomology. This trick is sometimes used for explicit computations of singular cohomology rings; see for instance Examples 3.7 and 3.8 in Hatcher, or Example 4.6 in Chapter VI of Bredon).*

[1] The formula we have derived here for the cochain $\alpha \cup \beta$ matches a formula in Bredon, but differs from Hatcher by a sign if k and ℓ are both odd. I'm sorry, I don't know why.