PROBLEM SET 12
To be discussed: 14.02.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

1. In Problem Set 2 #4, we established that for any closed topological n-manifold M with an **oriented triangulation**, one can define a homology class in the form $[M] = [\sum_i \epsilon_i \langle \sigma_i \rangle] \in H_n(M;\mathbb{Z})$, where the $\sigma_i : \Delta^n \to M$ are suitable parametrizations of the oriented simplices in the triangulation, and the signs $\epsilon_i \in \{1,-1\}$ are determined by the orientations so that $\sum_i \epsilon_i \langle \sigma_i \rangle$ is a cycle. At the time, we called $[M]$ a "fundamental class," but we did not actually prove that it is independent of the triangulation, nor that it is nontrivial. Now we can.
   (a) Prove that the class $[M] \in H_n(M;\mathbb{Z})$ described above in terms of a triangulation agrees with the general notion of a fundamental class for a closed topological n-manifold: for any point $x \in M$, the map induced by the inclusion $(M,\varnothing) \hookrightarrow (M, M\backslash\{x\})$ sends $[M]$ to a generator (i.e. a "local orientation") $[M]_x \in H_n(M, M\backslash\{x\};\mathbb{Z}) \cong \mathbb{Z}$.
   Hint: It suffices (why?) to consider only points $x \in M$ belonging to some dense subset, e.g. the points that are in the interiors of n-simplices in the triangulation. Compare Problem Set 5 #2(a).
   (b) Adapt the discussion for arbitrary (not necessarily oriented) triangulations using homology with $\mathbb{Z}_2$ coefficients.
   (c) Adapt the discussion for a compact triangulated n-manifold with boundary as in Problem Set 2 #4(e), i.e. show that the relatative class $[M] \in H_n(M, \partial M)$ (with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$) described there maps to a generator of $H_n(M, M\backslash\{x\})$ for any point $x \in M\backslash\partial M$ (cf. #4 on this sheet).
2. Prove that if M is a non-orientable connected topological manifold, then $\pi_1(M)$ contains a subgroup of index 2. (In particular, this implies that every simply connected manifold is orientable.)
3. Suppose M is any topological manifold of dimension $n \in \mathbb{N}$.
   (a) Prove that the torsion subgroup of $H_{n-1}(M)$ is $\mathbb{Z}_2$ if M is compact and non-orientable, and it is otherwise trivial.
   Hint: Use the universal coefficient theorem to compute $\mathrm{Tor}(H_{n-1}(M), \mathbb{Z}_p) = 0$ for every prime number p, and see what you can deduce from it. You may want to consider separately the cases where M is noncompact, compact and orientable, or compact and non-orientable. If it helps, feel free to assume also that $H_*(M)$ is finitely generated (though this is not strictly necessary).
   (b) Deduce that if $H_*(M)$ is finitely generated and M is orientable, then $H^n(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z})$.
4. In this problem, assume M is a topological n-manifold with boundary (see Problem Set 10 #4 for the definition). The interior $\mathring{M} = M\backslash\partial M$ is then an n-manifold without boundary, so given a commutative ring with unit R (typically $\mathbb{Z}$ or $\mathbb{Z}_2$), we can define an ***R*-orientation of** M to mean simply an R-orientation of $\mathring{M}$. We will need the following basic observation from point-set topology: if ∂M is compact, then it has a so-called **collar neighborhood** in M, meaning a neighborhood $\mathcal{U} \subset M$ of ∂M that is homeomorphic to $(-1,0] \times \partial M$ via a homeomorphism sending ∂M to $\{0\} \times \partial M$. This is not completely obvious, but the proof is not hard (see e.g. Hatcher, Proposition 3.42). It follows that M is homotopy equivalent to its interior, hence the latter has finitely generated homology if M is compact.
   Since $\mathring{M}$ is a manifold, a theorem we proved in lecture gives an isomorphism
   $$J_A : H_n(\mathring{M}, \mathring{M}\backslash A; G) \to \Gamma_c(\Theta^G|_A)$$

for any closed subset $A \subset \mathring{M}$ and abelian group G, where $\Theta^G = \bigcup_{x \in M} \Theta^G_x$ denotes the orientation bundle with fibers $\Theta^G_x = H_n(M, M \backslash \{x\}; G) \cong G$ for $x \in \mathring{M}$, and $\Gamma_c(\Theta^G|_A)$ is its group of compactly supported sections along A. Now set $G = R$ and assume M has an R-orientation, so a generator $[M]_x \in H_n(M, M\backslash\{x\}; R) \cong R$ is fixed for every $x \in \mathring{M}$. We will refer to a relative homology class

$$[M] \in H_n(M, \partial M; R)$$

as a **relative fundamental class** for M if the natural map $i_x : H_n(M, \partial M; R) \to H_n(M, M\backslash\{x\}; R)$ defined via the inclusion $(M, \partial M) \hookrightarrow (M, M\backslash\{x\})$ for every $x \in \mathring{M}$ sends $[M]$ to $[M]_x$. Notice that if $\partial M = \varnothing$, this matches our previous definition of fundamental classes for closed manifolds.

(a) Prove that if M is compact and an R-orientation of M is fixed, then a relative fundamental class $[M] \in H_n(M, \partial M; R)$ exists, is unique, and generates $H_n(M, \partial M; R) \cong R$.

(b) Under the same assumptions, show that if M and ∂M are both connected and ∂M is nonempty, then ∂M is also R-orientable, and the connecting homomorphism $\partial_* : H_n(M, \partial M; R) \to H_{n-1}(\partial M; R)$ in the long exact sequence of $(M, \partial M)$ is an isomorphism sending $[M]$ to a fundamental class $[\partial M]$ of ∂M (for a suitable choice of orientation of ∂M).
Hint: Focus on the case $R = \mathbb{Z}$. It is easy to prove that ∂_* is injective; show that if it were not surjective, then $H_{n-1}(M)$ would have torsion, contradicting the result of Problem 3(a).

(c) Generalize the result of part (b) to prove $\partial_*[M] = [\partial M]$ without assuming ∂M is connected.
Hint: For any connected component $N \subset \partial M$, consider the exact sequence of the triple $(M, \partial M, \partial M\backslash N)$ and notice that $H_{n-1}(\partial M, \partial M\backslash N) \cong H_{n-1}(N)$ by excision.

(d) Conclude that for any compact manifold M with boundary and an R-orientation, the map $H_{n-1}(\partial M; R) \to H_{n-1}(M; R)$ induced by the inclusion $\partial M \hookrightarrow M$ sends $[\partial M]$ to 0. In other words, "the boundary of a compact oriented n-manifold M represents the trivial homology class in $H_{n-1}(M)$."
Remark: Compare Problem Set 2 #4(g), which proved essentially the same thing in the presence of oriented triangulations.

5. In lecture we defined the **compactly supported cohomology** $H^*_c(X)$ of a space X via the direct limit

$$H^k_c(X; G) := \varinjlim \left\{ H^k(X, X\backslash K; G) \right\}_K$$

where K ranges over the set of all compact subsets of X, ordered by inclusion $K \subset K' \subset X$ and forming a direct system via the maps $H^k(X, X\backslash K; G) \to H^k(X, X\backslash K'; G)$ induced by inclusions $(X, X\backslash K') \hookrightarrow (X, X\backslash K)$.

(a) Show that there is a canonical isomorphism $H^*(X) = H^*_c(X)$ whenever X is compact.

(b) Prove that $H^n_c(\mathbb{R}^n; G) \cong G$ and $H^k_c(\mathbb{R}^n; G) = 0$ for all $k \neq n$.

(c) Construct a canonical isomorphism between $H^*_c(X; G)$ and the homology of the subcomplex $C^*_c(X; G) \subset C^*(X; G)$ consisting of every cochain $\lambda : C_k(X) \to G$ that vanishes on all simplices with images outside some compact subset $K \subset X$. (Note that K may depend on λ).

(d) Recall that a continuous map $f : X \to Y$ is called **proper** if for every compact set $K \subset Y$, $f^{-1}(K) \subset X$ is also compact. Show that proper maps $f : X \to Y$ induce homomorphisms $f^* : H^*_c(Y; G) \to H^*_c(X; G)$, making $H^*_c(\cdot; G)$ into a contravariant functor on the category of topological spaces with morphisms defined as proper maps.

(e) Deduce from part (d) that $H^*_c(\cdot; G)$ is a topological invariant, i.e. $H^*_c(X; G)$ and $H^*_c(Y; G)$ are isomorphic whenever X and Y are homeomorphic. Give an example showing that this need not be true if X and Y are only homotopy equivalent.

(f) In contrast to part (d), show that $H^*_c(\cdot; G)$ does *not* define a functor on the usual category of topological spaces with morphisms defined to be continuous (but not necessarily proper) maps.
Hint: Think about maps between $\mathbb{R}^n$ and the one-point space.

(g) We say that two proper maps $f, g : X \to Y$ are **properly homotopic** if there exists a homotopy $h : I \times X \to Y$ between them that is also a proper map. Show that under this assumption, the induced maps $f^*, g^* : H_c^*(Y; G) \to H_c^*(X; G)$ in part (d) are identical if X is Hausdorff and locally compact. In other words, $H_c^*(\cdot; G)$ defines a contravariant functor on the category whose objects are locally compact Hausdorff spaces and whose morphisms are proper homotopy classes of proper maps.
Hint: You might first want to remind yourself how one proves the homotopy axiom for $H^*(\cdot; G)$.[1] *It will then help to show that every compact subset $K \subset I \times X$ is contained in some set of the form $I \times K'$ for a compact subset $K' \subset X$. You will find a helpful lemma for this in Problem Set 3 #6 of last semester's Topologie I class.*

6. Let's prove a couple of results that have been used often in recent lectures but were skipped earlier in the semester. Suppose $A \subset X \subset Z$ and $B \subset Y \subset Z$ are subsets of a topological space Z, with the property that the chain maps $C_*(A) + C_*(B) \hookrightarrow C_*(A \cup B)$ and $C_*(X) + C_*(Y) \hookrightarrow C_*(X \cup Y)$ defined by inclusion all induce isomorphisms on homology. We've seen for instance that (by a subdivision argument) this is true whenever all of the sets involved are open.

   (a) Show that under the assumptions above, the induced map of quotients
   $$\frac{C_*(X) + C_*(Y)}{C_*(A) + C_*(B)} \to \frac{C_*(X \cup Y)}{C_*(A \cup B)} = C_*(X \cup Y, A \cup B)$$
   also descends to an isomorphism on homology.
   Hint: Compare the obvious short exact sequence $0 \to C_*(A) + C_*(B) \hookrightarrow C_*(X) + C_*(Y) \to \frac{C_*(X)+C_*(Y)}{C_*(A)+C_*(B)} \to 0$ with the short exact sequence of chain complexes for the pair $(X \cup Y, A \cup B)$. There is a natural morphism of short exact sequences from the first to the second, which therefore gives a morphism between the corresponding long exact sequences of homology groups. Write this down, then use the 5-lemma.

   (b) Show that the obvious sequence
   $$0 \longrightarrow C_*(X \cap Y, A \cap B) \xrightarrow{\alpha} C_*(X, A) \oplus C_*(Y, B) \xrightarrow{\beta} \frac{C_*(X) + C_*(Y)}{C_*(A) + C_*(B)} \longrightarrow 0$$
   defined by $\alpha([c]) = ([c], -[c])$ and $\beta([x], [y]) = [x + y]$ is a short exact sequence of chain complexes.[2] Using part (a), the induced long exact sequence is then the **relative Mayer-Vietoris** sequence[3]
   $$\ldots \to H_k(X \cap Y, A \cap B) \to H_k(X, A) \oplus H_k(Y, B) \to H_k(X \cup Y, A \cup B) \to H_{k-1}(X \cap Y, A \cap B) \to \ldots$$

   (c) Dualize this whole discussion to show that the cohomology of the chain complex $(C_*(X) + C_*(Y))/(C_*(A) + C_*(B))$ is naturally isomorphic to $H^*(X \cup Y, A \cup B)$, and there is a relative Mayer-Vietoris sequence
   $$\ldots \to H^k(X \cup Y, A \cup B) \to H^k(X, A) \oplus H^k(Y, B) \to H^k(X \cap Y, A \cap B) \to H^{k+1}(X \cup Y, A \cup B) \to \ldots$$

7. Last week (Problem Set 11 #3) we computed the ring structure of $H^*(\mathbb{T}^n)$ by exploiting the relationship between the cup and cross products on cohomology. This method suffices for a limited range of concrete examples, but if we are willing to restrict our attention to smooth manifolds, then Poincaré duality provides a much more powerful and geometrically revealing way to compute cup products. For a closed,

[1] ...which should in any case be good preparation for the final exam!

[2] Hatcher gives a clever proof of this on page 152, but it looks to me like a much more straightforward argument would suffice. I would be curious to see if any of you disagree. A question was asked about this on mathstackexchange about a year ago (see `https://math.stackexchange.com/questions/2156870/on-the-proof-of-mayer-vietoris-sequence-for-relative-homology-groups/21753`
but it wasn't satisfactorily answered.

[3] As you might guess, some version of this sequence can also be proved for arbitrary axiomatic homology theories, using something that I can only describe as "diagram-chase wizardry". If you really want to know the details, see §I.15 of the book by Eilenberg and Steenrod.

connected and oriented manifold M of dimension n with fundamental class $[M] \in H_n(M)$, denote the Poincaré duality isomorphism (with integer coefficients) by

$$\mathrm{PD} : H^k(M) \to H_{n-k}(M) : \alpha \mapsto \alpha \cap [M].$$

The **intersection product** on $H_*(M)$ is defined as the Poincaré dual of the cup product: for $A \in H_{n-k}(M)$ and $B \in H_{n-\ell}(M)$, define $A \cdot B \in H_{n-(k+\ell)}(M)$ by the relation

$$\mathrm{PD}^{-1}(A \cdot B) = \mathrm{PD}^{-1}(B) \cup \mathrm{PD}^{-1}(A).$$

Let us now take as a black box the following fact from intersection theory:[4] if M is a smooth manifold and A and B are closed and oriented smooth submanifolds that intersect each other transversely (so that $A \cap B$ is also a closed and oriented smooth submanifold), then

$$[A] \cdot [B] = [A \cap B].$$

We say that A and B are of **complementary dimension** if $\dim A + \dim B = n$, in which case the transverse intersection $A \cap B$ is a compact oriented 0-manifold, hence a finite set of points with signs attached. The signed count of these points then matches $[A] \cdot [B] \in H_0(M)$ under the canonical isomorphism $H_0(M) \to \mathbb{Z} : c \mapsto \langle 1, c \rangle$ defined by evaluating the unit $1 \in H^0(M)$, and it is conventional in this case to regard $[A] \cdot [B]$ as an integer instead of an element of $H_0(M)$.

(a) Use the standard relations between the cup and cap products (see the cheat sheet below) to derive the formula

$$\langle \alpha, c \rangle = c \cdot \mathrm{PD}(\alpha) \in \mathbb{Z}$$

for $\alpha \in H^k(M)$ and $c \in H_k(M)$.
Remark: In the absence of torsion, you can use this relation to fully characterize any cohomology class in terms of the intersections of its Poincaré dual class with other homology classes.

(b) Now consider $\mathbb{T}^n = (S^1)^{\times n} = \mathbb{R}^n/\mathbb{Z}^n$, with S^1 identified with $\mathbb{R}/\mathbb{Z}$, and for each $k = 1, \ldots, n$ and tuple of integers $1 \leqslant j_1 < \ldots < j_k \leqslant n$, define the submanifold

$$\mathbb{T}^n_{j_1,\ldots,j_k} = \{(x_1, \ldots, x_n) \in \mathbb{T}^n \mid x_{j_1} = \ldots = x_{j_k} = [0]\}.$$

This is a smooth submanifold diffeomorphic to $\mathbb{T}^{n-k}$, and after choosing an orientation (we shall not worry here about signs), it represents a homology class $[\mathbb{T}^n_{j_1,\ldots,j_k}] \in H_{n-k}(\mathbb{T}^n)$. Deduce from Problem Set 11 #3(a) that for each fixed $k = 1, \ldots, n$, the collection of all submanifolds of this form freely generates $H_{n-k}(\mathbb{T}^n)$, and that (in the notation of that problem), $[\mathbb{T}^n_{j_1,\ldots,j_k}]$ is Poincaré dual to the cohomology class $\lambda_{j_1,\ldots,j_k} \in H^k(\mathbb{T}^n)$, up to a sign.

(c) Use the intersection product to reprove the formula

$$\lambda_{j_1} \cup \ldots \cup \lambda_{j_k} = \pm\lambda_{j_1,\ldots,j_k}.$$

Cap product cheat sheet

For easy reference, here are the most important properties of the cap product $\cap : H^k(X; R) \otimes H_\ell(X; R) \to H_{\ell-k}(X; R)$. They are all straightforward to derive from the definition of $\cap$ in terms of diagonal approximations, see e.g. §VI.5 in Bredon. Each property also holds for the relative cap product $\cap : H^k(X, A; R) \otimes H_\ell(X, A \cup B; R) \to H_{\ell-k}(X, B; R)$ whenever it makes sense. The coefficient group R is assumed to be any commutative ring with unit.

- $1 \cap c = c$ for all $c \in H_*(X; R)$
- $\langle \alpha, c \rangle = \langle 1, \alpha \cap c \rangle$ for $\alpha \in H^k(X; R)$ and $c \in H_k(X; R)$
- $(\alpha \cup \beta) \cap c = \alpha \cap (\beta \cap c)$ for all $\alpha, \beta \in H^*(X; R)$ and $c \in H_*(X; R)$
- $f_*(f^*\alpha \cap c) = \alpha \cap f_*c$ for any continuous map $f : X \to Y$, $\alpha \in H^*(Y; R)$ and $c \in H_*(X; R)$

[4] to be discussed in lecture next week