Topology II — Humboldt-Universität zu Berlin
C. Wendl — Winter Semester 2017–18

PROBLEM SET 4
To be discussed: 15.11.2017

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

1. Recall that the **reduced homology** $\widetilde{H}_*(X;G)$ of a space X with coefficient group G is defined as the kernel of the *augmentation* homomorphism $\epsilon_* : H_*(X;G) \to H_*(P;G)$, where P is the one-point space and $\epsilon : X \to P$ is the unique map. For nonempty subsets $A \subset X$, we define $\widetilde{H}_*(X,A) = H_*(X,A)$. With these definitions in place, show that the usual long exact sequence $\ldots \to H_{k+1}(X,A) \stackrel{\partial_*}{\to} H_k(A) \to H_k(X) \to H_k(X,A) \stackrel{\partial_*}{\to} H_{k-1}(A) \to \ldots$ restricts to reduced homology as a long exact sequence

$$\ldots \to \widetilde{H}_{k+1}(X,A) \stackrel{\partial_*}{\to} \widetilde{H}_k(A) \to \widetilde{H}_k(X) \to \widetilde{H}_k(X,A) \stackrel{\partial_*}{\to} \widetilde{H}_{k-1}(A) \to \ldots$$

Hint: If $A = \varnothing$, there is nothing to prove. (Why not?) Otherwise, put both sequences into a diagram together with the long exact sequence of the pair (P,P). What is $H_*(P,P;G)$?

2. What would the week be without another diagram-chasing exercise? Suppose the following diagram commutes and that both of its rows are exact, meaning $\operatorname{im} f = \ker g$, $\operatorname{im} g' = \ker h'$ and so forth:

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E' \end{array}$$

   (a) Prove that if α, β, δ and ε are all isomorphisms, then so is γ.
   This result is known as the ***five-lemma****. It will be a useful tool to have at our disposal in the future.*

   (b) You may notice that for proving γ is an isomorphism, it suffices to have slightly weaker hypotheses than those stated in part (a). State a more general version of the lemma with weaker hypotheses.

   (c) Here is an application: given a map of pairs $f : (X,A) \to (Y,B)$, show that if any two of the induced maps $H_k(X;G) \stackrel{f_*}{\to} H_k(Y;G)$, $H_k(A;G) \stackrel{f_*}{\to} H_k(B;G)$ and $H_k(X,A;G) \stackrel{f_*}{\to} H_k(Y,B;G)$ are isomorphisms for every k, then so is the third.
   Hint: Use the functoriality of exact sequences (cf. Problem Set 2 #3).

   (d) In Problem Set 3 #2, we used excision and long exact sequences to prove that for two disjoint spaces X and Y, the inclusions $i^X : X \hookrightarrow X \amalg Y$ and $i^Y : Y \hookrightarrow X \amalg Y$ induce an isomorphism $i^X_* + i^Y_* : H_*(X;G) \oplus H_*(Y;G) \to H_*(X \amalg Y;G)$. Use the five-lemma to deduce from this that the same result holds for two disjoint pairs (X,A) and (Y,B), i.e. the map

$$i^X_* + i^Y_* : H_*(X,A;G) \oplus H_*(Y,B;G) \to H_*(X \amalg Y, A \amalg B;G)$$

   is an isomorphism.

3. Use Mayer-Vietoris sequences to compute $H_*(X;\mathbb{Z})$ and $H_*(X;\mathbb{Z}_2)$, where X is

   (a) The projective plane $\mathbb{RP}^2$.

   (b) The Klein bottle.

   Hint: $\mathbb{RP}^2$ is the union of a disk with a Möbius band, and the latter admits a deformation retraction to S^1. The Klein bottle, in turn, is the union of two Möbius bands, also known as $\mathbb{RP}^2 \# \mathbb{RP}^2$.

4. Recall that for a continuous map $f : X \to X$, one defines the **mapping torus** of f as the space

$$X_f = (X \times [0,1]) \Big/ (x,0) \sim (f(x),1).$$

Assume from now on that f is a homeomorphism. In this case, one can equivalently define X_f as

$$X_f = (X \times \mathbb{R}) \Big/ (x,t) \sim (f(x),t+1)$$

where the equivalence is defined for every $t \in \mathbb{R}$. Take a moment to convince yourself that these two quotients are homeomorphic. The second perspective has the advantage that one can view $\widetilde{X} := X \times \mathbb{R}$ as a covering space for X_f, with the quotient projection defining a covering map $\widetilde{X} \to X_f$ of infinite degree. Writing $S^1 := \mathbb{R}/\mathbb{Z}$, we also see a natural continuous surjective map $\pi : X_f \to S^1 : [(x,t)] \mapsto [t]$, whose **fibers** $\pi^{-1}(t)$ are homeomorphic to X for all $t \in S^1$. We shall denote by $i : X \hookrightarrow X_f$ the inclusion of the fiber $\pi^{-1}([0])$.

In lecture, we proved the existence of a long exact sequence

$$\ldots \longrightarrow H_{k+1}(X_f) \xrightarrow{\Phi} H_k(X) \xrightarrow{\mathbb{1}_* - f_*} H_k(X) \xrightarrow{i_*} H_k(X_f) \xrightarrow{\Phi} H_{k-1}(X) \longrightarrow \ldots,$$

and we briefly discussed the definition of the connecting homomorphisms $\Phi : H_{k+1}(X_f) \to H_k(X)$. The goal of this problem to gain a more concrete picture of the special case $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$. Assume X is path-connected, so there is a natural isomorphism $H_0(X;\mathbb{Z}) = \mathbb{Z}$, and notice that X_f is then also path-connected. Since $H_1(X_f;\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(X_f, x)$ for any choice of base point $x \in X_f$, we can identify X with $\pi^{-1}([0]) \subset X_f$, fix a base point $x \in X \subset X_f$ and represent any class in $H_1(X_f;\mathbb{Z})$ by a loop $\gamma : [0,1] \to X_f$ with $\gamma(0) = \gamma(1) = x$. Now let $\tilde{\gamma} : [0,1] \to \widetilde{X}$ denote the unique lift of γ to the cover $\widetilde{X} = X \times \mathbb{R}$ such that $\tilde{\gamma}(0) = (x,0)$. Since γ is a loop, it follows that $\tilde{\gamma}(1) = (f^m(x), m)$ for some $m \in \mathbb{Z}$.

(a) Prove that under the natural identification of $H_0(X;\mathbb{Z})$ with $\mathbb{Z}$, the connecting homomorphism $\Phi : H_1(X_f;\mathbb{Z}) \to \mathbb{Z}$ can be chosen[1] such that

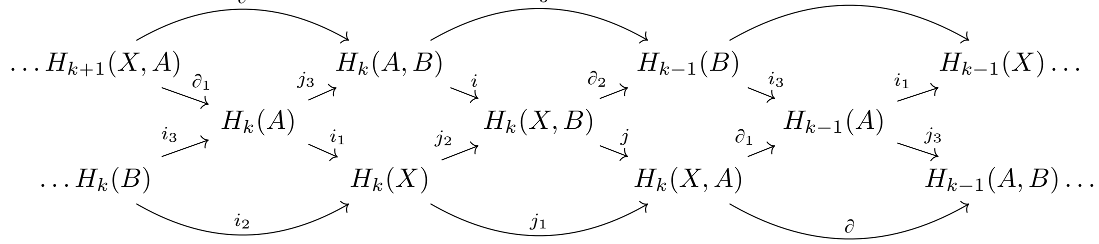
$$\Phi([\gamma]) = m,$$

so in particular, $[\gamma] \in \ker \Phi$ if and only if the lift of γ to the cover $\widetilde{X}$ is a loop.

(b) Prove directly from the characterization in part (a) that $\Phi : H_1(X_f;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is surjective. *Remark: Of course this can also be deduced less directly from the exact sequence.*

5. Here's another diagram-chasing exercise. It's probably the least essential problem on this sheet, but you might find it amusing and/or relaxing. (It kept me entertained during the flight to my conference last week.)

We saw in lecture that the long exact sequence of a triple (X, A, B) with $B \subset A \subset X$ is an immediate consequence of a rather obvious short exact sequence of relative singular chain complexes. It is interesting[2] to observe however that without mentioning singular chains at all, the exactness of the sequence for the triple can be deduced by purely algebraic means from the long exact sequences of pairs. Behold the following "braid" diagram:



[1] There is a bit of freedom allowed in the definition of Φ, e.g. we could replace it with $-\Phi$ and the sequence would still be exact since $\ker \Phi$ and $\operatorname{im} \Phi$ would not change.

[2] at least if you find axiomatic homology theories interesting. . .

The braid consists of four "strands," three of which you may recognize as the long exact sequences of the pairs (X, A), (X, B) and (A, B). The fourth strand is the sequence $\ldots \to H_{k+1}(X, A) \xrightarrow{\partial} H_k(A, B) \xrightarrow{i} H_k(X, B) \xrightarrow{j} H_k(X, A) \xrightarrow{\partial} H_{k-1}(A, B) \to \ldots$, which we would like to prove is exact. Here the map $\partial := j_3 \circ \partial_1$ is defined via the commutativity of the diagram, while all other maps are either induced by the obvious inclusions or are connecting homomorphisms from long exact sequences of pairs. The whole diagram commutes due to the commutativity of the obvious inclusions plus the naturality of the connecting homomorphisms.

(a) Deduce from the diagram that $i \circ \partial = 0$ and $\partial \circ j = 0$.
Hint: Each can be expressed as a different composition that includes two successive maps in an exact sequence.

(b) Prove that $j \circ i = 0$ by factoring it through the group $H_*(A, A)$, which is always zero. (Why?)

(c) Parts (a) and (b) together imply that in the sequence of maps $i, j, \partial, i, \ldots$, the image of each is contained in the kernel of the next. With this established, use a purely algebraic diagram-chasing argument to prove the converse: the kernel of each map is contained in the image of the previous one.