PROBLEM SET 5
To be discussed: 22.11.2017

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

1. Fill in the details of the proof that the obvious inclusion induces an isomorphism $H_*(\mathbb{D}^n, \partial\mathbb{D}^n) \to H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$. *Hint: $\mathbb{D}^n \hookrightarrow \mathbb{R}^n$ and $\partial\mathbb{D}^n \hookrightarrow \mathbb{R}^n\backslash\{0\}$ are homotopy equivalences. Use the five-lemma.*

2. Recall that for a topological n-manifold M, a **local orientation** at a point $x \in M$ is a choice of generator $[M]_x$ for the group $H_n(M, M\backslash\{x\};\mathbb{Z}) \cong \mathbb{Z}$.
   (a) Describe an example of a singular n-cycle that represents a local orientation $[M]_x$. How could you alter this cycle to make it represent the alternative local orientation $-[M]_x$?
   (b) Prove (e.g. via Mayer-Vietoris) that $H_2(\mathbb{T}^2;\mathbb{Z}) \cong \mathbb{Z}$. Show also that it has a generator $[\mathbb{T}^2] \in H_2(\mathbb{T}^2;\mathbb{Z})$ corresponding to an oriented triangulation of $\mathbb{T}^2$, such that the inclusion $i^x : (\mathbb{T}^2, \varnothing) \hookrightarrow (\mathbb{T}^2, \mathbb{T}^2\backslash\{x\})$ at each point $x \in \mathbb{T}^2$ induces an isomorphism $i^x_* : H_2(\mathbb{T}^2;\mathbb{Z}) \to H_2(\mathbb{T}^2, \mathbb{T}^2\backslash\{x\};\mathbb{Z})$.

3. For any integer $n \geqslant 2$, fix a generator $[S^{n-1}] \in H_{n-1}(S^{n-1};\mathbb{Z})$ and use it to determine local orientations $[\mathbb{R}^n]_x \in H_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{x\};\mathbb{Z})$ for every point $x \in \mathbb{R}^n$ via the natural isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n\backslash\{x\}) \cong H_n(\mathbb{D}^n_x, \partial\mathbb{D}^n_x) \cong H_{n-1}(\partial\mathbb{D}^n_x)$, where $\mathbb{D}^n_x$ denotes the closed unit disk centered at x, whose boundary is canonically identified with S^{n-1}. Recall that if f is any continuous map taking a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of x to a neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of y such that $f(x) = y$ but no other point in $\mathcal{U}$ maps to y, we define the **local degree** $\deg(f;x)$ of f at x as the unique $d \in \mathbb{Z}$ such that the homomorphism
$$f_* : H_n(\mathcal{U}, \mathcal{U}\backslash\{x\};\mathbb{Z}) \to H_n(\mathcal{V}, \mathcal{V}\backslash\{y\};\mathbb{Z})$$
sends $[\mathbb{R}^n]_x$ to $d[\mathbb{R}^n]_y$, where we use the obvious excision isomorphisms to view $[\mathbb{R}^n]_x$ and $[\mathbb{R}^n]_y$ as elements of these respective groups. Note that this does not depend on the choice of generator $[S^{n-1}]$.
   (a) Show that if $f : \mathbb{R}^n \supset \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^n$ is a smooth map whose derivative $df(x) : \mathbb{R}^n \to \mathbb{R}^n$ at $x \in \mathcal{U}$ is an isomorphism, then $\deg(f;x)$ is 1 or -1 depending on whether $\det df(x)$ is positive or negative respectively.
   Hint: The differentiability of f implies that after a homotopy that does not change its local degree, we can assume it is linear near x. Homotop it further to assume it is orthogonal.
   (b) For $n = 2$, identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider a map $f : \mathbb{C} \to \mathbb{C}$ of the form $f(z) = (z - z_0)^k g(z)$ for some $z_0 \in \mathbb{C}$, $k \in \mathbb{N}$ and g a continuous map with $g(z_0) \neq 0$. Show that $\deg(f;z_0) = k$.
   (c) Can you modify the example in part (b) to produce one with $\deg(f;z_0) = -k$ for $k \in \mathbb{N}$?
   Hint: Try complex conjugation.
   (d) Prove that if $f : \mathbb{R}^n \supset \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^n$ is continuous with $f(x) = y$ and $\deg(f;x) \neq 0$ for some $x \in \mathcal{U}$, then for any neighborhood $\mathcal{U}_x \subset \mathcal{U}$ of x, there exists an $\epsilon > 0$ such that every continuous map $\tilde{f} : \mathcal{U} \to \mathcal{V}$ satisfying $|\tilde{f} - f| < \epsilon$ maps some point in $\mathcal{U}_x$ to y.
   Hint: Consider the restriction of f to the boundary of some small ball about x, and normalize it so that it maps to the sphere surrounding a small ball about y. This map between spheres is not homotopic to a constant, right?
   (e) Find an example of a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^2$ that has an isolated zero at the origin with $\deg(f;0) = 0$ but admits arbitrarily small perturbations that are nowhere zero.
   (f) Suppose $f : S^n \to S^n$ is any continuous map, and $p_+ \in \Sigma S^n = C_+S^n \cup_{S^n} C_-S^n$ is the vertex of the top cone in the suspension $\Sigma S^n \cong S^{n+1}$, i.e. the point obtained by collapsing $S^n \times \{1\}$ to form $C_+S^n := (S^n \times [0,1])/(S^n \times \{1\})$. Recall that the suspended map $\Sigma f : \Sigma S^n \to \Sigma S^n$ is defined by $\Sigma f([(x,t)]) = [(f(x),t)]$. What is $\deg(\Sigma f;p_+)$? Use this to give a new proof (different from the one we saw in class) that $\deg(\Sigma f) = \deg(f)$.

(g) Let $f : S^2 \to S^2$ denote the natural continuous extension to $S^2 := \mathbb{C} \cup \{\infty\}$ of a complex polynomial $\mathbb{C} \to \mathbb{C}$ of degree n. What is $\deg(f)$?

(h) Pick a constant $t_0 \in S^1$ and let $A \cong S^1 \vee S^1$ denote the subset $\{(x, y) \mid x = t_0 \text{ or } y = t_0\} \subset S^1 \times S^1 = \mathbb{T}^2$. Show that $\mathbb{T}^2/A \cong S^2$, and that the quotient map $\mathbb{T}^2 \to \mathbb{T}^2/A$ has degree ± 1 (depending on choices of generators for $H_2(\mathbb{T}^2; \mathbb{Z})$ and $H_2(S^2; \mathbb{Z})$; both are $\mathbb{Z}$ due to Problem 2(b)).

4. Prove that for every positive even integer n, every continuous map $f : S^n \to S^n$ has at least one point $x \in S^n$ where either $f(x) = x$ or $f(x) = -x$. Deduce that every continuous map $\mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point if n is even. Construct counterexamples to this statement for every odd n.
Hint: Consider linear transformations with no real eigenvalues.

5. Consider a chain complex (C_*, ∂) with $C_k = 0$ for $k \notin \{0, 1, 2\}$, $C_2 = C_0 = \mathbb{Z}$ and $C_1 = \mathbb{Z}^2$, and boundary maps defined by $\partial : C_2 \to C_1 : m \mapsto (0, 2m)$ and $\partial = 0 : C_1 \to C_0$.

(a) Compute both the homology and the cohomology of this complex, and compare. What are the torsion subgroups[1] of $H_k(C_*, \partial)$ and $H^k(C_*, \partial)$ for each k?

(b) How does the answer change if you take coefficients in $\mathbb{Z}_2$ or $\mathbb{Q}$, i.e. replace (C_*, ∂) and its dual complex by their tensor products with $\mathbb{Z}_2$ or $\mathbb{Q}$?

6. In lecture, we defined the **reduced homology** of a space X as the subgroup $\widetilde{H}_*(X) := \ker \epsilon_* \subset H_*(X)$, where $\epsilon_* : H_*(X) \to H_*(P)$ is induced by the unique map $\epsilon : X \to P$ to the one-point space P. The homomorphism ϵ_* is surjective and admits right-inverses $i_* : H_*(P) \to H_*(X)$, induced by any choice of inclusion map $i : P \hookrightarrow X$, thus we have a split exact sequence $0 \to \widetilde{H}_*(X) \to H_*(X) \xrightarrow{\epsilon_*} H_*(P) \to 0$.

To define **reduced cohomology**, note that $\epsilon^* : H^*(P) \to H^*(X)$ is injective with left-inverse $i^* : H^*(X) \to H^*(P)$, thus we can define $\widetilde{H}^*(X)$ as the quotient $\operatorname{coker} \epsilon^* := H^*(X)/\operatorname{im} \epsilon^*$, so that the split exact sequence now takes the form

$$0 \longrightarrow H^*(P) \xrightarrow{\epsilon^*} H^*(X) \longrightarrow \widetilde{H}^*(X) \longrightarrow 0.$$

(a) Consider the following commutative diagram of abelian groups:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
\dots \arrow[r, dashed] & \widetilde{A}_{n-1} \arrow[u] \arrow[r, dashed, "\tilde{f}"] & \widetilde{A}_n \arrow[u] \arrow[r, dashed, "\tilde{f}"] & \widetilde{A}_{n+1} \arrow[u] \arrow[r, dashed] & \dots \\
\dots \arrow[r] & A_{n-1} \arrow[u] \arrow[r, "f"] & A_n \arrow[u] \arrow[r, "f"] & A_{n+1} \arrow[u] \arrow[r] & \dots \\
\dots \arrow[r] & B_{n-1} \arrow[u, "\epsilon"] \arrow[r, "g"] & B_n \arrow[u, "\epsilon"] \arrow[r, "g"] & B_{n+1} \arrow[u, "\epsilon"] \arrow[r] & \dots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Show by diagram-chasing that if all columns and the bottom two rows are exact, then the maps indicated by dashed arrows also exist and the resulting top row is exact.

(b) Use the algebraic result in part (a) to verify that the long exact sequence of a pair and the Mayer-Vietoris sequence in cohomology also make sense for *reduced* cohomology.

(c) Write down a formula for the augmentation map on singular 0-chains $\epsilon_* : C_0(X) \to C_0(P) = \mathbb{Z}$, and show that $\widetilde{H}_*(X)$ can also be defined as the homology of the *augmented* chain complex

$$\dots \longrightarrow C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon_*} \mathbb{Z} \longrightarrow 0,$$

defined so that $C_{-1}(X) := C_0(P) = \mathbb{Z}$. Then show that the cohomology of this complex is precisely $\widetilde{H}^*(X)$. Give explicit descriptions of $\widetilde{H}_0(X; G)$ and $\widetilde{H}^0(X; G)$ for any coefficient group G.

[1] Recall that the **torsion subgroup** of an abelian group G is the set of all $g \in G$ such that $ng = 0$ for some $n \in \mathbb{N}$.