# PROBLEM SET 6
**To be discussed: 29.11.2017**

## Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. We will discuss the solutions in the Übung beforehand.

1. Use the Eilenberg-Steenrod axioms to prove that if $h^* : \mathsf{Top}_{\text{rel}} \to \mathsf{Ab}_{\mathbb{Z}}$ is any axiomatic cohomology theory on the category of all pairs, there is a natural isomorphism of reduced cohomologies $\Phi_X : \tilde{h}^k(X) \to \tilde{h}^{k+1}(\Sigma X)$ for every $k \in \mathbb{Z}$, every space $X$ and its suspension $\Sigma X = C_+X \cup_X C_-X$. Here, the word "natural" is meant in the sense of "natural transformations": specifically, for any continuous map $f : X \to Y$ and the induced map $\Sigma f : \Sigma X \to \Sigma Y$ on suspensions, we have a commuting diagram:

$$\begin{array}{ccc} \tilde{h}^k(Y) & \xrightarrow{\Phi_Y} & \tilde{h}^{k+1}(\Sigma Y) \\ \downarrow f^* & & \downarrow (\Sigma f)^* \\ \tilde{h}^k(X) & \xrightarrow{\Phi_X} & \tilde{h}^{k+1}(\Sigma X) \end{array}$$

2. It is time to start getting acquainted with direct and inverse limits. These are essential for the definitions of Čech homology and cohomology, and they will play an increasing role in proofs of standard results about singular (co)homology as well, not to mention Problem 3(d) on this sheet. We shall start out with a general and somewhat abstract discussion that applies to arbitrary categories, but then specialize to our two favorite categories: topological spaces and abelian groups.

   Suppose $I$ is a set with a pre-order $\leqslant$, i.e. $\leqslant$ is reflexive ($\alpha \leqslant \alpha$) and transitive ($\alpha \leqslant \beta$ and $\beta \leqslant \gamma$ implies $\alpha \leqslant \gamma$), but the relations $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$ need not imply $\alpha = \beta$, so $\leqslant$ need not be a partial order. We call $(I, \leqslant)$ a **directed set** if for every pair $\alpha, \beta \in I$, there exists $\gamma \in I$ with $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$. Given a category $\mathscr{C}$, a **direct system** in $\mathscr{C}$ over $(I, \leqslant)$ associates to each $\alpha \in I$ an object $X_\alpha$ of $\mathscr{C}$ and to each pair $\alpha, \beta \in I$ with $\alpha \leqslant \beta$ a morphism $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$, such that $\varphi_{\alpha\alpha}$ is the identity morphism and $\varphi_{\gamma\beta} \circ \varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$ whenever $\alpha \leqslant \beta \leqslant \gamma$. A closely related notion is that of an **inverse system**, which is defined in the same way except that the morphism $\varphi_{\beta\alpha}$ defined for each $\beta \geqslant \alpha$ goes from $X_\beta$ to $X_\alpha$, and the composition law is correspondingly adjusted to $\varphi_{\beta\alpha} \circ \varphi_{\gamma\beta} = \varphi_{\gamma\alpha}$.[1]

   Notice that any covariant functor $\mathscr{A} \to \mathscr{B}$ transforms a direct/inverse system in $\mathscr{A}$ to a direct/inverse system in $\mathscr{B}$. If the functor is instead contravariant, then direct systems are transformed into inverse systems and vice versa. Important examples appear in Problems 2(h), 2(i) and 4(d) below.

   For a direct system $\{X_\alpha, \varphi_{\beta\alpha}\}$, an object $X_\infty$ of $\mathscr{C}$ with associated morphisms $\{\varphi_\alpha : X_\alpha \to X_\infty\}_{\alpha \in I}$ is called a *target* of the system $\{X_\alpha, \varphi_{\beta\alpha}\}$ if it satisfies $\varphi_\alpha = \varphi_\beta \circ \varphi_{\beta\alpha}$ whenever $\beta \geqslant \alpha$. Such a target is called a **direct limit**[2] of the system and written as

$$X_\infty = \varinjlim \{X_\alpha\}$$

   if it satisfies the following "universal" property: for all other targets $Y$ with associated morphisms $\{\psi_\alpha : X_\alpha \to Y\}_{\alpha \in I}$, there is a unique morphism $\psi : X_\infty \to Y$ such that $\psi \circ \varphi_\alpha = \psi_\alpha$ for all $\alpha \in I$. **Inverse limits** of an inverse system $\{X_\alpha, \varphi_{\beta\alpha}\}$ are defined in an analogous way with reversed arrows: we now call an object $X_\infty$ with morphisms $\{\varphi_\alpha : X_\infty \to X_\alpha\}_{\alpha \in I}$ a *target* if $\varphi_{\beta\alpha} \circ \varphi_\beta = \varphi_\alpha$ whenever $\beta \geqslant \alpha$, and write

$$X_\infty = \varprojlim \{X_\alpha\}$$

[1] If you recall Problem Set 2 #1, you may notice that there was a shorter way to say all this: denoting by $\mathscr{I}$ the category whose objects are the elements of $I$ with a unique morphism from $\alpha$ to $\beta$ whenever $\alpha \leqslant \beta$, a direct system in $\mathscr{C}$ over $(I, \leqslant)$ is simply a covariant functor $\mathscr{I} \to \mathscr{C}$, and an inverse system is a contravariant functor $\mathscr{I} \to \mathscr{C}$.

[2] Direct limits are also sometimes called **inductive limits**, and inverse limits are sometimes called **projective limits**.

if for every other target $Y$ with associated morphisms $\{\psi_\alpha : Y \to X_\alpha\}_{\alpha \in I}$, there is a unique morphism $\psi : Y \to X_\infty$ satisfying $\varphi_\alpha \circ \psi = \psi_\alpha$ for all $\alpha \in I$.

Note that from these definitions, there is generally no guarantee that a direct or inverse limit exists, and if it exists then it may not be unique, but the universal property provides a distinguished isomorphism between any two limits of the same system.

(a) Show that in the category of topological spaces with continuous maps, a direct limit of a system $\{X_\alpha, \varphi_{\beta\alpha}\}$ can always be defined as the space

$$\varinjlim\{X_\alpha\} = \coprod_{\alpha \in I} X_\alpha \Big/ \sim$$

with an equivalence relation defined by $x \sim \varphi_{\beta\alpha}(x)$ for every $x \in X_\alpha$ and $\beta \geqslant \alpha$, and the associated maps $\varphi_\alpha : X_\alpha \to \varinjlim\{X_\alpha\}$ are then the compositions of the natural inclusions $X_\alpha \hookrightarrow \coprod_\beta X_\beta$ with the quotient projection. Show also that the natural topology on this space is the strongest one for which the maps $\varphi_\alpha$ are all continuous.

(b) Show that for an inverse system of topological spaces, an inverse limit can always be defined as the space

$$\varprojlim\{X_\alpha\} = \left\{ (\dots, x_\alpha, \dots) \in \prod_{\alpha \in I} X_\alpha \;\middle|\; \varphi_{\beta\alpha}(x_\beta) = x_\alpha \text{ for all } \beta \geqslant \alpha \right\},$$

with the associated maps $\varphi_\alpha : \varprojlim\{X_\alpha\} \to X_\alpha$ defined as restrictions of the natural projections $\prod_\beta X_\beta \to X_\alpha$. Show moreover that the natural topology on this space is the weakest one for which the maps $\varphi_\alpha$ are all continuous.

(c) Consider the special case of part (b) in which the spaces $X_\alpha$ are all subsets (with the subspace topology) of some fixed space $X$, $\beta \geqslant \alpha$ if and only if $X_\beta \subset X_\alpha$ and the maps $\varphi_{\beta\alpha} : X_\beta \to X_\alpha$ are the natural inclusions. Show that $\bigcap_\alpha X_\alpha$ with the natural inclusions $\varphi_\alpha : \bigcap_\beta X_\beta \hookrightarrow X_\alpha$ defines an inverse limit of the system.

(d) Consider now the direct system analogue of part (c): all spaces are subsets of some fixed space $X$, $\beta \geqslant \alpha$ if and only if $X_\alpha \subset X_\beta$, and the $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$ are inclusion maps. It is easy to find a natural bijection between $\varinjlim\{X_\alpha\}$ and $\bigcup_\alpha X_\alpha$, but it is not always a homeomorphism. Take for example the family of sets $X_t = \{0\} \cup (t, 1] \in \mathbb{R}$, indexed by $t \in (0, 1)$ and ordered by inclusion: the union of these sets is the interval $[0, 1]$, but show that the topological space $\varinjlim\{X_t\}$ is not connected.

(e) Let $\{X_\alpha\}_{\alpha \in I}$ denote the family of all countable subsets of $S^1$, ordered by inclusion. Show that this forms a direct system of topological spaces whose direct limit is $S^1$, with its usual topology.

(f) Show that any direct system of abelian groups $\{G_\alpha, \varphi_{\beta\alpha}\}$ has a direct limit of the form

$$\varinjlim\{G_\alpha\} = \bigoplus_{\alpha \in I} G_\alpha \Big/ H,$$

were $H \subset \bigoplus_\alpha G_\alpha$ is the subgroup generated by all elements of the form $g - \varphi_{\beta\alpha}(g)$ for $g \in G_\alpha$ and $\beta \geqslant \alpha$, and the associated homomorphisms $\varphi_\alpha : G_\alpha \to \varinjlim\{G_\alpha\}$ are the compositions of the natural inclusions $G_\alpha \hookrightarrow \bigoplus_\beta G_\beta$ with the quotient projection.

(g) Show that any inverse system of abelian groups $\{G_\alpha, \varphi_{\beta\alpha}\}$ has an inverse limit of the form

$$\varprojlim\{G_\alpha\} = \left\{ (\dots, g_\alpha, \dots) \in \prod_{\alpha \in I} G_\alpha \;\middle|\; \varphi_{\beta\alpha}(g_\beta) = g_\alpha \text{ for all } \beta \geqslant \alpha \right\},$$

with the associated homomorphisms $\varphi_\alpha : \varprojlim\{G_\alpha\} \to G_\alpha$ defined as restrictions of the natural projections $\prod_\beta G_\beta \to G_\alpha$.

(h) Given a direct system of abelian groups $\{G_\alpha, \varphi_{\beta\alpha}\}$ and another abelian group $H$, one can dualize the direct system to define an inverse system $\{\mathrm{Hom}(G_\alpha, H), \varphi^*_{\beta\alpha}\}$, where $\varphi^*_{\beta\alpha} : \mathrm{Hom}(G_\beta, H) \to \mathrm{Hom}(G_\alpha, H) : \Phi \mapsto \Phi \circ \varphi_{\beta\alpha}$. Show that there is a natural isomorphism

$$\mathrm{Hom}(\varinjlim\{G_\alpha\}, H) \cong \varprojlim\{\mathrm{Hom}(G_\alpha, H)\}.$$

(i) Recall from Problem Set 1 #1 the category $\mathsf{Chain}$ of chain complexes of abelian groups, whose morphisms are chain maps, and the covariant functor $H_* : \mathsf{Chain} \to \mathsf{Ab}_{\mathbb{Z}}$ that assigns to each chain complex its homology groups. In analogy with parts (f) and (g), it is not hard to write down explicit descriptions of the direct/inverse limit of any direct/inverse system in the categories $\mathsf{Chain}$ and $\mathsf{Ab}_{\mathbb{Z}}$. In particular, if $\{(C^\alpha_*, \partial^\alpha), \varphi_{\beta\alpha}\}$ is a direct system of chain complexes and $\varinjlim\{C^\alpha_*\}$ denotes its direct limit as a system of $\mathbb{Z}$-graded abelian groups, you can check that the boundary maps $\partial^\alpha : C^\alpha_* \to C^\alpha_{*-1}$ naturally determine a boundary map $\partial^\infty$ on $\varinjlim\{C^\alpha_*\}$ that makes it into a chain complex. With this understood, prove that there is a natural isomorphism

$$H_*(\varinjlim\{C^\alpha_*\}, \partial^\infty) \cong \varinjlim\{H_*(C^\alpha_*, \partial^\alpha)\}.$$

Specializing to the case of chain complexes with trivial homology, this proves that any direct system of exact sequences has a direct limit which is also an exact sequence.
*Hint: The direct limit of a system of abelian groups $\{G_\alpha, \varphi_{\beta\alpha}\}$ has the convenient feature that every element in $\varinjlim\{G_\alpha\} = \bigoplus_\alpha G_\alpha / \sim$ can be represented by some element in one of the subgroups $G_\beta \subset \bigoplus_\alpha G_\alpha$ for "sufficiently large" $\beta \in I$. (Why?)*

(j) Try to prove the analogue of part (i) for inverse limits of chain complexes, but don't try very hard.
*Hint: It isn't true. In particular, there exist inverse systems of exact sequences whose inverse limits are not exact.[3] This embarrassing algebraic fact is the reason why Čech homology—which is defined as an inverse limit of homologies of chain complexes—does not satisfy the exactness axiom of Eilenberg and Steenrod, at least not without placing restrictions on the category of spaces or on the coefficient group. By contrast, the Čech* cohomology *is defined as a direct limit, and thus does not have this defect, due to part (i).*

3. The **Alexander-Spanier cohomology** $\bar{H}^*(X; G)$ of a space $X$ with coefficients in $G$ is defined as the homology of the following (co)chain complex. For integers $k \geqslant 0$, let $\bar{C}^k(X; G)$ denote the group of equivalence classes of (not necessarily continuous) functions $\varphi : X^{k+1} \to G$, where we say $\varphi \sim \psi$ whenever $\varphi$ and $\psi$ are identical on some neighborhood of the **diagonal** $\Delta := \{(x, \dots, x) \in X^{k+1} \mid x \in X\}$. Associate to each function $\varphi : X^{k+1} \to G$ the function $\delta\varphi : X^{k+2} \to G$ defined by

$$\delta\varphi(x_0, \dots, x_{k+1}) = \sum_{j=0}^{k+1} (-1)^j \varphi(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{k+1}).$$

This defines a homomorphism from the group of $(k+1)$-functions to the group of $(k+2)$-functions such that $\delta^2 = 0$, and it preserves the subgroup of functions that vanish near the diagonal, thus it descends to a coboundary homomorphism

$$\delta : \bar{C}^k(X; G) \to \bar{C}^{k+1}(X; G).$$

Extending this to all $k \in \mathbb{Z}$ by defining $\bar{C}^k(X; G) = 0$ for $k < 0$, we obtain a cochain[4] complex $(\bar{C}^*(X; G), \delta)$, and its homology is denoted by $\bar{H}^*(X; G)$. It is not hard to give $\bar{H}^*$ the structure of a

[3]For an explicit example, see Example 5.5 in Eilenberg and Steenrod, *Foundations of Algebraic Topology* (Princeton University Press 1952).

[4]I am calling $\bar{C}^*(X; G)$ a *cochain* complex instead of a chain complex simply because $\delta$ raises degree by 1 instead of lowering it, but this terminology is a debatable matter of convention. Notice in particular that $\bar{C}^*(X; G)$ is not in any obvious way the dual complex of any chain complex, thus it is far from obvious at this stage what the definition of "Alexander-Spanier homology" might be. We refer to $\bar{H}^*(X; G)$ as "cohomology" instead of "homology" because it is a *contravariant* functor, not covariant. A corresponding homology theory was defined in an appendix of Spanier's paper *Cohomology theory for general spaces* (Annals of Mathematics, 1948), but its definition is much more complicated, requiring inverse limits, and as a result it suffers from the same drawbacks as Čech homology, i.e. it fails to satisfy the exactness axiom of Eilenberg-Steenrod.

contravariant functor: given a continuous map $f : X \to Y$, one defines a chain map

$$f^* : \bar{C}^*(Y; G) \to \bar{C}^*(X; G) : \varphi \mapsto \varphi \circ (f \times \ldots \times f),$$

thus inducing homomorphisms $f^* : \bar{H}^*(Y; G) \to \bar{H}^*(X; G)$. With some more effort, one can also define relative groups $\bar{H}^*(X, A; G)$ and prove that $\bar{H}^*$ satisfies all of the Eilenberg-Steenrod axioms for a cohomology theory.[5] This implies in particular that $\bar{H}^*(X; G)$ is isomorphic to the singular cohomology $H^*(X; G)$ whenever $X$ is a sufficiently "nice" space such as a manifold or CW-complex. The goal of this problem is to gain some understanding of why $\bar{H}^*$, in spite of having a radically different definition from $H^*$, ends up seeing much of the same topological information for nice spaces, and how this ceases to be true for spaces that are less nice.

(a) Show that for any space $X$, $\bar{H}^0(X; G)$ is a direct product of copies of $G$, one for each connected component of $X$.

(b) Show by explicit computation that $\bar{H}^1(\mathbb{R}; G) = 0$ and $\bar{H}^1(S^1; G) \cong G$.
*Hint: Consider the map* $\bar{H}^1(X; G) \to G : [\varphi] \mapsto \sum_{j=1}^N \varphi(\gamma(t_j), \gamma(t_{j-1}))$ *for any loop* $\gamma : [0, 1] \to X$ *and partition* $0 = t_0 < t_1 < \ldots < t_N = 1$. *It is well defined because* $\varphi$ *represents a cocycle. (Why?)*

(c) Notice that in part (a), I said "connected component,", not "path-component". Let

$$X := (\{0\} \times [-1, 1]) \cup \left\{(x, y) \in \mathbb{R}^2 \mid y = \sin(1/x) \text{ and } 0 < x \leqslant 1\right\} \subset \mathbb{R}^2,$$

with its natural subspace topology as a subset of $\mathbb{R}^2$. This space is "nice" in the sense that it is compact and Hausdorff, but it is also a classic example of a connected space that has two path-components, so that its singular cohomology satisfies $H^0(X; \mathbb{Z}) \cong \mathbb{Z}^2$, while part (a) implies $\bar{H}^0(X; \mathbb{Z}) \cong \mathbb{Z}$. The corresponding reduced cohomologies are thus $\widetilde{H}^0(X; \mathbb{Z}) \cong \mathbb{Z}$ and $\widetilde{\bar{H}}^0(X, \mathbb{Z}) = 0$, thus Problem 1 implies that for the suspension $\Sigma X$,

$$H^1(\Sigma X; \mathbb{Z}) = \mathbb{Z}, \quad \text{but} \quad \bar{H}^1(\Sigma X; \mathbb{Z}) = 0.$$

Note that suspensions are always path-connected, so one of our standard results about singular (co)homology implies $H^1(\Sigma X; \mathbb{Z}) = \mathrm{Hom}(\pi_1(\Sigma X), \mathbb{Z})$, which is therefore not true for $\bar{H}^1(\Sigma X; \mathbb{Z})$. I suppose you're waiting for me to stop rambling and ask a question. Fine, here is one: since the above discussion implies that $\pi_1(\Sigma X)$ is not torsion, find an explicit loop in $\Sigma X$ that represents a non-torsion element of $\pi_1(\Sigma X)$.

(d) There is an "extra" axiom that Alexander-Spanier cohomology satisfies but singular cohomology does not. Observe that since $\bar{H}^*$ is a contravariant functor, any inverse system of spaces $\{X_\alpha, \varphi_{\beta\alpha}\}$ gives rise to a direct system of cohomology groups $\{\bar{H}^*(X_\alpha; G), \varphi^*_{\beta\alpha}\}$. The **continuity** property states that whenever the spaces $X_\alpha$ are all compact and Hausdorff, there is an isomorphism

$$\bar{H}^*(\varprojlim\{X_\alpha\}; G) \cong \varinjlim\{\bar{H}^*(X_\alpha; G)\}.$$

Find an inverse system of compact Hausdorff spaces whose inverse limit is the space $X$ in part (c), and verify that the continuity property holds in this example for $\bar{H}^*$, but not for singular cohomology $H^*$. *Hint: Problem 2(c) might be helpful.*
*Remark: One can show that every compact Hausdorff space is an inverse limit of some inverse system of compact Hausdorff spaces homotopy equivalent to CW-complexes. It follows that up to isomorphism, there is only one cohomology theory satisfying all of the Eilenberg-Steenrod axioms plus continuity. The Čech cohomology* $\check{H}^*(X; G)$ *also satisfies all of these properties, and is thus isomorphic to* $\bar{H}^*(X; G)$ *(but not necessarily to* $H^*(X; G)$*) for all compact Hausdorff spaces* $X$. *(This result can be generalized beyond compact spaces using sheaf cohomology; details are carried out in Chapter 6 of Spanier's book.)*
*One last remark: you may wonder whether there is also a continuity property involving the (co)homology of direct limits of spaces. The example in Problem 2(e) should make you suspect that no such result could hold without serious restrictions, but we will revisit this question in the context of singular homology when we prove Poincaré duality.*

[5]For a good exposition of the details, see §6.4–6.5 in Spanier, *Algebraic Topology* (Springer-Verlag 1966).