PROBLEM SET 1
To be discussed: 24.10.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Consider categories $\mathsf{Ab}_{\mathbb{Z}}$ and Chain, defined as follows:

   - Objects of $\mathsf{Ab}_{\mathbb{Z}}$ are $\mathbb{Z}$-graded abelian groups $G_* = \bigoplus_{n\in\mathbb{Z}} G_n$, and morphisms from G_* to H_* are group homomorphisms $\Phi : G_* \to H_*$ satisfying $\Phi(G_n) \subset H_n$ for every $n \in \mathbb{Z}$.
   - Objects of Chain are chain complexes (C_*, ∂), meaning $\mathbb{Z}$-graded abelian groups $C_* = \bigoplus_{n\in\mathbb{Z}} C_n$ endowed with homomorphisms $\partial : C_* \to C_*$ that satisfy $\partial(C_n) \subset C_{n-1}$ for each $n \in \mathbb{Z}$ and $\partial^2 = 0$. Morphisms from (A_*, ∂_A) to (B_*, ∂_B) are chain maps, meaning homomorphisms $\Phi : A_* \to B_*$ with $\Phi(A_n) \subset B_n$ for each $n \in \mathbb{Z}$ and $\Phi \circ \partial_A = \partial_B \circ \Phi$.

   Recall that the homology of a chain complex (C_*, ∂) is defined in general as the graded abelian group $H_*(C_*, \partial) = \bigoplus_{n\in\mathbb{Z}} H_n(C_*, \partial)$ where $H_n(C_*, \partial) = \ker \partial_n \big/ \operatorname{im} \partial_{n+1}$, with the restriction of $\partial : C_* \to C_*$ to $C_n \to C_{n-1}$ denoted by ∂_n.

   (a) Show that H_* defines a functor from Chain to $\mathsf{Ab}_{\mathbb{Z}}$ in a natural way. How does this functor act on morphisms of Chain?

   (b) Recall that two chain maps Φ and Ψ from (A_*, ∂_A) to (B_*, ∂_B) are called **chain homotopic** whenever there exists a homomorphism $h : A_* \to B_*$ satisfying $h(A_n) \subset B_{n+1}$ and

   $$\partial_B \circ h + h \circ \partial_A = \Phi - \Psi.$$

   This defines an equivalence relation on the set of chain maps, so we can define Chain^h as the category whose objects are the same as in Chain, but with morphisms defined as chain homotopy classes of chain maps. Show that H_* also defines a functor from Chain^h to $\mathsf{Ab}_{\mathbb{Z}}$.

2. One can speak of "functors of multiple variables" in much the same way as with functions. Show for instance that on the category Ab of abelian groups and homomorphisms,

   $$\operatorname{Hom} : \mathsf{Ab} \times \mathsf{Ab} \to \mathsf{Ab}$$

   defines a functor that is contravariant in the first variable and covariant in the second, assigning to each pair of abelian groups (G, H) the group $\operatorname{Hom}(G, H)$ of homomorphisms $G \to H$.

3. For a pointed space (X, p), recall that the *Hurewicz homomorphism*[1]

   $$h : \pi_1(X, p) \to H_1(X; \mathbb{Z})$$

   sends each element $[\gamma] \in \pi_1(X, p)$ represented by a path $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = p$ to the homology class represented by the singular 1-cycle $\gamma : \Delta^1 \to X$, defined by identifying $[0, 1]$ with the standard 1-simplex $\Delta^1 = \{(t_0, t_1) \in [0, 1]^2 \mid t_0 + t_1 = 1\}$. Let Top_* denote the category of pointed spaces with base-point preserving continuous maps, so that we can regard both π_1 and $H_1(\cdot; \mathbb{Z})$ as functors from Top_* to the category Grp of groups with homomorphisms. (Note that the base point is irrelevant for the definition of $H_1(\cdot; \mathbb{Z})$, which actually takes values in the smaller subcategory of *abelian* groups, but these details are unimportant for now.) In this context, show that the Hurewicz homomorphism defines a natural transformation from π_1 to $H_1(\cdot; \mathbb{Z})$.

[1] See Problem Set 10 #3 from last semester's *Topologie I* class.

4. Suppose $\mathscr{A}$ is a category whose objects form a set X, such that for each pair $x, y \in X$, the set of morphisms $\mathrm{Mor}(x, y)$ contains either exactly one element or none. We can turn this into a binary relation by writing $x \bowtie y$ for every pair such that $\mathrm{Mor}(x, y) \neq \varnothing$.

   (a) What properties does the relation $\bowtie$ need to have in order for it to define a category in the way indicated above?

   (b) If $\mathscr{B}$ is another category whose objects form a set Y with morphisms determined by a binary relation $\bowtie$ as indicated above, what properties does a map $f : X \to Y$ need to have in order for it to define a functor from $\mathscr{A}$ to $\mathscr{B}$?

5. In any category $\mathscr{C}$, each object X has an **automorphism group** (also called **isotropy** group) $\mathrm{Aut}(X)$, consisting of all the isomorphisms in $\mathrm{Mor}(X, X)$. A **groupoid** is a category in which all morphisms are also isomorphisms.

   (a) Show that if $\mathscr{G}$ is a groupoid and Grp denotes the usual category of groups with homomorphisms, there exists a contravariant functor from $\mathscr{G}$ to Grp that assigns to each object X of $\mathscr{G}$ its automorphism group $\mathrm{Aut}(X)$. How does this functor act on morphisms $X \to Y$? Could you alternatively define it as a *covariant* functor? Conclude either way that whenever X and Y are isomorphic objects in $\mathscr{G}$ (meaning there exists an isomorphism in $\mathrm{Mor}(X, Y)$), the groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are isomorphic.

   (b) Given a topological space X and two points x, y, let $\mathrm{Mor}(x, y)$ denote the set of homotopy classes (with fixed end points) of paths $[0, 1] \to X$ from x to y, and define a composition function $\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z) : (\alpha, \beta) \mapsto \alpha \cdot \beta$ by the usual notion of concatenation of paths. Show that this notion of morphisms defines a groupoid whose objects are the points in X.[2] In this case, what are the automorphism groups $\mathrm{Aut}(x)$ and the isomorphisms $\mathrm{Aut}(y) \to \mathrm{Aut}(x)$ given by the functor in part (a)?

6. For a fixed field $\mathbb{K}$, let $\mathsf{Vec}_{\mathbb{K}}$ denote the category of finite-dimensional vector spaces over $\mathbb{K}$ with $\mathbb{K}$-linear maps as morphisms.

   (a) Show that there is a covariant functor $\boldsymbol{\Delta}^2$ from $\mathsf{Vec}_{\mathbb{K}}$ to itself, assigning to each $V \in \mathsf{Vec}_{\mathbb{K}}$ the dual of its dual space $(V^*)^*$. Describe how this functor acts on morphisms.

   (b) Let Id denote the identity functor on $\mathsf{Vec}_{\mathbb{K}}$, which sends each object and morphism to itself. Construct a natural transformation from Id to $\boldsymbol{\Delta}^2$ that assigns to every $V \in \mathsf{Vec}_{\mathbb{K}}$ a vector space isomorphism $V \to (V^*)^*$.

   (c) Every complex vector space $V \in \mathsf{Vec}_{\mathbb{C}}$ has a **conjugate** space $\bar{V} \in \mathsf{Vec}_{\mathbb{C}}$, defined as the same set with the same notion of vector addition but with scalar multiplication conjugated: in other words, if for each $v \in V$ we denote the same element in $\bar{V}$ by $\bar{v}$, then scalar multiplication on $\bar{V}$ is defined for $\lambda \in \mathbb{C}$ by

   $$\lambda \bar{v} := \overline{\bar{\lambda} v}.$$

   Show that there is a covariant functor $\mathsf{Vec}_{\mathbb{C}} \to \mathsf{Vec}_{\mathbb{C}}$ sending each complex vector space to its conjugate, and describe how it acts on morphisms.

   (d) *(harder?)* Notice that for $V \in \mathsf{Vec}_{\mathbb{C}}$, the map $V \to \bar{V} : v \mapsto \bar{v}$ is not a morphism in $\mathsf{Vec}_{\mathbb{C}}$, as it is complex *antilinear*. Of course V and $\bar{V}$ are both complex vector spaces of the same dimension, so they are *always* isomorphic, but we claim that in contrast to the case of the double dual in part (b), there exists no natural transformation from the identity to the conjugation functor that provides a complex-linear isomorphism $V \to \bar{V}$ for every $V \in \mathsf{Vec}_{\mathbb{C}}$. See if you can convince yourself that this is true.

   Comments: While the problem sounds at first as if it involves only linear algebra, the only solution I can immediately think of requires some topology, and in particular some basic knowledge of vector bundles and characteristic classes. We may discuss these things near the end of the semester if there is time.

[2] It is called the **fundamental groupoid** of X.