PROBLEM SET 10
To be discussed: 9.01.2019

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Assume Σ is a compact oriented surface with nonempty boundary. Prove $H_2(\Sigma \times S^2) \cong \mathbb{Z}$. Can you describe (e.g. in terms of a closed oriented 2-dimensional submanifold) a specific homology class that generates $H_2(\Sigma \times S^2)$?
Hint: The classification of surfaces gives a countable list of possibilities of what Σ could be. Show that Σ is homotopy equivalent to a 1-dimensional cell complex, draw whatever conclusions you can from that, and then use the Künneth formula.

2. In Problem Set 6 #2, you computed $H_*(\mathbb{RP}^2)$ with integer coefficients: the nontrivial groups are $H_0(\mathbb{RP}^2) \cong \mathbb{Z}$ and $H_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.
   (a) Use this and the Künneth formula to prove
   $$H_n(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } n = 1, \\ \mathbb{Z}_2 & \text{for } n = 2, \\ \mathbb{Z}_2 & \text{for } n = 3, \\ 0 & \text{for all other } n \in \mathbb{Z}. \end{cases}$$
   (b) For $n = 0, 1, 2$, describe explicit oriented submanifolds that represent homology classes generating $H_n(\mathbb{RP}^2 \times \mathbb{RP}^2)$.
   Hint: As in Problem Set 9 #4, it is easiest to describe specific homology classes via submanifolds that are also subcomplexes in a cell decomposition, so that the inclusion is a cellular map.
   (c) Describe an explicit cell decomposition of $\mathbb{RP}^2 \times \mathbb{RP}^2$ and a specific element of the corresponding cellular chain complex that represents the nontrivial element of $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2)$.
   Remark: It is not so easy to describe this one as a submanifold, so do not get frustrated if you can't.

3. The goal of this exercise is to prove the associativity of the cross product:
$$(A \times B) \times C = A \times (B \times C) \in H_*(X \times Y \times Z; R)$$
for all $A \in H_*(X; R)$, $B \in H_*(Y; R)$ and $C \in H_*(Z; R)$. Here R may be any commutative ring with unit.
   (a) Use acyclic models to prove that for triples of spaces X, Y, Z, all natural chain maps
   $$\Phi : C_*(X; R) \otimes_R C_*(Y; R) \otimes_R C_*(Z; R) \to C_*(X \times Y \times Z; R)$$
   that act on 0-chains by $\Phi(x \otimes y \otimes z) = (x, y, z)$ are chain homotopic.
   Remark: The statement implicitly assumes that there is a well-defined notion of the tensor product of three chain complexes, which of course is true since there is a canonical chain isomorphism between $(C_*(X; R) \otimes_R C_*(Y; R)) \otimes_R C_*(Z; R)$ and $C_*(X; R) \otimes_R (C_*(Y; R) \otimes_R C_*(Z; R))$. Right?
   (b) Given $A \in H_*(X; R)$, $B \in H_*(Y; R)$ and $C \in H_*(Z; R)$, show that the products $(A \times B) \times C$ and $A \times (B \times C) \in H_*(X \times Y \times Z; R)$ can each be expressed via natural chain maps as in part (a), and conclude that they are identical.

4. We defined inverse limits in lecture in terms of a universal property. Prove each of the following:[1]

[1] All inverse systems in this problem are over an arbitrary directed set $(I, \prec)$.

(a) For any inverse system $\{X_\alpha, \varphi_{\alpha\beta}\}$ of topological spaces,

$$\varprojlim\{X_\alpha\} \cong \left\{ \{x_\alpha\}_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \;\middle|\; \varphi_{\alpha\beta}(x_\beta) = x_\alpha \text{ for all } \beta > \alpha \right\},$$

with the associated maps $\varphi_\alpha : \varprojlim\{X_\beta\} \to X_\alpha$ defined for each $\alpha \in I$ as restrictions of the natural projections $\prod_\beta X_\beta \to X_\alpha$. Show moreover that the topology on $\varprojlim\{X_\alpha\}$ is the weakest one for which the maps φ_α are all continuous.

(b) Consider the special case of part (a) in which the spaces X_α are all subsets (with the subspace topology) of some fixed space X, $\beta > \alpha$ if and only if $X_\beta \subset X_\alpha$ and the maps $\varphi_{\alpha\beta} : X_\beta \to X_\alpha$ are the natural inclusions. Show that $\bigcap_\alpha X_\alpha$ with the natural inclusions $\varphi_\alpha : \bigcap_\beta X_\beta \hookrightarrow X_\alpha$ defines an inverse limit of the system.

(c) Prove that for any inverse system $\{X_\alpha, \varphi_{\alpha\beta}\}$ of nonempty compact Hausdorff spaces, $\varprojlim\{X_\alpha\}$ is a nonempty space.
Hint: This depends on Tychonoff's theorem, which implies that $\prod_{\alpha \in I} X_\alpha$ in this case is a compact space. One possible approach is to construct a net $\{x^\beta \in \prod_\alpha X_\alpha\}_{\beta \in I}$ such that for each $\beta \in I$, the coordinates $x^\beta_\alpha \in X_\alpha$ of x^β satisfy $x^\beta_\alpha = \varphi_{\alpha\beta}(x^\beta_\beta)$ for every $\alpha < \beta$. One can then prove that every cluster point of this net is an element of $\varprojlim\{X_\alpha\}$.

(d) Show that any inverse system of abelian groups $\{G_\alpha, \varphi_{\alpha\beta}\}$ has an inverse limit of the form

$$\varprojlim\{G_\alpha\} = \left\{ \{g_\alpha\}_{\alpha \in I} \in \prod_{\alpha \in I} G_\alpha \;\middle|\; \varphi_{\alpha\beta}(g_\beta) = g_\alpha \text{ for all } \beta \geqslant \alpha \right\},$$

with the associated homomorphisms $\varphi_\alpha : \varprojlim\{G_\alpha\} \to G_\alpha$ defined as restrictions of the natural projections $\prod_\beta G_\beta \to G_\alpha$.

(e) Extend part (d) to describe an inverse limit $(C^\infty_*, \partial^\infty)$ of an inverse system of chain complexes $\{(C^\alpha_*, \partial^\alpha)\}_{\alpha \in I}$, where the $\varphi_{\alpha\beta} : C^\beta_* \to C^\alpha_*$ for $\alpha < \beta$ are chain maps. In particular, how is the boundary map ∂^∞ on C^∞_* defined?

(f) A subset $I_0 \subset I$ is called a **cofinal set** if for every $\alpha \in I$ there exists some $\beta \in I_0$ such that $\beta > \alpha$. Suppose $\{X_\alpha, \varphi_{\alpha\beta}\}$ is an inverse system over $(I, \prec)$ in any category, and $I_0 \subset I$ is a cofinal set with the property that for every $\alpha, \beta \in I_0$ with $\alpha < \beta$, $\varphi_{\alpha\beta} \in \mathrm{Mor}(X_\beta, X_\alpha)$ is an isomorphism. Prove that $\varprojlim\{X_\alpha\}$ is then isomorphic to X_γ for any $\gamma \in I_0$, and describe the associated morphisms $\varprojlim\{X_\beta\} \xrightarrow{\varphi_\alpha} X_\alpha$ for every $\alpha \in I$.
Advice: This problem becomes a bit easier if you work in any of the categories Top, Ab *or* Chain *so that you can use the results of parts (a), (d) or (e). But it can also be done without that assumption, just by using the universal property and playing with commutative diagrams.*

(g) Using part (f), go back to the proof sketched in lecture that $\check{H}_*(X; G) \cong H^\Delta_*(K; G)$ for any compact polyhedron $X = |K|$ and fill in more details.

5. Using the results stated (but not necessarily proved) in lecture about Čech homology, find an example of a path-connected space X for which $\check{H}_1(X; \mathbb{Z}_2) = 0$ but $H_1(X; \mathbb{Z}_2) \neq 0$. Can you also describe a specific nontrivial element of $\pi_1(X)$?
Hint: Take the suspension of something that is connected but not path-connected.

6. Find an example of a compact space X that is connected but not path-connected and is the inverse limit of a system $\{X_\alpha\}$ of path-connected spaces. Conclude that for this example,

$$H_*\Big(\varprojlim\{X_\alpha\}\Big) \not\cong \varprojlim\{H_*(X_\alpha)\}.$$

Hint: Use Problem 4(b).