PROBLEM SET 12
To be discussed: 6.02.2019

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Prove that if M is a non-orientable connected topological manifold, then $\pi_1(M)$ contains a subgroup of index 2. (In particular, this implies that every simply connected manifold is orientable.)

2. Suppose M is any topological manifold of dimension $n \in \mathbb{N}$.

   (a) Prove that the torsion subgroup of $H_{n-1}(M)$ is $\mathbb{Z}_2$ if M is compact and non-orientable, and it is otherwise trivial.
   Hint: Use the universal coefficient theorem to compute $\mathrm{Tor}(H_{n-1}(M), \mathbb{Z}_p) = 0$ *for every prime number* p*, and see what you can deduce from it. You may want to consider separately the cases where* M *is noncompact, compact and orientable, or compact and non-orientable. If it helps, feel free to assume also that* $H_*(M)$ *is finitely generated (though this is not strictly necessary).*

   (b) Deduce that if $H_*(M)$ is finitely generated and M is orientable, then $H^n(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z})$.

3. Recall that if M is a compact n-manifold with boundary, an **R-orientation** of M is defined to be an R-orientation of its interior, i.e. a section $s \in \Gamma(\Theta^R|_{\mathring{M}})$ such that $s(x)$ generates $\Theta^R_x = H_n(M, M\backslash\{x\}; R) \cong R$ for every $x \in \mathring{M}$. The **relative fundamental class** of M is then the unique class $[M] \in H_n(M, \partial M; R)$ such that the map induced by the inclusion $(M, \partial M) \hookrightarrow (M, M\backslash\{x\})$ sends $[M]$ to $s(x)$ for every $x \in \mathring{M}$.

   (a) Show that if M and ∂M are both connected and ∂M is nonempty, then ∂M is also R-orientable, and the connecting homomorphism $\partial_* : H_n(M, \partial M; R) \to H_{n-1}(\partial M; R)$ in the long exact sequence of $(M, \partial M)$ is an isomorphism sending $[M]$ to the fundamental class $[\partial M]$ of ∂M (for a suitable choice of orientation of ∂M).
   Hint: Focus on the case $R = \mathbb{Z}$*. It is easy to prove that* ∂_* *is injective; show that if it were not surjective, then* $H_{n-1}(M)$ *would have torsion, contradicting the result of Problem 3(a).*

   (b) Generalize the result of part (a) to prove $\partial_*[M] = [\partial M]$ without assuming ∂M is connected.
   Hint: For any connected component $N \subset \partial M$*, consider the exact sequence of the triple* $(M, \partial M, \partial M\backslash N)$ *and notice that* $H_{n-1}(\partial M, \partial M\backslash N) \cong H_{n-1}(N)$ *by excision.*

   (c) Conclude that for any compact manifold M with boundary and an R-orientation, the map $H_{n-1}(\partial M; R) \to H_{n-1}(M; R)$ induced by the inclusion $\partial M \hookrightarrow M$ sends $[\partial M]$ to 0. In other words, "the boundary of a compact oriented n-manifold M represents the trivial homology class in $H_{n-1}(M)$."
   Remark: We discussed a similar result in the setting of triangulable manifolds in Lecture 29, but here we are not assuming that any of our manifolds admit triangulations.

4. There is an interesting application of Čech cohomology to the question of orientability of manifolds. Fix a space X and abelian group G, and recall that the set $\mathcal{O}(X)$ of all open coverings of X admits an ordering relation $\prec$ that makes it into a directed set: we write $\mathfrak{U} \prec \mathfrak{U}'$ whenever $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$. There is a direct system of $\mathbb{Z}$-graded abelian groups over $\mathcal{O}(X)$ whose direct limit is Čech cohomology, namely
$$\check{H}^*(X;G) := \varinjlim \left\{ H^*_o\big(\mathcal{N}(\mathfrak{U}); G\big) \right\}_{\mathfrak{U} \in \mathcal{O}(X)},$$
where $\mathcal{N}(\mathfrak{U})$ is the so-called **nerve** of the open covering $\mathfrak{U} \in \mathcal{O}(X)$, defining a simplicial complex, and $H^*_o\big(\mathcal{N}(\mathfrak{U}); G\big)$ is the cohomology with coefficients in G of its ordered simplicial complex (cf. Problem 2 on the take-home midterm). Concretely, $H^*_o\big(\mathcal{N}(\mathfrak{U}); G\big)$ is the homology of a cochain complex

$\check{C}^*(\mathfrak{U};G) := C^*_o\big(\mathcal{N}(\mathfrak{U});G)\big)$, where $\check{C}^n(\mathfrak{U};G) = 0$ for $n < 0$ and, for each $n \geqslant 0$, $\check{C}^n(\mathfrak{U};G)$ is the additive abelian group of all functions φ that assign an element of G to each ordered $(n+1)$-tuple of sets $\mathcal{U}_0, \dots \mathcal{U}_n \in \mathfrak{U}$ with nonempty intersection:

$$\varphi(\mathcal{U}_0, \dots, \mathcal{U}_n) \in G \quad \text{assuming} \quad \mathcal{U}_0 \cap \dots \cap \mathcal{U}_n \neq \varnothing.$$

The coboundary map $\delta : \check{C}^n(\mathfrak{U};G) \to \check{C}^{n+1}(\mathfrak{U};G)$ is defined by

$$(\delta\varphi)(\mathcal{U}_0, \dots, \mathcal{U}_{n+1}) := (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \varphi(\mathcal{U}_0, \dots, \widehat{\mathcal{U}}_k, \dots, \mathcal{U}_{n+1}),$$

where the hat over $\widehat{\mathcal{U}}_k$ means that that term is skipped. The homologies of these cochain complexes form a direct system over $(\mathcal{O}(X), \prec)$ because, as mentioned in Lecture 44, refinements $\mathfrak{U}' \succ \mathfrak{U}$ give rise to chain maps $C^o_*\big(\mathcal{N}(\mathfrak{U}')\big) \to C^o_*\big(\mathcal{N}(\mathfrak{U})\big)$ that are canonical up to chain homotopy, so dualizing these gives chain maps $\check{C}^*(\mathfrak{U};G) \to \check{C}^*(\mathfrak{U}';G)$ that are also canonical up to chain homotopy and therefore induce canonical maps on the cohomology groups (see the notes for Lecture 46).

Let us call an open covering $\mathfrak{U}$ *admissible* if intersections between two sets in $\mathfrak{U}$ are always connected; this will be a useful technical condition in the following, and one can show that at least if X is a smooth manifold, every open covering of X has an admissible refinement, so assume this from now on.[1] We are going to consider covering[2] maps $f : Y \to X$ of degree 2. Recall that two such covering maps (Y_i, f_i) for $i = 1, 2$ are called **isomorphic** if there exists a homeomorphism $\varphi : Y_1 \to Y_2$ such that the diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{\ \varphi\ } & Y_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & X & \end{array}$$

commutes. We will say that a covering map (Y, f) is **trivial** if it is isomorphic to the **trivial double cover**

$$X \times \mathbb{Z}_2 \to X : (x, i) \mapsto x.$$

Given $f : Y \to X$, any open covering $\mathfrak{U} \in \mathcal{O}(X)$ can be replaced with a refinement such that every $\mathcal{U} \in \mathfrak{U}$ is **evenly covered** by $f : Y \to X$, meaning $f^{-1}(\mathcal{U})$ is the union of two disjoint subsets $\mathcal{V}_0, \mathcal{V}_1 \subset Y$ such that $f|_{\mathcal{V}_i} : \mathcal{V}_i \to \mathcal{U}$ is a homeomorphism for $i = 0, 1$. After a further refinement, assume $\mathfrak{U}$ is also admissible. We can now choose for each $\mathcal{U} \in \mathfrak{U}$ a so-called **local trivialization**, meaning a homeomorphism

$$\Phi_{\mathcal{U}} : f^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{Z}_2$$

that sends $f^{-1}(x)$ to $\{x\} \times \mathbb{Z}_2$ for each $x \in \mathcal{U}$. This determines a set of continuous **transition functions** $g_{\mathcal{U},\mathcal{V}} : \mathcal{U} \cap \mathcal{V} \to \mathbb{Z}_2$ for each intersecting pair $\mathcal{U}, \mathcal{V} \in \mathfrak{U}$, defined such that the map

$$(\mathcal{U} \cap \mathcal{V}) \times \mathbb{Z}_2 \xrightarrow{\Phi_{\mathcal{V}} \circ \Phi_{\mathcal{U}}^{-1}} (\mathcal{U} \cap \mathcal{V}) \times \mathbb{Z}_2$$

takes the form $(x, i) \mapsto (x, i + g_{\mathcal{U},\mathcal{V}}(x))$. Note that since $\mathcal{U} \cap \mathcal{V}$ is always assumed connected, the transition functions are all constant, i.e. they associate to each ordered pair $(\mathcal{U}, \mathcal{V})$ of sets in $\mathfrak{U}$ with $\mathcal{U} \cap \mathcal{V} \neq \varnothing$ an element $\varphi(\mathcal{U}, \mathcal{V}) := g_{\mathcal{U},\mathcal{V}} \in \mathbb{Z}_2$. See if you can prove the following:

(a) $\varphi \in \check{C}^1(\mathfrak{U};\mathbb{Z}_2)$ is a cocycle, and choosing different local trivializations changes φ by a coboundary.

(b) Feeding $[\varphi] \in H^1_o(\mathcal{N}(\mathfrak{U});\mathbb{Z}_2)$ into the canonical map to the direct limit produces a class $w_1(f) \in \check{H}(X;\mathbb{Z}_2)$ that is independent of the choice of admissible open covering.

(c) If X is an n-manifold and $f : Y \to X$ is its orientation double cover, then $w_1(X) := w_1(f)$ is zero if and only if X is orientable. (We call $w_1(X)$ the first **Stiefel-Whitney class** of X.)

[1] Alternatively, one could avoid the need for connected intersections by using Čech cohomology with sheaf coefficients, cf. Chapter 6 in Spanier's book.

[2] Caution! This problem now contains two distinct meanings of the word "cover": one in the sense of "open covering" *(Überdeckung)* and the other in the sense of "covering map" *(Überlagerung)*. I am trying very hard to ensure that it would be clear in each instance which meaning is intended.