PROBLEM SET 5
To be discussed: 21.11.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. In lecture we used the suspension to show that every integer can be the degree of a map $f : S^n \to S^n$ for $n \geqslant 1$. This cannot be true for $n = 0$ since there are only four possible maps $f : S^0 \to S^0$.

   (a) Compute $\deg(f) \in \mathbb{Z}$ for each of the four maps $f : S^0 \to S^0$.
   Hint: You will need a fairly concrete picture of $\widetilde{H}_0(S^0;\mathbb{Z})$, which can be found in Problem Set 3 #2.

   (b) Prove directly that if $f : S^0 \to S^0$ has degree $d \in \mathbb{Z}$ and h_* is any axiomatic homology theory, then the induced map $f_* : \widetilde{h}_0(S^0) \to \widetilde{h}_0(S^0)$ takes the form $f_*c = dc$.
   Hint: Same hint as for part (a).

   (c) Write down examples of maps $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(0) = 0$ with 0 as an isolated point of $f^{-1}(0)$ and achieve all of the possible values of the local degree $\deg(f; 0) \in \mathbb{Z}$. Also compute $\deg_2(f; 0) \in \mathbb{Z}_2$ for these maps.

2. Viewing S^1 as the unit circle in $\mathbb{C}$, fix a generator of $[S^1] \in H_1(S^1;\mathbb{Z})$ and use it to determine local orientations $[\mathbb{C}]_z \in H_n(\mathbb{C}, \mathbb{C}\backslash\{z\};\mathbb{Z})$ for every point $z \in \mathbb{C}$ via the natural isomorphisms $H_2(\mathbb{C}, \mathbb{C}\backslash\{z\};\mathbb{Z}) \cong H_2(\mathbb{D}_z, \partial\mathbb{D}_z;\mathbb{Z}) \cong H_1(\partial\mathbb{D}_z;\mathbb{Z})$, where $\mathbb{D}_z \subset \mathbb{C}$ denotes the closed unit disk centered at z, whose boundary is canonically identified with S^1. This choice will be used in the following for the definition of local degrees of maps $f : \mathcal{U} \to \mathbb{C}$ defined on open subsets $\mathcal{U} \subset \mathbb{C}$; note that changing the generator $[S^1] \in H_1(S^1;\mathbb{Z})$ does not change the definition of $\deg(f; z)$ since it changes *both* $[\mathbb{C}]_z$ and $[\mathbb{C}]_{f(z)}$ by a sign.

   (a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = (z - z_0)^k g(z)$ for some $z_0 \in \mathbb{C}$, $k \in \mathbb{N}$ and g a continuous map with $g(z_0) \neq 0$, then $\deg(f; z_0) = k$.

   (b) Can you modify the example in part (a) to produce one with $\deg(f; z_0) = -k$ for $k \in \mathbb{N}$?
   Hint: Try complex conjugation.

   (c) Suppose $\mathcal{U} \subset \mathbb{C}$ is open and $f : \mathcal{U} \to \mathbb{C}$ is continuous with $f(z_0) = w_0$ and $\deg(f; z_0) \neq 0$ for some $z_0 \in \mathcal{U}$, $w_0 \in \mathbb{C}$. Prove that for any neighborhood $\mathcal{V} \subset \mathcal{U}$ of z_0, there exists an $\epsilon > 0$ such that every continuous map $\widehat{f} : \mathcal{U} \to \mathbb{C}$ satisfying $|\widehat{f} - f| < \epsilon$ maps some point in $\mathcal{V}$ to w_0.
   Hint: Consider the loop around w_0 defined by restricting f to a small circle $C \subset \mathbb{C}$ about z_0, and normalize so that you can view it as a map $S^1 \cong C \to S^1$. What can you say about the degree of this map if f sends all the points enclosed by C to $\mathbb{C}\backslash\{w_0\}$?
   Remark: I have stated this problem for maps on $\mathbb{C}$ just for convenience, but one can do something similar with maps on open subsets of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

   (d) Find an example of a continuous map $f : \mathbb{C} \to \mathbb{C}$ that has an isolated zero at the origin with $\deg(f; 0) = 0$ and admits arbitrarily small continuous perturbations that are nowhere zero.
   Hint: Parts (a) and (b) tell you that the map you're looking for cannot be holomorphic, nor even antiholomorphic. You should probably not think in complex terms, but instead identify $\mathbb{C}$ with $\mathbb{R}^2$.

   (e) Let $f : S^2 \to S^2$ denote the natural continuous extension to $S^2 := \mathbb{C} \cup \{\infty\}$ of a complex polynomial $\mathbb{C} \to \mathbb{C}$ of degree n. What is $\deg(f)$?

   (f) Pick a constant $t_0 \in S^1$ and let $A \cong S^1 \vee S^1$ denote the subset $\{(x, y) \mid x = t_0 \text{ or } y = t_0\} \subset S^1 \times S^1 = \mathbb{T}^2$. Show that $\mathbb{T}^2/A \cong S^2$, and that the quotient map $\mathbb{T}^2 \to \mathbb{T}^2/A$ has degree ± 1 (depending on choices of generators for $H_2(\mathbb{T}^2;\mathbb{Z})$ and $H_2(S^2;\mathbb{Z})$).

3. Suppose $f : S^n \to S^n$ is any continuous map, and $p_+ \in SS^n = C_+S^n \cup_{S^n} C_-S^n$ is the vertex of the top cone in the suspension $SS^n \cong S^{n+1}$. What is $\deg(Sf; p_+)$? Use this to give a new proof (different from the one we saw in lecture) that $\deg(Sf) = \deg(f)$.

4. (a) Show that every map $S^n \to \mathbb{T}^n$ has degree 0 if $n \geqslant 2$.
   Hint: Lift $S^n \to \mathbb{T}^n$ to the universal cover of $\mathbb{T}^n$.

   (b) Show that for every $d \in \mathbb{Z}$ and every $\mathbb{Z}$-admissible n-dimensional manifold M with $n \geqslant 1$, there exists a map $M \to S^n$ of degree d.
   Hint: Try a map that is interesting only on some n-ball in M and constant everywhere else.

5. For these problems you need to use the mod 2 degree, since $\mathbb{RP}^2$ and the Klein bottle are $\mathbb{Z}_2$-admissible but not $\mathbb{Z}$-admissible.

   (a) Find an example of a map $\mathbb{RP}^2 \to S^2$ that cannot be homotopic to a constant.

   (b) Same problem but with $\mathbb{RP}^2$ replaced by the Klein bottle.
   Hint: Problem 2(f) might provide some useful inspiration.

6. (a) Prove that for every positive even integer n, every continuous map $f : S^n \to S^n$ has at least one point $x \in S^n$ where either $f(x) = x$ or $f(x) = -x$. Deduce that every continuous map $\mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point if n is even.

   (b) Construct counterexamples to the statement in part (a) for every odd n.
   Hint: Consider linear transformations with no real eigenvalues.