Topology II — Humboldt-Universität zu Berlin
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PROBLEM SET 6
To be discussed: 28.11.2018

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. The set $\mathbb{R}^\infty := \bigoplus_{j\in\mathbb{N}} \mathbb{R}$ consists of all sequences of real numbers $(x_1, x_2, x_3, \dots)$ such that at most finitely many terms are nonzero. Identifying $\mathbb{R}^n$ for each $n \in \mathbb{N}$ with the subset

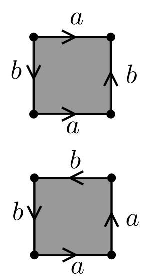
$$\left\{(x_1, x_2, x_3, \dots) \in \mathbb{R}^\infty \;\middle|\; x_j = 0 \text{ for all } j > n\right\},$$

we can define a topology on $\mathbb{R}^\infty$ such that a set $\mathcal{U} \subset \mathbb{R}^\infty$ is open if and only if $\mathcal{U} \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ (with its standard topology) for all $n \in \mathbb{N}$.[1] Notice that every element $\mathbf{x} \in \mathbb{R}^\infty$ belongs to $\mathbb{R}^n$ for sufficiently large $n \in \mathbb{N}$. Prove that for any convergent sequence $\mathbf{x}^k \to \mathbf{x} \in \mathbb{R}^\infty$, there exists a (possibly larger) number $N \in \mathbb{N}$ such that $\mathbf{x}^k \in \mathbb{R}^N$ for all k. Deduce from this that every compact subset $K \subset \mathbb{R}^\infty$ is contained in $\mathbb{R}^N$ for some N sufficiently large (depending on K).

2. Recall that $\mathbb{RP}^n$ has a cell decomposition with one k-cell for every $k = 0, \dots, n$, derived by starting from the cell decomposition of S^n with two cells in each dimension and dividing the whole thing by a $\mathbb{Z}_2$-action. Use this to compute $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Z})$, $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Z}_2)$ and $H_*^{\mathrm{CW}}(\mathbb{RP}^n; \mathbb{Q})$.

3. For integers $g \geqslant 0$ and $m \geqslant 1$, let $\Sigma_{g,m}$ denote the compact surface with boundary obtained by deleting m open disks from the closed oriented surface Σ_g of genus g. Assuming the isomorphism between singular and cellular homology, compute $H_*(\Sigma_{g,m}; G)$ with G an arbitrary coefficient group. *Hint: Since singular homology is homotopy invariant, you are free to replace $\Sigma_{g,m}$ by a CW-complex that is homotopy equivalent to it.*

4. The picture at the right shows two spaces that you may recall from *Topologie I* are both homeomorphic to the Klein bottle. Each also defines a cell complex $X = X^0 \cup X^1 \cup X^2$ consisting of one 0-cell, two 1-cells (labeled a and b) and one 2-cell.



   (a) Compute $H_*^{\mathrm{CW}}(X; \mathbb{Z})$, $H_*^{\mathrm{CW}}(X; \mathbb{Z}_2)$ and $H_*^{\mathrm{CW}}(X; \mathbb{Q})$ for both complexes. (You'll know you've done something wrong if the answers you get from the two complexes are not isomorphic!)

   (b) Recall that the **rank** (*Rang*) of a finitely generated abelian group G is the unique integer $k \geqslant 0$ such that $G \cong \mathbb{Z}^k \oplus T$ for some finite group T. Verify for both cell decompositions of the Klein bottle above that

$$\sum_k (-1)^k \operatorname{rank} H_k^{\mathrm{CW}}(X; \mathbb{Z}) = \sum_k (-1)^k \dim_{\mathbb{Z}_2} H_k^{\mathrm{CW}}(X; \mathbb{Z}_2) = \sum_k (-1)^k \dim_{\mathbb{Q}} H_k^{\mathrm{CW}}(X; \mathbb{Q}) = 0.$$

   (Congratulations, you've just computed the ***Euler characteristic*** *of the Klein bottle!)*

[1] This is the right topology so that the subspace topology on the set of unit vectors $S^\infty \subset \mathbb{R}^\infty$ matches the topology defined on S^∞ via the infinite-dimensional cell decomposition we discussed in lecture.