Topology II | Humboldt-Universität zu Berlin
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TAKE-HOME MIDTERM

Instructions

To receive credit for this assignment, you must hand it in by **Wednesday, January 23** before the Übung. The solutions will be discussed in the Übung on that day.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

There are 100 points in total; a score of 75 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. Note that the number of points assigned to each part of each problem is usually proportional to its conceptual importance/difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. in cases where you want to show that two spaces are homotopy equivalent and can justify it with a very convincing picture (use your own judgement). You are free to make use of all results we've proved in lectures or problem sets, without reproving them. (When using a result from a problem set or the lecture notes, say explicitly which one.)

One more piece of general advice: if you get stuck on one part of a problem, it may often still be possible to move on and do the next part.

You are free to ask for clarification or hints via e-mail or in office hours; of course I reserve the right not to answer such questions.

Problems

1. [25 pts total] Consider a knot $K \subset \mathbb{R}^3$, i.e. the image of a topological embedding[1] $S^1 \hookrightarrow \mathbb{R}^3$. For technical reasons, it is conventional in knot theory to assume that K is not too "wild," for instance it is good enough to assume that the embedding $S^1 \hookrightarrow \mathbb{R}^3$ is smooth (meaning C^∞).

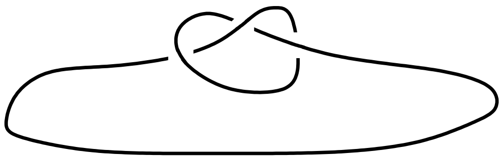

Figure 1: A smooth knot.

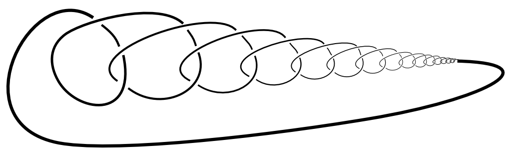

Figure 2: A "wild" knot, which is continuous, but not smooth. We will not consider these.

The smoothness condition has the following advantage: if K is the image of $f : S^1 \hookrightarrow \mathbb{R}^3$, we can always assume there exists an extension of f to a topological embedding $S^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ that matches f along $S^1 \times \{0\}$. (Take a moment to convince yourself that no such extension exists for the knot in Figure 2.) We shall denote the image of this extension by $N \subset \mathbb{R}^3$, so

$$K \subset N \subset \mathbb{R}^3 \quad \text{where} \quad N \cong S^1 \times \mathbb{D}^2.$$

[1] Recall that a map $f : X \to Y$ between two topological spaces is called a **topological embedding** if it is continuous and injective and the inverse $f^{-1} : f(X) \to X$ is also continuous with respect to the subspace topology on $f(X) \subset Y$.

One way to distinguish topologically between two knots is via their **knot groups**, meaning the group $\pi_1(\mathbb{R}^3 \setminus K)$. As you might recall from *Topology I*, one can equivalently extend $\mathbb{R}^3$ to its one-point compactification S^3 and replace $\pi_1(\mathbb{R}^3 \setminus K)$ with $\pi_1(S^3 \setminus K)$, as it is easy to show via the Seifert-van Kampen theorem that these two groups are isomorphic. With this in mind, we lose nothing by regarding all knots as subsets of S^3.

(a) [15 pts] Prove that for every knot K, the abelianization of its knot group is isomorphic to $\mathbb{Z}$.
Hint: Consider the Mayer-Vietoris sequence for $S^3 = N \cup (S^3 \setminus K)$.

(b) [10 pts] Draw a picture of the knot K in Figure 1 together with a loop in $S^3 \setminus K$ representing a generator of the abelianization of $\pi_1(S^3 \setminus K)$.
Advice: For the purposes of this problem, you should imagine S^3 as $\mathbb{R}^3$ with an extra "point at infinity" that cannot be be shown in the picture.

Remark: Note that since the result of Problem 1(a) does not depend on the knot K, it is bad news if your goal is to distinguish inequivalent knots—you cannot do so by distinguishing the abelianizations of their knot groups. One has to find cleverer algebraic tricks for distinguishing two non-isomorphic knot groups, e.g. one such trick involving the center *of $\pi_1(S^3 \setminus K)$ is used for torus knots in Example 1.24 of Hatcher.*

2. [50 pts total] Let $K = (V, S)$ be a simplicial complex, where V denotes the set of vertices and S the set of simplices. Recall that the usual simplicial chain complex $C_*^\Delta(K)$ is generated by so-called *oriented* simplices

$$[v_0, \ldots, v_n] \in C_n^\Delta(K),$$

where $\{v_0, \ldots, v_n\} \in S$ is an n-simplex of K and an equivalence relation is imposed on $C_n^\Delta(K)$ such that for any permutation $\pi : \{0, \ldots, n\} \to \{0, \ldots, n\}$ with even/odd parity $|\pi| \in \{0, 1\}$,

$$[v_0, \ldots, v_n] = (-1)^{|\pi|}[v_{\pi(0)}, \ldots, v_{\pi(n)}].$$

Simplicial homology $H_*^\Delta(K)$ is conventionally defined as the homology of the chain complex $C_*^\Delta(K)$. It was mentioned in the notes for Lecture 44 (though we did not have time for this in the lecture itself) that there is an alternative way to define simplicial homology, via the so-called *ordered* simplicial complex $C_*^o(K)$. We define $C_n^o(K)$ for each $n \ge 0$ as the free abelian group generated by the set of all ordered tuples

$$(v_0, \ldots, v_n) \in V^{n+1}$$

that satisfy the following condition: there exists $\sigma \in S$ such that $v_0, \ldots, v_n \in \sigma$. Note that in this definition, the vertices $v_0, \ldots, v_n$ in the tuple need not all be distinct, whereas it is implicitly assumed whenever we write down an oriented simplex $[v_0, \ldots, v_n]$ that the v_i are all distinct. The boundary map on $C_*^o(K)$ is defined via the same formula as in $C_*^\Delta(K)$, namely

$$\partial(v_0, \ldots, v_n) := \sum_{k=0}^{n} (-1)^k (v_0, \ldots, \widehat{v}_k, \ldots, v_n),$$

where the hat notation is used to indicate that v_k is skipped, i.e.

$$(v_0, \ldots, \widehat{v}_k, \ldots, v_n) := (v_0, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n).$$

Any simplicial map $f : K \to L$ then induces a chain map $f_* : C_*^o(K) \to C_*^o(L)$ taking $(v_0, \ldots, v_n)$ to $(f(v_0), \ldots, f(v_n))$, and we shall denote the homology of the chain complex $C_*^o(K)$ by $H_*^o(K)$.[2]

The goal of this problem is to prove that there is a natural isomorphism

$$H_*^o(K) \cong H_*^\Delta(K).$$

[2] I am using integer coefficients in this whole discussion, but one can also introduce an arbitrary abelian coefficient group G in the usual way by applying the functor $\otimes G$ to the chain complexes.

The proof starts with the observation that there is a canonical chain map

$$\Phi : C^o_*(K) \to C^\Delta_*(K) : (v_0, \dots, v_n) \mapsto \begin{cases} 0 & \text{if } v_i = v_j \text{ for some } i \neq j, \\ [v_0, \dots, v_n] & \text{otherwise.} \end{cases}$$

I suggest taking a moment to convince yourself that Φ really is a chain map. Moreover, it has the property that for any subcomplex $L \subset K$, Φ maps $C^o_*(L)$ to $C^\Delta_*(L)$. This can be interpreted as a form of *naturality* if we view C^o_* and C^Δ_* as functors on the category of subcomplexes of K, with morphisms $L \to L'$ defined by inclusion: indeed, any nested pair of subcomplexes $L \subset L' \subset K$ gives rise to a commutative diagram

$$\begin{tikzcd} C^o_*(L) \arrow[r, "\Phi"] \arrow[d] & C^\Delta_*(L) \arrow[d] \\ C^o_*(L') \arrow[r, "\Phi"] & C^\Delta_*(L'), \end{tikzcd}$$

where the two vertical maps are the chain maps induced by the inclusion $L \hookrightarrow L'$.

To prove that $\Phi_* : H^o_*(K) \to H^\Delta_*(K)$ is an isomorphism, we would like to show that $\Phi : C^o_*(K) \to C^\Delta_*(K)$ is a chain homotopy equivalence, which means finding a chain homotopy inverse $\Psi : C^\Delta_*(K) \to C^o_*(K)$. This inverse will *not* be canonically defined, but we shall prove its existence and uniqueness up to chain homotopy using the method of acyclic models (cf. Lecture 43).

As a preliminary step, we need to introduce a *reduced* version of simplicial homology. The definition should seem familiar: assume P is a simplicial complex with only one vertex, let $\epsilon : K \to P$ denote the unique simplicial map, and define

$$\widetilde{H}^o_*(K) := \ker\left(H^o_*(K) \xrightarrow{\epsilon_*} H^o_*(P) \right), \qquad \widetilde{H}^\Delta_*(K) := \ker\left(H^\Delta_*(K) \xrightarrow{\epsilon_*} H^\Delta_*(P) \right).$$

(a) [10 pts] Prove

$$H^o_n(K) \cong \begin{cases} \widetilde{H}^o_n(K) \oplus \mathbb{Z} & \text{if } n = 0, \\ \widetilde{H}^o_n(K) & \text{if } n \neq 0, \end{cases}$$

and that the analogous relation between $\widetilde{H}^\Delta_*(K)$ and $H^\Delta_*(K)$ also holds.

(b) [10 pts] Show that $\widetilde{H}^o_*(K)$ is also the homology of an *augmented* chain complex $\widetilde{C}^o_*(K)$ of the form

$$\dots \longrightarrow C^o_2(K) \xrightarrow{\partial} C^o_1(K) \xrightarrow{\partial} C^o_0(K) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots,$$

i.e. $\widetilde{C}^o_n(K) = C^o_n(K)$ for all $n \neq -1$ but $\widetilde{C}^o_{-1}(K) = \mathbb{Z}$. Describe the *augmentation* map $\epsilon : C^o_0(K) \to \mathbb{Z}$ in this complex explicitly, and show that the analogous statement also holds for $\widetilde{H}^\Delta_*(K)$.

We next define a simplicial analogue of the *cone* of a topological space. Let $CK = (CV, CS)$ denote the simplicial complex whose vertex set CV is the union of V with one extra element $v_\infty \notin V$, and whose simplices consist of all sets of the form $\sigma \cup \{v_\infty\}$ for $\sigma \in S$, plus all their subsets. It is not hard to show that the polyhedron $|CK|$ is then homeomorphic to the cone of $|K|$, thus it is contractible, and the isomorphism $H^\Delta_*(CK) \cong H_*(|CK|)$ implies $\widetilde{H}^\Delta_*(CK) = 0$. But this does not immediately imply $\widetilde{H}^o_*(CK) = 0$ since we haven't yet proved H^o_* and H^Δ_* are isomorphic.

(c) [10 pts] For integers $n \geq 0$, consider the homomorphism $h : C^o_n(CK) \to C^o_{n+1}(CK)$ defined by

$$h(v_0, \dots, v_n) := (v_\infty, v_0, \dots, v_n).$$

Find a definition of $h : \mathbb{Z} = \widetilde{C}^o_{-1}(CK) \to C^o_0(CK)$ that makes

$$\widetilde{C}^o_*(CK) \xrightarrow{h} \widetilde{C}^o_{*+1}(CK)$$

into a chain homotopy between the chain maps $\mathbb{1} : \widetilde{C}^o_*(CK) \to \widetilde{C}^o_*(CK)$ and $0 : \widetilde{C}^o_*(CK) \to \widetilde{C}_*(CK)$, and deduce that $\widetilde{H}^o_*(CK) = 0$.

(d) [20 pts] for a given simplicial complex K, let us say that a chain map $\Psi : C_*^\Delta(K) \to C_*^o(K)$ is *natural* if $C_0^\Delta(K) \xrightarrow{\Psi} C_0^o(K)$ takes the form

$$\Psi[v] := (v)$$

and for every subcomplex $L \subset K$, Ψ sends $C_*^\Delta(L)$ into $C_*^o(L)$. It follows that any nested pair of subcomplexes $L \subset L' \subset K$ gives rise to a commutative diagram

$$\begin{tikzcd} C_*^\Delta(L) \arrow[r, "\Psi"] \arrow[d] & C_*^o(L) \arrow[d] \\ C_*^\Delta(L') \arrow[r, "\Psi"] & C_*^o(L'), \end{tikzcd}$$

where the vertical maps are again the chain maps induced by the inclusion $L \hookrightarrow L'$. Use the method of acyclic models to prove that a natural chain map $\Psi : C_*^\Delta(K) \to C_*^o(K)$ exists and is unique up to chain homotopy.
Hint: You need to construct $\Psi : C_n^\Delta(K) \to C_n^o(K)$ by induction on the degree n, and for the inductive step, the main task is to define it on "model" subcomplexes $L \subset K$ that consist of a single n-simplex and all its faces. Deduce from part (c) that this can always be done because $\widetilde{H}_*^o(L) = 0$ for all such subcomplexes. Then show that the definition of $\Psi : C_n^\Delta(K) \to C_n^o(K)$ follows uniquely from this via the naturality property.

If you've gotten this far, then you can probably guess how the rest of the proof that $H_*^\Delta(K) \cong H_*^o(K)$ goes: one must similarly show the uniqueness up to chain homotopy of natural chain maps $C_*^o(K) \to C_*^\Delta(K)$, $C_*^o(K) \to C_*^o(K)$ and $C_*^\Delta(K) \to C_*^\Delta(K)$. The existence of these chain maps does not need to be proved, because we already have examples, namely Φ, $\mathbb{1}$ and $\mathbb{1}$ respectively, thus the uniqueness implies that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are each chain homotopic to the identity map. I suggest you work out the details the next time you get bored on a long train ride.

3. [25 pts total] Assume X is a path-connected space. Under the usual identification of singular 1-simplices $\sigma : \Delta^1 \to X$ with paths $\gamma : I \to X$, a singular 1-cochain $\varphi \in C^1(X; G)$ can be regarded as a function assigning to each path γ an element $\varphi(\gamma) \in G$.

   (a) [15 pts] Show that $\varphi \in C^1(X; G)$ is a cocycle if and only if it satisfies both of the following:

   (i) For all paths $\gamma : I \to X$, $\varphi(\gamma) \in G$ depends only on the homotopy class of γ with fixed end points;

   (ii) For every pair of paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0)$, $\varphi(\alpha \cdot \beta) = \varphi(\alpha) + \varphi(\beta)$.

   (b) [10 pts] Show that $\varphi \in C^1(X; G)$ is a coboundary if and only if there exists a function[3] $\psi : X \to G$ such that for all paths $\gamma : I \to X$, $\varphi(\gamma) = \psi(\gamma(1)) - \psi(\gamma(0))$. Deduce from this that for any $x \in X$ and $[\varphi] \in H^1(X; G)$, there is a well-defined homomorphism $\Psi_\varphi : \pi_1(X, x) \to G$ given by

$$\Psi_\varphi([\gamma]) = \varphi(\gamma) \quad \text{for paths } x \overset{\gamma}{\rightsquigarrow} x,$$

   which depends only on the cohomology class $[\varphi] \in H^1(X; G)$.

Remark: One can show in fact that the map $\Psi : H^1(X; G) \to \mathrm{Hom}(\pi_1(X, x), G) : [\varphi] \mapsto \Psi_\varphi$ is an isomorphism. For further hints on this, see the notes for Lecture 45.

[3] Note that since G is not assumed to have any topology in this discussion, there is no continuity assumption on the function $\psi : X \to G$.