Recall: for $f \in L^1(\mathbb{R}^n)$, defn for each $r > 0$,

$$f^r(x) := \frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f(x)| \, dm, \qquad f^0(x) := \limsup_{r \to 0} f^r(x)$$

Want to prove: Given $\varepsilon > 0$, $f^0(x) < \varepsilon$ $\forall x$ outside a set of measure $< \varepsilon$.

($\Rightarrow f^0 = 0$ a.e., i.e. Lebesgue diff. thm.)

Choose seq of continuous fns $f_k \xrightarrow{L^1} f$, then

$$f^r(x) \le \underbrace{\frac{1}{m(B_r(x))} \int_{B_r(x)} |f - f_k| \, dm}_{=:A} + \underbrace{\frac{1}{m(B_r(x))} \int_{B_r(x)} |f_k - f_k(x)| \, dm}_{=:B} + \underbrace{|f_k(x) - f(x)|}_{=:C}$$

Continuity $\Rightarrow$ given $k$, $\forall x$ & all $r > 0$ suff. small, $B < \frac{\varepsilon}{3}$.

claim: If $k$ large enough s.t. $\|f_k - f\|_{L^1}$ is suff. small, then $\forall r > 0$, $A, C < \frac{\varepsilon}{3}$ $\forall x$ outside a set of measure $< \varepsilon$.

pf for $C$: Let $g := f_k - f \in L^1(\mathbb{R}^n)$; for $t > 0$, $A_t := \{x \in \mathbb{R}^n \mid |g(x)| > t\}$,

then $\|g\|_{L^1} = \int_{\mathbb{R}^n} |g(x)| \, dx \ge \int_{A_t} |g(x)| \, dx > t \, m(A_t)$

$$\Rightarrow \boxed{m(\{x ; |g(x)| > t\}) \le \frac{\|g\|_{L^1}}{t}} \quad \text{"Chebyshev's ineq."}$$

Now set $t := \frac{\varepsilon}{3}$, then $|f_k(x) - f(x)| \le \frac{\varepsilon}{3}$ outside a set of measure $\le \frac{3\|f_k - f\|_{L^1}}{\varepsilon} < \varepsilon$ if we choose $k$ large s.t. $\|f_k - f\|_{L^1} < \frac{\varepsilon^2}{3}$.

pf for $A$:

defn: For $g \in L^1_{loc}(\mathbb{R}^n)$, the maximal fn of $g$ is $Mg : \mathbb{R}^n \to [0, \infty]$,

$$Mg(x) := \sup_{r > 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g| \, dm$$

defn: For $g : \mathbb{R}^n \to V$ measurable,

$$\|g\|_{L^1_{weak}} := \sup_{t > 0} t \cdot m(\{x ; |g(x)| > t\}) = \inf \{C > 0 \mid m(\{x ; |g(x)| > t\}) \le \tfrac{C}{t} \ \forall t > 0\}.$$

If $\|g\|_{L^1_{weak}} < \infty$, we say $g$ is weakly integrable.

Caution: $\|\cdot\|_{L^1_{weak}}$ does not satisfy $\Delta$-ineq; not a norm!

thm (Hardy-Littlewood maximal ineq.): $\exists$ $C > 0$ dep. only on $n$ s.t.

$$\|Mg\|_{L^1_{weak}} \le C \|g\|_{L^1} \quad \forall g \in L^1(\mathbb{R}^n).$$

Cor: $\|f_k - f\|_{L^1}$ suff. small $\Rightarrow$ $\|M(f_k - f)\|_{L^1_{weak}}$ arbitrarily small

$\Rightarrow$ $M(f_k - f)(x) < \frac{\varepsilon}{3}$ $\forall x$ outside a set of measure $\varepsilon$. $\square$

remaining goal ("step 2" in FTC): $F : [a,b] \to V$ abs. contin. $\Rightarrow$

$$F(x) = c + \int_a^x f(t)\,dt \quad \text{for some const. } c \in V \ \& \ f \in L^1([a,b]).$$

related Q: On a measure space $(X, \mu)$, which measures $\lambda$ on the same $\sigma$-algebra can be expressed as $\lambda(A) = \int_A f\,d\mu$ for some measurable $f : X \to [0, \infty]$?

defn: When this holds, call $f =: \frac{d\lambda}{d\mu}$ the "Radon-Nikodym derivative" of $\lambda$ w.r.t. $\mu$.

necessary cond.: $\mu(A) = 0 \Rightarrow \lambda(A) = 0 \quad \forall A.$

defn: If this holds, we say $\lambda$ is absolutely continuous w.r.t. $\mu$ ("$\lambda \ll \mu$")

ex. On $\mathbb{R}$, Dirac measure $\delta(A) := \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$, then $\int_{\mathbb{R}} f\,d\delta = f(0) \ \forall f$

then $\delta \not\ll m \Rightarrow \nexists$ "Dirac $\delta$-fn" satisfying $\int_{\mathbb{R}} f\,d\delta = f(0) = \int_{\mathbb{R}} f(x)\,\underbrace{\delta(x)}_{\substack{\text{R-N deriv}\\ \text{of } \delta \text{ w.r.t. } m.}}\,dx$

Radon-Nikodym Thm: If $\mu$ & $\lambda$ are $\sigma$-finite measures on the same $\sigma$-algebra, then $\lambda \ll \mu \Leftrightarrow \exists$ a R.-N. deriv. $\frac{d\lambda}{d\mu}$.

<u>relation to FTC</u>: Assume $F : [a,b] \to V$ abs. contin.

<u>case 1</u>: $F : [a,b] \to \mathbb{R}$ is <u>strictly</u> <u>increasing</u>.

<u>EX</u>: $\forall\ A \subseteq [a,b]$ with $m(A) = 0$, $m(F(A)) = 0$. (<u>recall</u>: false for Cantor fn.)

$\Rightarrow \lambda(A) := m(F(A))$ defines a measure on $[a,b]$ s.t. $\lambda \ll m$

(e.g. strictly increasing $\Rightarrow$ for $A_1, A_2, \ldots \subseteq [a,b]$ disjoint, $F(A_1), F(A_2), \ldots$ also disjoint

$$\Rightarrow \lambda\left(\bigcup_{j=1}^{\infty} A_j\right) = m\left(F\left(\bigcup_j A_j\right)\right) = m\left(\bigcup_j F(A_j)\right) = \sum_j m(F(A_j)) = \sum_j \lambda(A_j).)$$

R.-N. $\Rightarrow \exists$ measurable fn. $f : [a,b] \to [0,\infty]$ s.t. $\forall\ A \subseteq [a,b]$,

$\lambda(A) = m(F(A)) = \int_A f\, dm$. In particular, $A := [a,x] \Rightarrow$

$$m(F([a,x])) = F(x) - F(a) = \int_{[a,x]} f\, dm = \int_a^x f(t)\, dt.$$

Since $\int_a^b f(t)\, dt = F(b) - F(a)$, $f \in L^1([a,b])$.

<u>case 2</u>: $F : [a,b] \to \mathbb{R}$ increasing but not strictly.

Let $G(x) := x + F(x)$, then $G$ is strictly incr. & abs. contin., apply case 1.

<u>case 3</u>: <u>Lemma</u>: $F : [a,b] \to \mathbb{R}$ is abs. contin. $\Leftrightarrow$ $F = F_+ - F_-$ for 2 increasing abs. contin. fn. $F_\pm$.

(key word: "bounded variation")

<u>general</u> <u>case</u>: $F : [a,b] \to V$ reduces to case $V = \mathbb{R}$ (case 3) by choosing a basis of $V$.

$\square$

pf of Radon-Nikodym: Assume $\lambda \ll \mu$ on $X$.

idea: If $\exists\ f := \frac{d\lambda}{d\mu}$, then $g \in L^1(X,\lambda) \Rightarrow \boxed{\int_X g\, d\lambda = \int_X g f\, d\mu}$ (1).

For $\mathbb{R}$-valued fns. $g : X \to \mathbb{R}$, $\lambda + \mu$ is also a measure on $X$, $\exists$ a bdd linear fnal $\Lambda : L^1(X, \lambda+\mu) \to \mathbb{R}$, $\Lambda(g) := \int_X g\, d\lambda$ : bdd since

$$|\Lambda(g)| = \left|\int_X g\, d\lambda\right| \leq \int_X |g|\, d\lambda \leq \int_X |g|\, d\lambda + \int_X |g|\, d\mu = \int_X |g|\, d(\lambda+\mu) = \|g\|_{L^1(\lambda+\mu)}$$

$\xRightarrow{\text{Riesz}}$ $\exists\ h \in L^\infty(X, \lambda+\mu)$ s.t. $\|h\|_{L^\infty} = 1$ & $\boxed{\int_X g\, d\lambda = \int_X h g\, d(\lambda+\mu)}$ (2)

$\forall\ g \in L^1(X, \lambda+\mu)$

If (1) also holds, then $\forall\ g \in L^1(X, \lambda+\mu)$,

$$\int_X g f\, d\mu = \int_X g\, d\lambda = \int_X h g\, d\lambda + \int_X h g\, d\mu \overset{(1)}{=} \int_X (h g f + h g)\, d\mu = \int_X h g (1+f)\, d\mu$$

this suggests defining $f$ s.t. $f = h(1+f)$, i.e. $\boxed{f = \frac{h}{1-h}}$

claim: If $\lambda \ll \mu$, then $\frac{h}{1-h} = \frac{d\lambda}{d\mu}$.

Lemma: $0 \leq h < 1$ a.e. wrt. $\mu$, hence $0 \leq f < \infty$ a.e. wrt. $\mu$. ∎

Set $\mu_f(A) := \int_A f\, d\mu$, so want to show $\mu_f = \lambda$.

(2) $\Longleftrightarrow \int_X g\, d\lambda - \int_X h g\, d\lambda = \boxed{\int_X g(1-h)\, d\lambda = \int_X h g\, d\mu}$ (3) $\forall\ g \in L^1(X, \lambda+\mu)$.

Now for $A \subseteq X$ measurable, set $g := \frac{1}{1-h}\chi_A$, so

$$\lambda(A) = \int_X \chi_A\, d\lambda = \int_X g(1-h)\, d\lambda \overset{(3)}{=} \int_X h g\, d\mu = \int_A f\, d\mu = \mu_f(A)$$

holds whenever $\frac{1}{1-h}\chi_A \in L^1(X, \lambda+\mu)$. NOT ALWAYS TRUE

$\sigma$-finiteness $\Rightarrow X = \bigcup_{n\in\mathbb{N}} X_n$ for $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ s.t. $\lambda(X_n), \mu(X_n) < \infty$.

Given $A \subseteq X$, let $A_n := X_n \cap \{x \in A \mid 1 - h(x) \geq \frac{1}{n}\} \subseteq A$, so

$(\lambda+\mu)(A_n) < \infty$ & $\frac{1}{1-h}\chi_{A_n} \leq n \Rightarrow \frac{1}{1-h}\chi_{A_n} \in L^1(X, \lambda+\mu) \Rightarrow$

$\lambda(A_n) = \mu_f(A_n)$. Let $A_0 := A \setminus \bigcup_{n=1}^\infty A_n$, so $A = \underbrace{A_0 \cup \overbrace{\bigcup_{n\in\mathbb{N}} A_n}^{\text{nested seq.}}}_{\text{disjoint union}}$

Lemma $\Rightarrow \mu(A_0) = 0$, $\Rightarrow \mu_f(A_0) = 0$

$$\Rightarrow \mu_f(A) = \mu_f(A_0) + \lim_{n\to\infty} \mu_f(A_n) = \lim_{n\to\infty} \mu_f(A_n) = \lim_{n\to\infty} \lambda(A_n).$$

$$\lambda(A) = \lambda(A_0) + \lim_{n\to\infty} \lambda(A_n) = \lambda(A_0) + \mu_f(A).$$

We've proved: Under no assumption on $\lambda$ & $\mu$, $\exists$ a distinguished measurable fn. $f : X \to [0,\infty]$ s.t. $\forall\ A \subseteq X$ measurable,

$\boxed{\lambda(A) \geq \int_A f\, d\mu}$

Since $\mu(A_0) = 0$, if $\lambda \ll \mu$, then $\lambda(A_0) = 0$

$\Rightarrow$ inequality becomes equality.

∎