$$f : \mathbb{T}^n \longrightarrow V \quad \rightsquigarrow \quad \mathcal{F} f = \hat{f} : \mathbb{Z}^n \to V : k \longmapsto \hat{f}_k := \int_{\mathbb{T}^n} e^{-2\pi i k \cdot x} f(x)\, dx$$

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$$

$$g : \mathbb{Z}^n \longrightarrow V : k \longmapsto g_k \quad \rightsquigarrow \quad \mathcal{F}^* g = \check{g} : \mathbb{T}^n \longrightarrow V : x \longmapsto \sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot x} g_k$$

<u>proved so far</u>: $\mathcal{F} : L^1(\mathbb{T}^n) \longrightarrow \ell^\infty(\mathbb{Z}^n)$ bdd

$\mathcal{F}^* : \ell^1(\mathbb{Z}^n) \longrightarrow C^0(\mathbb{T}^n)$ bdd

$\mathcal{F}(C^\infty(\mathbb{T}^n)) \subseteq \mathcal{S}(\mathbb{Z}^n)$, $\quad \mathcal{F}^*(\mathcal{S}(\mathbb{Z}^n)) \subseteq C^\infty(\mathbb{T}^n)$.

Orthonormality of $\{e^{2\pi i k \cdot x}\}_{k \in \mathbb{Z}^n} \Rightarrow \mathcal{F}\mathcal{F}^* = \mathrm{Id}$ on $\ell^2(\mathbb{Z}^n)$.

<u>to prove next</u>: $\mathcal{F}^*\mathcal{F} = \mathrm{Id}$ on $C^\infty(\mathbb{T}^n)$.

"<u>Lemma</u>" (science fiction): For the Dirac δ-fn $\delta : \mathbb{T}^n \longrightarrow [0,\infty]$, $\mathcal{F}^*\mathcal{F}\delta = \delta$.

"<u>cor</u>": Since $\hat{\delta}_k = \int_{\mathbb{T}^n} e^{-2\pi i k \cdot x} \delta(x)\, dx = 1 \ \forall k \Rightarrow \boxed{\delta(x) = \sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot x}}$

<u>"physicist's proof" that $\mathcal{F}^*\mathcal{F} f = f$ for $f \in C^\infty(\mathbb{T}^n)$</u>

$$(\mathcal{F}^* \hat{f})(x) = \sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot x} \hat{f}_k = \sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot x} \left(\int_{\mathbb{T}^n} e^{-2\pi i k \cdot y} f(y)\, dy \right)$$

$$= \int_{\mathbb{Z}^n} e^{2\pi i k \cdot x} \left(\int_{\mathbb{T}^n} e^{-2\pi i k \cdot y} f(y)\, dm(y) \right) d\nu(k)$$

$$= \int_{\mathbb{Z}^n \times \mathbb{T}^n} \boxed{e^{2\pi i k \cdot (x-y)} f(y)}\, d(\nu(k) \otimes m(y)) = \int_{\mathbb{T}^n} \left(\int_{\mathbb{Z}^n} e^{2\pi i k \cdot (x-y)} f(y)\, d\nu(k) \right) dy$$

not in $L^1(\mathbb{Z}^n \times \mathbb{T}^n)$!

$$= \int_{\mathbb{T}^n} \left(\sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot (x-y)} \right) f(y)\, dy = \int_{\mathbb{T}^n} \delta(x-y) f(y)\, dy = \delta * f(x)$$

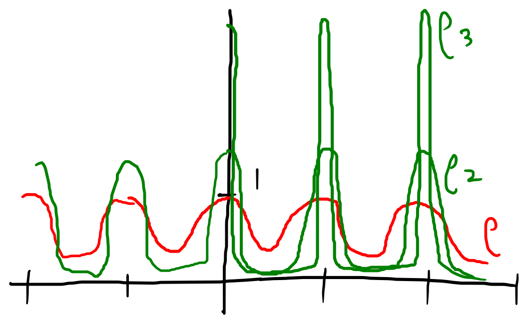
$$= f * \delta(x) = \int_{\mathbb{T}^n} f(x-y)\, \delta(y)\, dy = f(x).$$

$\square$

defn: An *approximate identity* on $\mathbb{T}^n$ is a seq. $\rho_j : \mathbb{T}^n \to [0,\infty)$ of C^∞-fns s.t. $\forall \varphi \in C^\infty(\mathbb{T}^n)$, $\int_{\mathbb{T}^n} \rho_j \varphi \, dm \to \varphi(0)$ as $j \to \infty$.

$\Rightarrow$ $\rho_j * f \to f$ pointwise $\forall f \in C^\infty(\mathbb{T}^n)$.

prop: Spse $\rho : \mathbb{T}^n \xrightarrow{C^\infty} [0,\infty)$ s.t. $\rho(0) = 1$ & $\rho(x) < 1 \ \forall x \neq 0$.



Set $\rho_j(x) := \frac{1}{c_j}[\rho(x)]^j$ for $j \in \mathbb{N}$, where $c_j := \int_{\mathbb{T}^n} \rho^j \, dm > 0$.

Then ρ_j is an approx. id. $\boxtimes$

Rigorous interpretation of $\delta(x) = \sum_{k \in \mathbb{Z}^n} e^{2\pi i k \cdot x}$:

Lemma: If ρ_j is an approx. id. on $\mathbb{T}^n$, then $(\hat{\rho}_j)_k \in \mathbb{C}$ satisfy

$|(\hat{\rho}_j)_k| \le |(\hat{\rho}_j)_0|$ & $\lim_{j \to \infty} (\hat{\rho}_j)_k = 1 \ \forall k \in \mathbb{Z}^n$.

pf: $|(\hat{\rho}_j)_k| \le \int_{\mathbb{T}^n} |e^{-2\pi i k \cdot x} \rho_j(x)| \, dx = \int_{\mathbb{T}^n} \rho_j(x) \, dx = \int_{\mathbb{T}^n} e^{-2\pi i 0 \cdot x} \rho_j(x) \, dx$

$= (\hat{\rho}_j)_0$.

$(\hat{\rho}_j)_k = \int e^{-2\pi i k \cdot x} \rho_j(x) \, dx \xrightarrow{j \to \infty} e^{-2\pi i k \cdot 0} = 1$. $\boxtimes$

Lemma: $\exists$ an approx. id. ρ_j s.t. $\mathcal{F}^* \mathcal{F} \rho_j = \rho_j \ \forall j$.

(rigorous interp. of "$\mathcal{F}^* \mathcal{F} \delta = \delta$".)

pf: Defn. $\beta : \mathbb{T}^1 \to [0,\infty)$ by $\beta(t) := \frac{\cos(2\pi t) + 1}{2}$

Then $\rho(x_1, \dots, x_n) := \beta(x_1) \dots \beta(x_n)$ satisfies $\rho(x) \le 1$, = iff $x = 0 \in \mathbb{T}^n$.

$\Rightarrow$ $\exists$ approx. id. of the form $\rho_j = \frac{\rho^j}{c_j}$ (c_j = const > 0).

β is a $\mathbb{C}$-lin. combin. of $e^{2\pi i t}$ & $e^{-2\pi i t}$.

$\Rightarrow$ each ρ_j is a fin. lin. combin. of fns. of form $e^{2\pi i k \cdot x}$

$\Rightarrow$ $\rho_j = \mathcal{F} \mathcal{F}^* \rho_j$. $\boxtimes$

Pf that $\mathcal{F}^*\mathcal{F}f = f$ for $f \in C^\infty(\mathbb{T}^n)$

Fix ϱ_j with properties discussed above.

$F(y,k) := e^{2\pi i k\cdot(x-y)}(\hat\varrho_j)_k f(y)$ is an L^1-fn. on $\mathbb{T}^n \times \mathbb{Z}^n$ since $\hat\varrho_j \in \mathcal{S}(\mathbb{Z}^n) \subseteq \ell^1(\mathbb{Z}^n)$.

Fubini $\Rightarrow$ $\int_{\mathbb{Z}^n}\left(\int_{\mathbb{T}^n} F(y,k)\,dy\right)d\nu(k)$

$$= \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}(\hat\varrho_j)_k \int_{\mathbb{T}^n} e^{-2\pi i k\cdot y} f(y)\,dy = \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\underbrace{(\hat\varrho_j)_k}_{\text{bdd by } (\hat\varrho_j)_0 \to 1} \hat f_k$$

since $(\hat\varrho_j)_k \xrightarrow{j\to\infty} 1$, dominated conv. $\Rightarrow$
as $j\to\infty$, integral $\to \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\hat f_k = \mathcal{F}^*\mathcal{F}f(x)$.

That also equals $\int_{\mathbb{T}^n}\left(\int_{\mathbb{Z}^n} F(y,k)\,d\nu(k)\right)dy = \int_{\mathbb{T}^n}\left(\sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot(x-y)}(\hat\varrho_j)_k\right) f(y)\,dy$
$(\mathcal{F}^*\mathcal{F}\varrho_j = \varrho_j)$

$$= \int_{\mathbb{T}^n} \varrho_j(x-y) f(y)\,dy = \varrho_j * f(x) \xrightarrow{j\to\infty} f(x) \quad \forall\ x. \qquad \square$$

We've proved: $C^\infty(\mathbb{T}^n) \underset{\mathcal{F}^* = \mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{S}(\mathbb{Z}^n)$.

prop: If $f \in C^\infty(\mathbb{T}^n)$, then $\sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\hat f_k$ converges in C^∞ to f.

Pf: $f \in C^\infty \Rightarrow \hat f \in \mathcal{S} \subseteq \ell^1 \Rightarrow$ convergence to f is uniform.

Recall: $\partial^\alpha f(x) = \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x}\underbrace{(2\pi i k)^\alpha \hat f_k}_{\text{also in } \mathcal{S}(\mathbb{Z}^n) \subseteq \ell^1(\mathbb{Z}^n)}$

$\Rightarrow$ this series also conv. unif. to $\partial^\alpha f$. $\square$

Lemma: $\forall\ f \in C^\infty(\mathbb{T}^n)$ & $g \in \mathcal{S}(\mathbb{Z}^n)$, $\langle g, \mathcal{F}f\rangle_{\ell^2} = \langle \mathcal{F}^* g, f\rangle_{L^2}$.

Pf: $\langle g, \mathcal{F}f\rangle_{\ell^2} = \sum_{k\in\mathbb{Z}^n}\left\langle g(k), \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} f(x)\,dx\right\rangle$

$$= \sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\langle g(k), e^{-2\pi i k\cdot x} f(x)\rangle\,dx = \int_{\mathbb{T}^n\times\mathbb{Z}^n}\underbrace{e^{-2\pi i k\cdot x}\langle g(k), f(x)\rangle\, d(m(x)\otimes\nu(k))}_{\in L^1(\mathbb{T}^n\times\mathbb{Z}^n) \text{ since } f\in C^\infty \subseteq L^1 \text{ \& } g \in \mathcal{S} \subseteq \ell^1}$$

$$= \int_{\mathbb{T}^n}\sum_{k\in\mathbb{Z}^n}\langle e^{2\pi i k\cdot x} g(k), f(x)\rangle\,dx$$

$$= \int_{\mathbb{T}^n}\left\langle \sum_{k\in\mathbb{Z}^n} e^{2\pi i k\cdot x} g(k), f(x)\right\rangle dx = \langle \mathcal{F}^* g, f\rangle_{L^2}. \qquad \square$$

cor (Parseval's identity): $\forall\ f, g \in C^\infty(\mathbb{T}^n)$, $\langle \hat f, \hat g\rangle_{\ell^2} = \langle f, g\rangle_{L^2}$

Pf: $\langle \hat f, \hat g\rangle_{\ell^2} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{\ell^2} = \langle \mathcal{F}^*\mathcal{F}f, g\rangle_{L^2} = \langle f, g\rangle_{L^2}$. $\square$

Since $C^\infty \subseteq L^2$ & $\mathcal{S} \subseteq \ell^2$ are dense & $\|\mathcal{F}f\|_{\ell^2} = \|f\|_{L^2}$, $\|\mathcal{F}^* g\|_{L^2} = \|g\|_{\ell^2}$ on these dense subspaces, $\mathcal{F}$ extends to a bdd lin. op. $L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$, $\mathcal{F}^*$ extends to $\ell^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n)$ s.t. $\mathcal{F}\mathcal{F}^* = \mathrm{Id}$ on ℓ^2, $\mathcal{F}^*\mathcal{F} = \mathrm{Id}$ on L^2, both are unitary isomorphisms of Hilbert spaces.

cor (for $\mathbb{C}$-valued fns): The O-N set $\{e^{2\pi i k\cdot x}\}_{k\in\mathbb{Z}^n}$ is a basis of $L^2(\mathbb{T}^n)$.

Fourier transform:

"position space"

Q: Which fns. $f : \mathbb{R}^n \to V$ (not periodic) can be written as "lin. combins." of $e^{2\pi i p\cdot x}$ for $p \in \mathbb{R}^n$ (not just $\mathbb{Z}^n$)?

"frequency space" / "momentum space"

defn: For fns. $f : \mathbb{R}^n \to V$ s.t. the following integral is def'd, the Fourier transform of f is the fn. $\mathcal{F} f = \hat{f} : \mathbb{R}^n \to V$ given by
$$\hat{f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} f(x)\, dx.$$

For $g : \mathbb{R}^n \to V$, we also dfn. $\mathcal{F}^* g = \check{g} : \mathbb{R}^n \to V$ by
$$\check{g}(x) = \int_{\mathbb{R}^n} e^{2\pi i p\cdot x} g(p)\, dp.$$

Observe: If $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ is def'd $\forall\, p \in \mathbb{R}^n$ & deps contin. on p, & $|\hat{f}(p)| \leq \|f\|_{L^1}$ $\forall\, p$ $\Rightarrow$

prop: $\mathcal{F}$ (& also $\mathcal{F}^*$) defns. a bdd lin. op. $L^1(\mathbb{R}^n) \to C^0_b(\mathbb{R}^n)$. $\boxed{}$

We will prove: $\mathcal{F}$ & $\mathcal{F}^*$ extend uniquely to unitary isomorphisms $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ & $\mathcal{F}^* = \mathcal{F}^{-1}$.