functional calculus (part 2)

$A \in \mathcal{L}(\mathcal{H})$ self-adj. $\rightsquigarrow$ bdd lin. "$*$-algebra hom." $C(\sigma(A)) \longrightarrow \mathcal{L}(\mathcal{H})$
$f \longmapsto f(A)$

Write $UAU^{-1} = T_F : u \longmapsto Fu$ for unitary $U : \mathcal{H} \longrightarrow L^2(X,\mu)$,
$F : X \longrightarrow \mathbb{R}$, $\sigma(A) = \sigma(T_F) =$ essential range of F (PSET 11).

Lemma: $F(x) \in \sigma(A)$ for almost all x ($\Rightarrow$ WLOG $F(X) \subseteq \sigma(A)$).

pf: $\lambda \notin \sigma(A) = \text{ess ran}(F) \Rightarrow \exists$ nbhd $\lambda \in U_\lambda \subseteq \mathbb{C}$ s.t. $\mu(F^{-1}(U_\lambda)) = 0$.
$\mathbb{C} \setminus \sigma(A)$ can be covered by countably many such nbhds.
$\Rightarrow \mu(F^{-1}(\mathbb{C} \setminus \sigma(A))) = 0$. $\square$

For $P =$ cpx polyn. on $\mathbb{R}$, $P(A) = P(U^{-1} T_F U) = U^{-1} P(T_F) U$
$= U^{-1} T_{P \circ F} U \Rightarrow$ by density, $f(A) = U^{-1} T_{f \circ F} U \quad \forall f \in C(\sigma(A))$.

observe: RHS makes sense $\forall f \in \mathcal{B}(\sigma(A)) = \{$bdd Borel measurable fns $\sigma(A) \to \mathbb{C}\}$

$\Rightarrow$ thm: $f \longmapsto f(A)$ extends to a $*$-alg. hom. $\mathcal{B}(\sigma(A)) \longrightarrow \mathcal{L}(\mathcal{H})$ s.t.

(iv)' $Ax = \lambda x \Rightarrow f(A)x = f(\lambda)x$ (note: $f(\sigma(A))$ maybe not closed $\Rightarrow \neq \sigma(f(A))$.)

(v)' $f \geq 0 \Rightarrow \langle x, f(A)x \rangle \geq 0 \quad \forall x$.

(vi)' $\mathcal{B}(\sigma(A)) \ni f_n \xrightarrow{p.w.} f$ & $|f_n| \leq C \;\forall n \Rightarrow f_n(A)x \longrightarrow f(A)x$
$\forall x \in \mathcal{H}$.

(vii)' If $AB = BA$ for some $B \in \mathcal{L}(\mathcal{H})$, then $f(A)B = Bf(A) \quad \forall f \in \mathcal{B}(\sigma(A))$.

rk: $\mathcal{B}(\sigma(A))$ is the smallest class of fns. $\sigma(A) \to \mathbb{C}$ that contains
$C(\sigma(A))$ & is closed under limits as in (vi)'.
$\Rightarrow$ most of these properties follow from polynomials via approximation using (vi)'.

pf of (vi)': Suff. to show if $f_n \xrightarrow{p.w.} f$ & $|f_n|$ unif. bdd,
$\forall u \in L^2(X,\mu)$, $(f_n \circ F) \cdot u \xrightarrow{L^2} (f \circ F)u$, i.e.

$$\int_X |f_n(F(t)) - f(F(t))|^2 \cdot |u(t)|^2 \, d\mu(t) \longrightarrow 0.$$

dominated conv.! $\square$

rk: $f(A)$ is indep. of choice of spectral repr.; true for $f =$ poly.,
rest by approximation.

applications

(1) If $\lambda \in \mathbb{C} \setminus \sigma(A)$ for A self-adj., then $g(x) := \frac{1}{\lambda - x}$ is in $C(\sigma(A))$,

$g(\lambda - x) = 1 \Rightarrow g(A)(\lambda - A) = \mathrm{Id} \Rightarrow g(A) = R_\lambda(T)$, $\Rightarrow \|R_\lambda(T)\| =$

$\|g\|_{C^0(\sigma(A))} = \sup_{\mu \in \sigma(A)} \frac{1}{|\lambda - \mu|} = \frac{1}{\mathrm{dist}(\lambda, \sigma(A))}$.

(2) $A \in \mathcal{L}(\mathcal{H})$ is called positive ("$A \geq 0$") if $\langle x, Ax \rangle \geq 0 \quad \forall x \in \mathcal{H}$.

EX: $A \geq 0 \Rightarrow A$ is self-adj. Hint: $\langle x, Ax \rangle \in \mathbb{R} \Rightarrow \langle x, Ax \rangle = \langle Ax, x \rangle$.

compute $\langle x+y, A(x+y) \rangle$ & $\langle x+iy, A(x+iy) \rangle$ for $x, y \in \mathcal{H}$ arbitrary.

Lemma (PSET 12): For $A \in \mathcal{L}(\mathcal{H})$ self-adj., $A \geq 0 \Leftrightarrow \sigma(A) \subseteq [0, \infty)$.

Caution: A can be pos.-def. $(\langle x, Ax \rangle > 0 \;\forall x \neq 0)$ but $0 \in \sigma(A)$.

$A \geq 0 \Rightarrow \sqrt{\cdot}$ is a contin. fn. on $\sigma(A) \Rightarrow$ can defn. $\sqrt{A} \geq 0$ s.t.

$\sqrt{A}\sqrt{A} = A$. observe: $\forall A \in \mathcal{L}(\mathcal{H})$, $A^* A \geq 0$.

motivation: $\mathbb{C} \ni z = \rho e^{i\theta}$ for unique $\rho \geq 0$, $e^{i\theta} \in S^1 := \{|z| = 1\} \subseteq \mathbb{C}$.

thm (polar decomposition): $\forall A \in \mathcal{L}(\mathcal{H})$, $A = UP$ for $P := \sqrt{A^* A}$

& a unique $U \in \mathcal{L}(\mathcal{H})$ s.t. $\ker U = \ker A$, $\operatorname{im} U \subseteq \mathcal{H}$ is closed

& $(\ker U)^\perp \xrightarrow{U} \operatorname{im} U$ is a unitary iso. (we call U a "partial isometry").

Moreover, $\operatorname{im} U = \overline{\operatorname{im} A}$. In particular, U is unitary if A is invertible.

pf: See Reed + Simon.

(3) <u>defn</u>: For $A \in \mathcal{L}(\mathcal{H})$ self-adj., & a Borel set $\Omega \subseteq \mathbb{R}$, $\exists$ a <u>spectral projection</u> $P_\Omega := \chi_\Omega(A) \in \mathcal{L}(\mathcal{H})$.

In spectral repr., $A = T_F : u \longmapsto Fu$, $P_\Omega(u) = (\chi_\Omega \circ F) u$

(<u>compare</u>: for $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, every $\lambda \in \sigma(A) \rightsquigarrow$ ortho. proj. $P_\lambda : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ w/ $\mathrm{im}\, P_\lambda = \ker(\lambda - A)$.)

P_Ω = ortho. proj. onto the "approximate eigenspace" corresponding to $\lambda \in \sigma(A) \cap \Omega$.

<u>EX</u>: $\lambda \in \mathbb{R}$ is in $\sigma(A) \iff \forall\, \varepsilon > 0$, $P_{(\lambda-\varepsilon, \lambda+\varepsilon)} \neq 0$.

<u>Q</u>: When can multiple self-adj. ops $\{A_i\}$ be diagonalized simultaneously?

i.e. $\exists$ unitary $\mathcal{H} \xrightarrow{U} L^2(X, \mu)$ s.t. $UA_iU^{-1} = T_{F_i}$ for some fns $F_i : X \longrightarrow \mathbb{R}$?

<u>observe</u>: $T_{F_i} T_{F_j} = T_{F_i F_j} = T_{F_j} T_{F_i} \implies$ only possible if $A_i A_j = A_j A_i$.

<u>thm</u>: This is possible for any finite collection of <u>commuting</u> self-adj. ops.

<u>cor</u>: Normal ops. $A \in \mathcal{L}(\mathcal{H})$ admit spectral reprs.

<u>pf</u>: PSET 11 $\implies A = B + iC$ for self-adj. B, C s.t. $BC = CB$, $\implies$

$UBU^{-1} = T_F$, $UCU^{-1} = T_G$ for $F, G : X \longrightarrow \mathbb{R}$, $\implies UAU^{-1} = T_{F+iG}$. $\square$

<u>pf of thm for 2 commuting ops. A, B</u>

<u>idea</u>: Construct a fnal. calculus $C(\sigma(A) \times \sigma(B)) \longrightarrow \mathcal{L}(\mathcal{H}) : f \longmapsto f(A, B)$

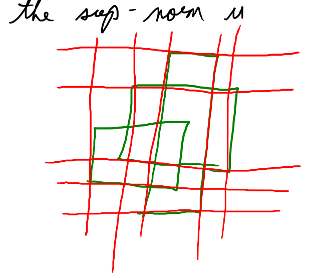
<u>trouble</u>: For polyn. P in 2 variables, $\sigma(P(A,B)) = P(\sigma(A) \times \sigma(B))$. harder to prove.

<u>alternative</u>: Approximate contin. fns. by fns. <u>con</u>stant <u>on</u> <u>rectangle</u>.

<u>Lemma</u>: $\forall\, \Omega, \Omega' \subseteq \mathbb{R}$, $\chi_\Omega(A)$ & $\chi_{\Omega'}(B)$ commute. $\square$

Let $\mathcal{R} := \{$finite $\mathbb{C}$-linear combinations of fns. $\chi_{\Omega \times \Omega'}$ for intervals $\Omega, \Omega' \subseteq \mathbb{R}\}$.

<u>Easy lemma</u>: The closure of $\{ f|_{\sigma(A) \times \sigma(B)} \mid f \in \mathcal{R} \}$ in the sup-norm is dense in $C(\sigma(A) \times \sigma(B))$.



<u>Note</u>: Every $f \in \mathcal{R}$ can be written in the form

$$f(x,y) = \sum_{i,j} c_{ij}\, \chi_{\Omega_i^A}(x)\, \chi_{\Omega_j^B}(y)$$ for some collection of disjoint intervals $\Omega_1^A, \dots, \Omega_M^A \subseteq \mathbb{R}$, $\Omega_1^B, \dots, \Omega_N^B \subseteq \mathbb{R}$ & $c_{ij} \in \mathbb{C}$.

For $f = \sum_j c_j\, \chi_{\Omega_j \times \Omega_j'} \in \mathcal{R}$, defn. $f(A,B) = \sum_j c_j\, \chi_{\Omega_j}(A)\, \chi_{\Omega_j'}(B) \in \mathcal{L}(\mathcal{H})$.

<u>Lemma</u>: $\| f(A,B) \| = \| f \|_{L^\infty(\sigma(A) \times \sigma(B))}$.

<u>pf</u>: Given $f(x,y) = \sum_{i,j} c_{ij}\, \chi_{\Omega_i^A}(x)\, \chi_{\Omega_j^B}(y)$ as above,

$\implies \|f\|_{L^\infty} = \max_{i,j} |c_{ij}|$. $\implies \|f(A,B)\| \le \|f\|_{L^\infty} \cdot \Big\| \underbrace{\sum_{i,j} \chi_{\Omega_i^A}(A)\, \chi_{\Omega_j^B}(B)} \Big\|$

$= \left(\sum_i \chi_{\Omega_i^A}(A) \right) \left(\sum_j \chi_{\Omega_j^B}(B) \right) = \chi_{\Omega_1^A \cup \dots \cup \Omega_M^A}(A)\, \chi_{\Omega_1^B \cup \dots \cup \Omega_N^B}(B)$

= composition of 2 ortho. projs. $\implies \| \cdot \| \le 1$. $\implies \|f(A,B)\| \le \|f\|_{L^\infty}$.

Can show is = by evaluating $f(A,B)$ on a suitable element of $\mathcal{H}$. $\square$

$\Rightarrow f \longmapsto f(A,B)$ has a ! contin. ext. to a bdd lin. op.
$C(\sigma(A)\times\sigma(B)) \longrightarrow \mathcal{L}(\mathcal{H})$ s.t. $\|f(A,B)\| = \|f\|_{C^0}$.

Now $f \geq 0$, $\langle x, f(A,B)x\rangle = \langle g(A,B)x, g(A,B)x\rangle \geq 0 \Rightarrow$

$\Rightarrow f = g^2$ for some $g \in C(\sigma(A)\times\sigma(B))$

Riesz-Markov gives a finite regular Borel measure μ_x on $\sigma(A)\times\sigma(B)$ s.t. $\langle x, f(A,B)x\rangle = \int_{\sigma(A)\times\sigma(B)} f \, d\mu_x \quad \forall f \in C(\sigma(A)\times\sigma(B))$.

Defn. $T : C(\sigma(A)\times\sigma(B)) \longrightarrow \mathcal{H} : f \longmapsto f(A,B)x$, show $\|Tf\| = \|f\|_{L^2(\mu_x)} \Rightarrow T$ extends to an isometry $L^2(\sigma(A)\times\sigma(B), \mu_x) \longrightarrow \mathcal{H}$

with image = closure of the ~~set~~ span of $\{A^m B^n x \mid m,n \geq 0 \text{ integers}\}$.

Say x is cyclic for A & B if that closure is $\mathcal{H}$.

$\Rightarrow U := T^{-1}$ this identifies A with mult. by $f_1(x,y) := x$
B with mult. by $f_2(x,y) := y$

on $L^2(\sigma(A)\times\sigma(B))$.

Can always decompose $\mathcal{H}$ into orthogonal subspaces that admit cyclic vectors

$\Rightarrow (X,\mu) =$ disjoint of copies of $\sigma(A)\times\sigma(B)$ w/ various spectral measures μ_x.

$\square$