mollification

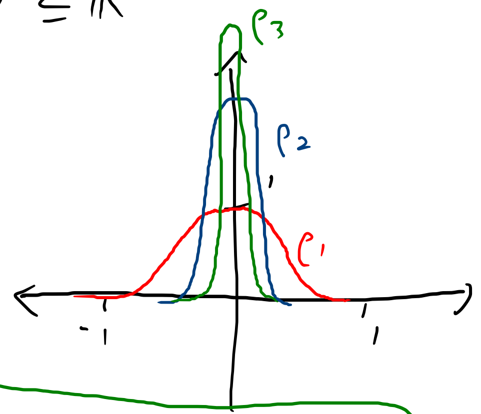
integration w.r.t. Lebesgue measure: $\Omega \subseteq \mathbb{R}^n$

$$\int_\Omega f\,dm := \int_\Omega f(x)\,dx = \int_\Omega f(x_1,\dots,x_n)\,dx_1\dots dx_n$$

Choose: $\rho : \mathbb{R}^n \xrightarrow{C^\infty} [0,1]$ with $\operatorname{supp}(\rho) \subseteq B_1(0) :=$ unit ball $\subseteq \mathbb{R}^n$

s.t. $\int_{\mathbb{R}^n} \rho\,dm = 1$. Let $\rho_j(x) := j^n \rho(jx)$ for $j \in \mathbb{N}$,

so $\operatorname{supp}(\rho_j) \subseteq B_{1/j}(0)$ & $\int_{\mathbb{R}^n} \rho_j\,dm = 1$ $\forall j$.



Given $f : \mathbb{R}^n \longrightarrow V$, define $f_j : \mathbb{R}^n \longrightarrow V$ by $\boxed{f_j(x) := \int_{\mathbb{R}^n} f(x-y)\rho_j(y)\,dy.}$

"mollifier"

main thm: If $f \in L^p(\mathbb{R}^n)$ for some $p \in [1,\infty)$, then

(1) $f_j \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R})$ & $\|f_j\|_{L^p} \le \|f\|_{L^p}$ $\forall j$,

(2) $f_j \xrightarrow{L^p} f$ as $j \longrightarrow \infty$.

cor: $\forall$ $p \in [1,\infty)$, $C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

cor: $\forall$ $p \in [1,\infty)$ & $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $C_0^\infty(\Omega) := \{ f \in C^\infty(\Omega) \mid \operatorname{supp}(f) \text{ cpt in } \Omega \}$
is dense in $L^p(\Omega)$.

pf: Given $f \in L^p(\Omega)$, extend to $\mathbb{R}^n$ as 0 on $\mathbb{R}^n \setminus \Omega$, then $f_j \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$
s.t. $\|f_j - f\|_{L^p} < \frac{\varepsilon}{2}$ for large j (given $\varepsilon > 0$ small),
then choose open sets $\Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega_2 \subseteq \overline{\Omega}_2 \subseteq \dots \subseteq \bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$
where $\Omega_j \overset{\text{open}}{\subseteq} \Omega$, $\overline{\Omega}_j$ cpt. $\exists$ C^∞-fns. $\beta_j : \Omega \longrightarrow [0,1]$ s.t.
$\beta_j|_{\Omega_j} \equiv 1$ & $\operatorname{supp}(\beta_j) \overset{\text{cpt}}{\subseteq} \Omega$. EX: for large j, $\|\beta_j f_j - f_j\|_{L^p} < \frac{\varepsilon}{2}$.

$\Rightarrow$ $\|\underset{C_0^\infty}{\beta_j f_j} - f\|_{L^p} < \varepsilon$. ∎

preparation: $f : \mathbb{R}^n \to V$, $v \in \mathbb{R}^n$ $\rightsquigarrow$ translation operator

$$(\tau_v f)(x) := f(x+v).$$

thm: If $1 \le p < \infty$ & $f \in L^p(\mathbb{R}^n)$, then the map

$$\mathbb{R}^n \longrightarrow L^p(\mathbb{R}^n) : v \longmapsto \tau_v f \quad \text{is continuous.}$$

pf: $\|\tau_v f\|_{L^p} = \|f\|_{L^p} \Rightarrow \tau_v$ is a bdd lin. op. $L^p \to L^p$ with $\|\tau_v\| = 1$.

Then $\|\tau_{v+w} f - \tau_w f\|_{L^p} = \|\tau_w(\tau_v f - f)\|_{L^p} = \|\tau_v f - f\|_{L^p}$,

need to prove this is small when $|v|$ is small.

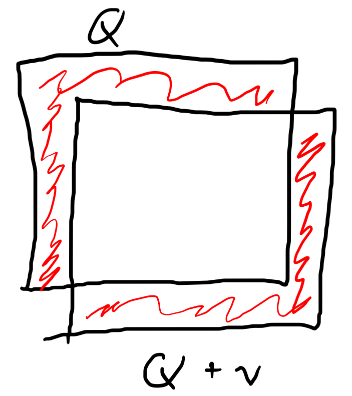
Suff. to prove this $\forall f$ in some dense subspace of $L^p(\mathbb{R}^n)$.

Let $\hat{Q}(\mathbb{R}^n) := \{$ fin. linear combin. of char. fns. of cubes $Q_j \subseteq \mathbb{R}^n$, i.e. $\sum_{j=1}^{N} \chi_{Q_j} v_j$ for $N \in \mathbb{N}$, $v_j \in V$, $Q_j \subseteq \mathbb{R}^n$ cubes $\}$

Last Tuesday $\Rightarrow$ $\hat{Q}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

For any cube $Q \subseteq \mathbb{R}^n$,

$$\|\tau_v \chi_Q - \chi_Q\|_{L^p}^p = \int_{\mathbb{R}^n} |\chi_Q(x+v) - \chi_Q(x)|^p \, dx = m$$

(

)

$$\longrightarrow 0 \quad \text{as} \quad v \longrightarrow 0.$$

Now $f = \sum_j \chi_{Q_j} v_j \in \hat{Q}(\mathbb{R}^n)$, satisfies

$$\|\tau_v f - f\|_{L^p} = \Big\|\sum_j (\tau_v \chi_{Q_j} - \chi_{Q_j}) v_j\Big\|_{L^p} \overset{\text{(Minkowski)}}{\le} \sum_j \underbrace{\|\tau_v \chi_{Q_j} - \chi_{Q_j}\|_{L^p}}_{\text{small for } |v| \text{ small}} \cdot \underbrace{|v_j|}_{\text{bdd}}$$

arbitrarily small for $|v|$ small. $\blacksquare$

convolution: Assume f, g are fns. on $\mathbb{R}^n$, one valued in V, the other in $\mathbb{K}$.

The convolution of f & g is the fn. $f * g$ def'd by

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Domain of $f * g$ is $\{ x \in \mathbb{R}^n \mid$ the fn. $y \longmapsto f(x-y)g(y)$ is in $L^1(\mathbb{R}^n) \}$

Usually: true for almost all $x \in \mathbb{R}^n \Rightarrow f * g$ is a well-def'd fn. on $\mathbb{R}^n$ up to equality a.e.

EX: $f * g = g * f$ (change of variables)

observe: Spse f is diff-able,

$$\partial_k (f * g)(x) = \frac{\partial}{\partial x_k} \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^n} \left[\frac{\partial}{\partial x_k} f(x-y) \right] g(y)\, dy$$

$$= \int_{\mathbb{R}^n} \partial_k f(x-y)\, g(y) = (\partial_k f * g)(x)$$

whenever diff. under the integral sign can be justified.

e.g. if $f \in C^1$ & $\operatorname{supp}(f) \subseteq K$ cpct $\subseteq \mathbb{R}^n$, also $g \in L^1_{loc}(\mathbb{R}^n)$ $:= \{ L^1$ on cpct subsets$\}$

then $\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = \int_{x-K} f(x-y)\, g(y)\, dy$; on $x - K \underset{\text{cpct}}{\subseteq} \mathbb{R}^n$,

$|f(x-y)\, g(y)| \leq \|f\|_{C^0} \cdot |g(y)|$ an L^1-fn. indep. of x

$\Rightarrow$ integral deps. contin. on x.

This also works for $\partial_k f(x-y)\, g(y)$ if $f \in C^1$.

$\Rightarrow$ thm: For any $f \in C_0^\infty(\mathbb{R}^n)$ & $g \in L^1_{loc}(\mathbb{R}^n)$, the fn. $f * g$ is smooth on $\mathbb{R}^n$ & $\forall$ multi-indices α, $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

pf: Case $|\alpha| = 1$ is above; rest by induction. ☒

Young's inequality:

thm: $\forall\ f \in L^1(\mathbb{R}^n)$ & $g \in L^p(\mathbb{R}^n)$ for some $p \in [1,\infty]$, then $f * g$ is def'd a.e. & belongs to $L^p(\mathbb{R}^n)$, with $\|f*g\|_{L^p} \le \|f\|_{L^1} \cdot \|g\|_{L^p}$.

pf: Case $p=\infty$ is an easy exercise. Consider $p<\infty$.

Let $q \in (1,\infty]$ s.t. $\frac{1}{p}+\frac{1}{q}=1$, then

$$|f(x-y)\,g(y)| = \left(|f(x-y)|^{1/p} \cdot |g(y)|\right) \cdot |f(x-y)|^{1/q},$$ so Hölder $\Rightarrow$

$$\forall\ x \in \mathbb{R}^n,\quad \varphi(x) := \int_{\mathbb{R}^n} |f(x-y)\,g(y)|\,dy \le \left(\int_{\mathbb{R}^n} |f(x-y)|\cdot|g(y)|^p\,dy\right)^{1/p} \cdot$$

$$\underbrace{\left(\int_{\mathbb{R}^n} |f(x-y)|\,dy\right)}_{=\|f\|_{L^1}}{}^{1/q} = \|f\|_{L^1}^{1/q}\left(\int_{\mathbb{R}^n} |f(x-y)|\cdot|g(y)|^p\,dy\right)^{1/p}.$$

Now $$\int_{\mathbb{R}^n} \varphi(x)^p\,dx = \|f\|_{L^1}^{p/q}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y)|\cdot|g(y)|^p\,dy\right)dx$$

$$\overset{\text{Fubini}}{=} \|f\|_{L^1}^{p/q}\int_{\mathbb{R}^n}\Big(\underbrace{\int_{\mathbb{R}^n} |f(x-y)|\,dx}_{=\|f\|_{L^1}}\Big)|g(y)|^p\,dy = \|f\|_{L^1}^{\frac{p}{q}+1}\cdot\|g\|_{L^p}^p$$

$\frac{p}{q}+1 = p\left(1-\frac{1}{p}\right)+1 = p$

$$= \|f\|_{L^1}^p \cdot \|g\|_{L^p}^p.$$

$\Rightarrow \varphi(x)^p < \infty$ for almost all x $\Rightarrow$ $f*g$ is def'd almost everywhere,

& since $|f*g(x)| = \left|\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy\right| \le \int_{\mathbb{R}^n} |f(x-y)|\cdot|g(y)|\,dy = \varphi(x)$,

$\Rightarrow \|f*g\|_{L^p} \le \|f\|_{L^1}\cdot\|g\|_{L^p}$. ☒

__approximate identities__

"__physicists' defn.__": The Dirac δ-fn. $\delta : \mathbb{R}^n \longrightarrow \mathbb{R}$ characterized by $\delta(x) = 0 \ \forall\ x \neq 0$, $\delta(0) = \infty$ & $\int_{\mathbb{R}^n} f(x)\,\delta(x)\,dx = f(0) \ \forall f$.

__defn__: An __approximate identity__ on $\mathbb{R}^n$ is a seq. of smooth fns. $\rho_j : \mathbb{R}^n \longrightarrow [0,\infty)$ s.t. $\forall\ \varphi \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \rho_j \varphi\, dm \xrightarrow{j\to\infty} \varphi(0)$.

__defn__: We say ρ_j has __shrinking support__ if $\exists$ seq $r_j > 0$ s.t. $r_j \longrightarrow 0$ & $\operatorname{supp}(\rho_j) \subseteq B_{r_j}(0)$.

__Lemma__: A seq. of C^∞ fns. $\rho_j : \mathbb{R}^n \longrightarrow [0,\infty)$ w/ shrinking support is an approx. id. $\iff \int_{\mathbb{R}^n} \rho_j\, dm \longrightarrow 1$ as $j \to \infty$.

__pf of__ $\Leftarrow$: Assume $\int_{\mathbb{R}^n} \rho_j\, dm \to 1$ & $\varphi \in C_0^\infty(\mathbb{R}^n)$.

$$\left| \varphi(0) - \int_{\mathbb{R}^n} \varphi \rho_j\, dm \right| = \left| \varphi(0)\left(1 - \int_{\mathbb{R}^n} \rho_j\, dm\right) + \int_{\mathbb{R}^n} [\varphi(0) - \varphi(x)]\,\rho_j(x)\, dx \right|$$

$$\leq |\varphi(0)| \underbrace{\left(1 - \int_{\mathbb{R}^n} \rho_j\, dm\right)}_{\to 0 \text{ as } j\to\infty} + \underbrace{\sup_{x \in B_{r_j}(0)} |\varphi(0) - \varphi(x)|}_{\substack{\text{small as } j\to\infty \\ \text{since } \varphi \text{ contin. at } 0}} \underbrace{\int_{\mathbb{R}^n} \rho_j\, dm}_{\text{bdd}} \qquad \boxtimes$$

main thm now follows from:

__thm__. For any $f \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ & any approx. id. ρ_j w/ shrinking support, $\rho_j * f \xrightarrow{L^p} f$ as $j \to \infty$.

__pf__: Assume $\operatorname{supp}(\rho_j) \subseteq B_{r_j}(0)$, $\left| \int_{\mathbb{R}^n} \rho_j\, dm - 1 \right| < \varepsilon_j$, $r_j, \varepsilon_j \to 0$.

__Assumption R__: $\|f\|_{L^\infty} \leq R$ & $f \equiv 0$ on $\mathbb{R}^n \setminus B_R(0)$.

__claim__: $f_j := \rho_j * f \xrightarrow{L^1} f$.

__pf__: $f_j(x) = f * \rho_j(x) = \int_{\mathbb{R}^n} f(x-y)\,\rho_j(y)\,dy$, then

$$|f_j(x) - f(x)| = \left| \int_{\mathbb{R}^n} [f(x-y) - f(x)]\,\rho_j(y)\,dy + f(x)\left(\int_{\mathbb{R}^n} \rho_j\, dm - 1\right) \right|$$

$$\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|\,\rho_j(y)\,dy + |f(x)|\,\varepsilon_j.$$

$$\|f_j - f\|_{L^1} \leq \underbrace{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y) - f(x)|\,\rho_j(y)\,dy\right)dx}_{= \ \text{Fubini}} + \underbrace{\varepsilon_j \|f\|_{L^1}}_{\to 0 \text{ as } j\to\infty}.$$

$\int_{\mathbb{R}^n} \rho_j($

TO BE CONTINUED