Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi,
L. Upmeier zu Belzen

Funktionalanalysis
WiSe 2020–21

Problem Set 9

Due: Thursday, 21.01.2021 (23pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Note: *Several results relevant to this problem set were stated but not proved (at least not with all details) in lecture, and you may feel free to use them in your solutions unless otherwise indicated. These include:*

- *For any distribution Λ and test function φ, $\varphi * \Lambda$ is a smooth function satisfying $\partial^\alpha(\varphi * \Lambda) = (\partial^\alpha \varphi) * \Lambda = \varphi * \partial^\alpha \Lambda$ for all multi-indices α. (Theorem 10.27 in the notes)*
- *If $\Lambda \in \mathscr{D}'(\Omega)$ has first derivatives $\partial_1 \Lambda, \dots, \partial_n \Lambda \in \mathscr{D}'(\Omega)$ that are all representable by continuous functions on $\Omega \subset \mathbb{R}^n$, then Λ is representable by a C^1-function on Ω. (Theorem 10.33 in the notes)*

Problem 1 $(*)$
Consider the locally integrable real-valued function $f(x) := |x|$ on $\mathbb{R}$.

(a) Prove that f has weak derivative $f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0 \end{cases}$. [3pts][1]

(b) Prove that f' is not weakly differentiable, but its derivative in the sense of distributions is $2\delta \in \mathscr{D}'(\mathbb{R})$. [3pts]

Problem 2
Consider the real-valued function $f(x) := \ln|x|$ on $\mathbb{R}$.

(a) $(*)$ Show that f is in $L^1_{\mathrm{loc}}(\mathbb{R})$ and its distributional derivative $\Lambda_f' \in \mathscr{D}'(\mathbb{R})$ is[2]

$$\Lambda_f'(\varphi) = \mathrm{p.\,v.} \int_{\mathbb{R}} \frac{\varphi(x)}{x}\,dx := \lim_{\epsilon \to 0^+} \int_{|x| \geqslant \epsilon} \frac{\varphi(x)}{x}\,dx \quad \text{for} \quad \varphi \in \mathscr{D}(\mathbb{R}). \qquad [6\text{pts}]$$

(b) Show that for any smooth compactly supported function $\psi : \mathbb{R} \to \mathbb{R}$, the smooth function $\psi * f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(\psi * f)'(x) = \int_{-\infty}^{\infty} \psi'(x - y) \ln|y|\,dy = \lim_{\epsilon \to 0^+} \int_{|y-x| \geqslant \epsilon} \frac{\psi(y)}{x - y}\,dy$$

for all $x \in \mathbb{R}$.

---

[1] Note that there is no need to define $f'(0)$ in Problem 1(a) since $\{0\} \subset \mathbb{R}$ is a set of measure zero.

[2] The notation p. v. in Problem 2 stands for "Cauchy principal value" and is defined as the limit on the right hand side. The limit is necessary since $1/x$ is not a locally integrable function and thus $x \mapsto \varphi(x)/x$ is not always in $L^1(\mathbb{R})$ for $\varphi \in \mathscr{D}(\mathbb{R})$.

Problem 3
Let $W^{m,p}_{\mathrm{loc}}(\Omega)$ denote the space of functions on $\Omega \subset \mathbb{R}^n$ whose restrictions to every open subset $\mathcal{U} \subset \Omega$ with compact closure are in $W^{m,p}(\mathcal{U})$. Prove:

(a) $(*)$ If f is an absolutely continuous function on an interval $[a,b]$, then its classical derivative f' (defined almost everywhere) is also its weak derivative on the domain (a,b), hence $f \in W^{1,1}((a,b))$. [3pts]
Hint: For any $\varphi \in \mathscr{D}((a,b))$, φf defines an absolutely continuous function on $[a,b]$ that vanishes at the end points.

(b) If $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for an open subset $\Omega \subset \mathbb{R}$, then on every compact subinterval $[a,b] \subset \Omega$, f is equal almost everywhere to an absolutely continuous function.
Hint: Compare the weak derivatives of f and the function $g(x) := \int_a^x f'(t)\,dt$ on $[a,b]$.

(c) $(*)$ Part (b) implies that every $f \in W^{1,1}(\Omega)$ on an open interval $\Omega \subset \mathbb{R}$ can be assumed continuous after changing its values on a set of measure zero. Assuming this modification has been made, prove that there exists a constant $c > 0$ independent of f such that

$$\|f\|_{C^0} \leqslant c\|f\|_{W^{1,1}} \quad \text{for all} \quad f \in W^{1,1}(\Omega).$$

In other words, there is a continuous inclusion $W^{1,1}(\Omega) \hookrightarrow C^0_b(\Omega)$.
Hint: Prove that $|f(x) - f(y)| \leqslant \|f'\|_{L^1}$ for all $x, y \in \Omega$, and deduce from this that $|f(x)| \geqslant \|f\|_{C^0} - \|f'\|_{L^1}$ for all $x \in \Omega$. [5pts]

(d) Show that for $\Omega = (-1,1)$, the continuous inclusion $W^{1,1}(\Omega) \hookrightarrow C^0(\Omega)$ in part (c) is not compact.
Hint: Describe (by drawing a picture) an L^1-convergent sequence of smooth functions $f_j : (-1,1) \to \mathbb{R}$ such that $\|f_j'\|_{L^1}$ is bounded but the L^1-limit is discontinuous.

Comment: The Sobolev embedding theorem gives continuous inclusions $W^{k,p} \hookrightarrow C^0$ when $kp > n$ with domains $\Omega \subset \mathbb{R}^n$, but no such inclusion exists in general for the so-called "Sobolev borderline cases" where $kp = n$, of which $W^{1,1}$ on $\Omega \subset \mathbb{R}$ is an example. For this reason, the result of part (c) is slightly surprising, though part (d) implies that there is no improved inclusion $W^{1,1} \hookrightarrow C^{0,\alpha}$ for any $\alpha > 0$. If there were, then $W^{1,1} \hookrightarrow C^0$ would be compact on bounded intervals $\Omega \subset \mathbb{R}$ due to the compactness of $C^{0,\alpha} \hookrightarrow C^0$, which follows from Arzelà-Ascoli.

Problem 4
When Ω is a nonempty bounded interval $(a,b) \subset \mathbb{R}$, the Sobolev embedding theorem gives continuous inclusions

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{if} \quad 0 < \alpha < 1,\ 1 < p \leqslant \infty \text{ and } \alpha \leqslant 1 - \frac{1}{p}$$
$$W^{2,1}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{if} \quad 0 < \alpha < 1.$$

Without citing the theorem, prove this as follows:

(a) Deduce the inclusions $W^{2,1} \hookrightarrow C^{0,\alpha}$ for $\alpha \in (0,1]$ from a continuous inclusion $W^{2,1} \hookrightarrow C^1$ using Problem 3.

(b) Deduce the inclusion $W^{1,p} \hookrightarrow C^0$ for every $p \geqslant 1$ from Problem 3.

(c) $(*)$ For $a \leqslant x < y \leqslant b$, the fundamental theorem of calculus implies $|f(x) - f(y)| \leqslant \|f'\|_{L^1([x,y])}$ for $f \in W^{1,p}(\Omega)$ since (by Problem 3) f can be assumed absolutely continuous. Use Hölder's inequality to deduce a Hölder-type estimate $|f(x) - f(y)| \leqslant c\|f'\|_{L^p} \cdot |x-y|^\alpha$ for $0 < \alpha \leqslant 1 - 1/p$ whenever $p > 1$. [3pts]

Problem Set 9

Problem 1 (∗)
Consider the locally integrable real-valued function $f(x) := |x|$ on $\mathbb{R}$.

(a) Prove that f has weak derivative $f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0 \end{cases}$. [3pts][1]

(b) Prove that f' is not weakly differentiable, but its derivative in the sense of distributions is $2\delta \in \mathscr{D}'(\mathbb{R})$. [3pts]

<u>Sol^n</u> $\quad f(x) = |x|$

a) Let $\varphi \in \mathscr{D}(\mathbb{R})$

$$\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = -\int_{\mathbb{R}} \operatorname{sgn}(x)\,\varphi(x)\,dx$$

This will prove the assertion.

$$\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = \int_{-\infty}^{0} -x\,\varphi'(x)\,dx + \int_{0}^{\infty} x\,\varphi'(x)\,dx$$

$$\text{IBP} \quad = \underbrace{\left[-x\,\varphi(x)\right]_{-\infty}^{0}} - \int_{-\infty}^{0} -\varphi(x)\,dx$$

$$+ \underbrace{\left[x\,\varphi(x)\right]_{0}^{\infty}} - \int_{0}^{\infty} \varphi(x)\,dx$$

$$\varphi \in C_0^\infty(\mathbb{R}) \implies \lim_{x \to \pm\infty} \varphi(x) = 0$$

$$= -\left(\int_{-\infty}^{0} -\varphi(x)\,dx + \int_{0}^{\infty} \varphi(x)\,dx \right)$$

$$= - \int_{\mathbb{R}} \operatorname{sgn}(x)\, \varphi(x)\, dx$$

$\Rightarrow$ $\operatorname{sgn}(x)$ is the weak derivative of $f(x)$. $\boxtimes$

(b) $\operatorname{sgn}(x)$ is not weakly differentiable.

Suppose $g(x) = \operatorname{sgn}(x)'$ (weak derivative)

on $\{x>0\}$ and when $\{x<0\}$, $\operatorname{sgn}(x)$

has classical derivative 0. $\Rightarrow$ $g(x)=0$ on

$(-\infty,0) \cup (0,\infty)$

$$\int_{\mathbb{R}} \operatorname{sgn}(x)\, \varphi'(x)\, dx = - \int g(x)\varphi(x)\, dx$$

$$\forall\, \varphi \in C_0^\infty(\mathbb{R})$$

$$\int_{-\infty}^{0} -\varphi'(x)\, dx + \int_{0}^{\infty} \varphi'(x)\, dx = \left[-\varphi(x)\right]_{-\infty}^{0} + \left[\varphi(x)\right]_{0}^{\infty}$$

$$= -\varphi(0) + \varphi(-\infty) + \varphi(\infty) - \varphi(0)$$

$$= -2\varphi(0)$$

$\Rightarrow \quad -2\varphi(0) = 0 \quad \Rightarrow \quad \varphi(0) = 0 \quad \forall \quad \varphi \in C_0^\infty(\mathbb{R})$

contradiction $\Rightarrow$ $sgn(x)$ has no weak derivative.

$$(\partial_i \Lambda_{sgn(x)}, \varphi) = -(\Lambda_{sgn(x)}, \partial_i \varphi)$$
$$= -(\Lambda_{sgn(x)}, \varphi') = -\Lambda_{sgn(x)}(\varphi)$$
$$= -\int_{\mathbb{R}} sgn(x)\varphi(x)\,dx$$
$$= 2\varphi(0) = 2\delta(\varphi)$$

$\Rightarrow$ 2δ is the distributional derivative of $f' = sgn(x)$.

$\boxed{}$

Problem 2

Consider the real-valued function $f(x) := \ln|x|$ on $\mathbb{R}$.

(a) $(*)$ Show that f is in $L^1_{\text{loc}}(\mathbb{R})$ and its distributional derivative $\Lambda'_f \in \mathscr{D}'(\mathbb{R})$ is[2]

$$\Lambda'_f(\varphi) = \text{p.v.} \int_{\mathbb{R}} \frac{\varphi(x)}{x}\,dx := \lim_{\epsilon \to 0^+} \int_{|x| \geqslant \epsilon} \frac{\varphi(x)}{x}\,dx \quad \text{for} \quad \varphi \in \mathscr{D}(\mathbb{R}). \qquad [6\text{pts}]$$

(b) Show that for any smooth compactly supported function $\psi : \mathbb{R} \to \mathbb{R}$, the smooth function $\psi * f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(\psi * f)'(x) = \int_{-\infty}^{\infty} \psi'(x-y) \ln|y|\,dy = \lim_{\epsilon \to 0^+} \int_{|y-x| \geqslant \epsilon} \frac{\psi(y)}{x-y}\,dy$$

for all $x \in \mathbb{R}$.

a) The only problematic compact sets are those containing 0.

$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{b} \ln|x|\,dx \xrightarrow{IBP} \lim_{\epsilon \to 0^+} \left[x \ln x - x\right]_{\epsilon}^{b}$$

$$= \underbrace{b \ln b - b}_{<\infty} + \lim_{\epsilon \to 0^+} (\underbrace{x \ln x}_{0} - x)$$

$$\lim_{x \to 0} x \ln x = 0 \quad \text{(L'Hospitals rule)}$$

$$\frac{\ln x}{\frac{1}{x}} \rightsquigarrow \frac{-x^2}{x} = -x$$

$$\Rightarrow \quad f(x) = \ln|x| \in L^1_{loc}(\mathbb{R}).$$

Consider $\Lambda_f \in \mathcal{S}'(\mathbb{R})$

$$\Lambda_f'(\varphi) = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x}\,dx \quad \forall \varphi \in \mathcal{S}(\mathbb{R})$$

$$\Lambda_f'(\varphi) = -\int_{\mathbb{R}} \ln|x|\,\varphi(x)\,dx$$

$$= \lim_{\epsilon \to 0^+} - \int_{|x| \geq \epsilon} \ln|x|\,\varphi'(x)\,dx$$

$$= -\lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} \ln|x|\,\varphi'(x)\,dx + \int_{\epsilon}^{\infty} \ln|x|\,\varphi'(x)\,dx \right)$$

$$= -\lim_{\epsilon\to 0^+}\left(\ln|\epsilon|\,\varphi(-\epsilon) - \ln|\epsilon|\,\varphi(\epsilon) - \int_{|x|>\epsilon} \frac{\varphi(x)}{x}\,dx \right)$$

$$\lim_{\epsilon\to 0} \ln|\epsilon|\left(\varphi(\epsilon)-\varphi(-\epsilon)\right) = 0$$

thus $$\Lambda_f'(\varphi) = \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \frac{\varphi(x)}{x}\,dx$$

(b) $\psi : \mathbb{R}\to\mathbb{R}$, $\psi * f : \mathbb{R}\to\mathbb{R}$

$$(\psi * f)'(x) = \int_{-\infty}^{\infty} \psi'(x-y)\ln|y|\,dy$$

$$= \lim_{\epsilon\to 0^+}\int_{|y-x|\geq\epsilon} \frac{\psi(y)}{x-y}\,dy$$

If $\varphi\in\mathcal{S}(\mathbb{R})$

$$(\varphi * f)(x) = (\Lambda_f , \tau_{-x}\sigma\varphi)$$

$$= \Lambda_f(\tau_{-x}\sigma\varphi) = (\varphi * \Lambda_f)(x)$$

$$(\varphi * \Lambda_f)' = \varphi * \Lambda_f' = \varphi' * \Lambda_f$$

$$\varphi' * \Lambda_f(x) = \Lambda_f(\tau_{-x}\sigma\varphi)$$

$$= \int_{-\infty}^{\infty} \varphi'(x-y)\ln|y|\, dy$$

put $z = x - y \Rightarrow dz = -dy$

$$= -\lim_{\epsilon \to 0^+} \int_{|z|\geq\epsilon} \varphi'(z)\ln|x-y|\, dz$$

IBP

$$= \lim_{\epsilon \to 0^+} \int_{|z|\geq\epsilon} \frac{\varphi(z)}{x-z}\, dz \qquad \square$$

Problem 3

Let $W^{m,p}_{\mathrm{loc}}(\Omega)$ denote the space of functions on $\Omega \subset \mathbb{R}^n$ whose restrictions to every open subset $\mathcal{U} \subset \Omega$ with compact closure are in $W^{m,p}(\mathcal{U})$. Prove:

(a) ($*$) If f is an absolutely continuous function on an interval $[a,b]$, then its classical derivative f' (defined almost everywhere) is also its weak derivative on the domain (a,b), hence $f \in W^{1,1}((a,b))$. [3pts]
Hint: For any $\varphi \in \mathscr{D}((a,b))$, φf defines an absolutely continuous function on $[a,b]$ that vanishes at the end points.

(b) If $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for an open subset $\Omega \subset \mathbb{R}$, then on every compact subinterval $[a,b] \subset \Omega$, f is equal almost everywhere to an absolutely continuous function.
Hint: Compare the weak derivatives of f and the function $g(x) := \int_a^x f'(t)\,dt$ on $[a,b]$.

(c) ($*$) Part (b) implies that every $f \in W^{1,1}(\Omega)$ on an open interval $\Omega \subset \mathbb{R}$ can be assumed continuous after changing its values on a set of measure zero. Assuming this modification has been made, prove that there exists a constant $c > 0$ independent of f such that

$$\|f\|_{C^0} \leqslant c\|f\|_{W^{1,1}} \quad \text{for all} \quad f \in W^{1,1}(\Omega).$$

In other words, there is a continuous inclusion $W^{1,1}(\Omega) \hookrightarrow C^0_b(\Omega)$.
Hint: Prove that $|f(x) - f(y)| \leqslant \|f'\|_{L^1}$ for all $x, y \in \Omega$, and deduce from this that $|f(x)| \geqslant \|f\|_{C^0} - \|f'\|_{L^1}$ for all $x \in \Omega$. [5pts]

(d) Show that for $\Omega = (-1,1)$, the continuous inclusion $W^{1,1}(\Omega) \hookrightarrow C^0(\Omega)$ in part (c) is not compact.
Hint: Describe (by drawing a picture) an L^1-convergent sequence of smooth functions $f_j : (-1,1) \to \mathbb{R}$ such that $\|f_j'\|_{L^1}$ is bounded but the L^1-limit is discontinuous.

$W^{1,1}$ in dim 1, $n=1$

$1.1 = 1$, $kp > n$ ✗ $kp = n$

(a) Let $\varphi \in \mathcal{D}((a,b)) \Rightarrow \varphi f(a) = \varphi f(b) = 0$

φ is smooth compactly supported fnc on (a,b)

note that φf is an absolutely continuous function on $[a,b]$.

$$a \leq a_1 < b_1 \leq a_2 \leq b_2 \dots \leq a_n \leq b_n \leq b$$

$\forall \epsilon > 0 \quad \exists \delta > 0$ s.t.

$$\sum |b_i - a_i| < \delta \Rightarrow \sum |f(b_i) - f(a_i)| < \epsilon$$

$$|\varphi f(a_i) - \varphi f(b_i)| \leq M |f(b_i) - f(a_i)|$$

where $M = \sup \varphi$.

for any $x \in (a,b)$

$$\varphi f(x) = \varphi f(a) + \int_a^x (\varphi f)'(t)\, dt$$

$$= 0 + \int_a^x (\varphi' f)(t)\, dt + \int_a^x (\varphi f')(t)\, dt$$

put $x=b$

$$\varphi f(b) = \int_a^b \varphi' f \, dx + \int_a^b \varphi f' \, dx$$

$\overset{\shortparallel}{0}$

$$\Rightarrow \int_a^b \varphi' f \, dx = - \int_a^b \varphi f' \, dx$$

$\Rightarrow$ f' is also the weak derivative of f.

∇

(b) $f \in W^{1,1}_{loc}(\Omega)$, $\Omega \overset{\text{open}}{\subset} \mathbb{R}$, $[a,b] \subset \Omega$

$f =$ some absolutely continuous function.

$$f = g(x) = \int_a^x f'(t) \, dt \quad \text{on } [a,b]$$

We know g is an absolutely continuous function. $\Rightarrow$ it has derivative a.e.

$$g' = f'$$

$$\Rightarrow g(x) = \int_a^x f'(t) \, dt = f(x) \quad \text{a.e. on } [a,b].$$

any $f \in W^{1,1}([a,b])$ or $f \in W^{1,1}_{loc}(\Omega)$

can be represented by an absolutely cont. func.

(c) $\|f\|_{C^0} \le c\|f\|_{W^{1,1}}$ $\forall f \in W^{1,1}(\Omega)$

$W^{1,1}(\Omega) \hookrightarrow C^0_b(\Omega)$.

By the FTC

$$|f(x)-f(y)| = \left|\int_y^x f'(t)\,dt\right|$$

$$\le \int_y^x |f'(t)|\,dt \le c\|f'\|_{L^1} \quad \text{— (1)}$$

$$f(y) = f(x) + f(y) - f(x)$$

$$\Rightarrow |f(y)| = |f(x) + f(y) - f(x)|$$

$$\le |f(x)| + |f(y) - f(x)|$$

$$\Rightarrow \|f\|_{C^0} \le |f(x)| + |f(y) - f(x)|$$

$$\le |f(x)| + \|f'\|_{L^1}$$

$\Rightarrow \quad |f(x)| \geq \|f\|_{C^0} - \|f'\|_{L^1}$

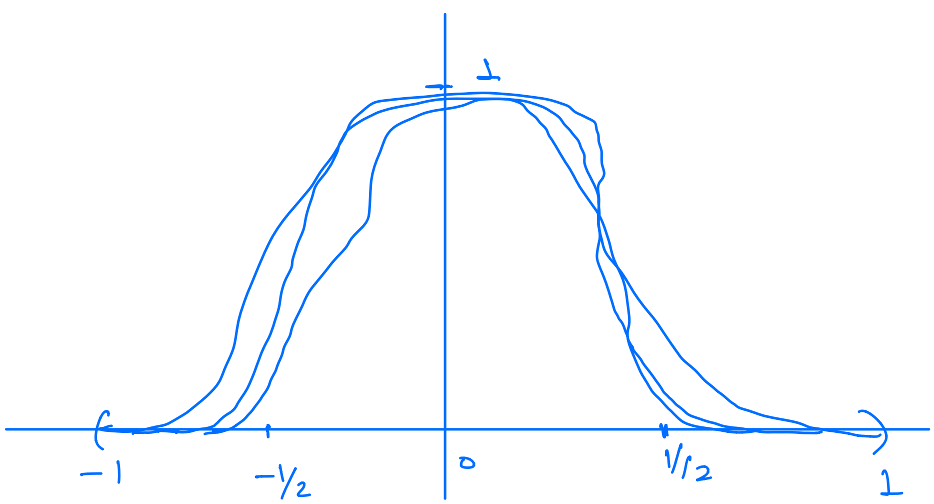
$$\Downarrow$$

$$\|f\|_{C^0} \leq C\left(\|f(x)\|_{L^1} + \|f'\|_{L^1}\right)$$
$$= C\|f\|_{W^{1,1}}$$

$\Rightarrow \quad \|f\|_{C^0} \leq C\|f\|_{W^{1,1}}$, C independent of f and just depend on Ω.

$\Rightarrow \quad W^{1,1} \hookrightarrow C^0$.

(d)

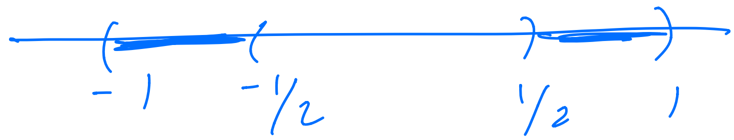


$\mathrm{supp}(f_\varepsilon) \subset [-1/2, 1/2]$

$\mathrm{Area}(f_\varepsilon) = 1 \quad \Rightarrow \quad \chi_{[-1/2,1/2]}$

$\therefore$ L^1-limit of $f_j = \chi_{[-1/2, 1/2]}$

$\chi_{[-1/2, 1/2]}$ is not continuous on $(-1, 1)$



4

Problem 4

When Ω is a nonempty bounded interval $(a, b) \subset \mathbb{R}$, the Sobolev embedding theorem gives continuous inclusions

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{if} \quad 0 < \alpha < 1,\ 1 < p \leqslant \infty \text{ and } \alpha \leqslant 1 - \frac{1}{p}$$

$$W^{2,1}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{if} \quad 0 < \alpha < 1.$$

Without citing the theorem, prove this as follows:

(a) Deduce the inclusions $W^{2,1} \hookrightarrow C^{0,\alpha}$ for $\alpha \in (0, 1]$ from a continuous inclusion $W^{2,1} \hookrightarrow C^1$ using Problem 3.

(b) Deduce the inclusion $W^{1,p} \hookrightarrow C^0$ for every $p \geqslant 1$ from Problem 3.

(c) $(*)$ For $a \leqslant x < y \leqslant b$, the fundamental theorem of calculus implies $|f(x) - f(y)| \leqslant \|f'\|_{L^1([x,y])}$ for $f \in W^{1,p}(\Omega)$ since (by Problem 3) f can be assumed absolutely continuous. Use Hölder's inequality to deduce a Hölder-type estimate $|f(x) - f(y)| \leqslant c\|f'\|_{L^p} \cdot |x - y|^\alpha$ for $0 < \alpha \leqslant 1 - 1/p$ whenever $p > 1$. [3pts]

in prob. 3) $W^{1,1} \hookrightarrow C^0$

$\Rightarrow$ $W^{2,1} \hookrightarrow C^1$ continuous

$\alpha \in (0, 1]$

$$\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + \sup_{\Omega} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$$

$$\leq C\left[\|Df\|_{C^0} + \sup_{\Omega} \frac{|f(x)-f(y)|}{|x-y|}\right]$$

$$\leq C\|f\|_{C^1} \leq C\|f\|_{W^{2,1}}$$

continuous.

$$\|f\|_{C^{0,\alpha}} \leq C\,\|f\|_{W^{2,1}}$$

$$\Longrightarrow \quad W^{2,1} \xrightarrow[\text{continuous}]{} C^{0,\alpha}.$$

☒

(b) $W^{1,p} \hookrightarrow C^0$ continuous $\forall\, p$.

By the FTC

$$|f(x)-f(y)| = \left|\int_y^x f'(t)\,dt\right|$$

$$\leq \int_y^x |f'(t)|\,dt$$

$$\leq \left(\int_y^x |f'(t)|^p\,dt\right)^{1/p} \cdot \left(\int_y^x 1\,dt\right)^{1/q}$$

$\leq C \, \|f'\|_{L^p}$

just depends on the domain Ω.

estimate $|f(y)| = |f(y) + f(x) - f(x)|$

$\leq |f(x)| + \underbrace{|f(x) - f(y)|}_{\leq C \|f'\|_{L^p}}$

doing the same thing as in Prob. 3

$\Rightarrow \quad \|f\|_{C^0} \leq C' \|f\|_{L^p} + C \|f'\|_{L^p}$

just depend on Ω

$\leq C \, \|f\|_{W^{1,p}}$

$\Rightarrow \quad W^{1,p} \hookrightarrow C^0$.

(c) exactly similar to part (b) $\rightsquigarrow$ 3 (c).

$\boxtimes$