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Funktionalanalysis
WiSe 2020–21

Problem Set 11

Due: Thursday, 18.02.2021 (24pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *Unless stated otherwise, $\mathcal{H}$ is a complex Hilbert space, and functions on domains in $\mathbb{R}^n$ or $\mathbb{T}^n$ take values in a fixed finite-dimensional complex inner product space $(V, \langle\ ,\ \rangle)$.*

Problem 1

An operator $T \in \mathscr{L}(\mathcal{H})$ is called *normal* if it commutes with its adjoint T^*. Prove:

(a) The following conditions on $T \in \mathscr{L}(\mathcal{H})$ are equivalent:

  (i) T is normal;
  (ii) $T = A + iB$ for two self-adjoint operators $A, B \in \mathscr{L}(\mathcal{H})$ that commute with each other;
  (iii) $\|Tx\| = \|T^*x\|$ for every $x \in \mathcal{H}$.

  Hint: Consider $\|T(x+y)\|^2$ and $\|T(x+iy)\|^2$ for arbitrary $x, y \in \mathcal{H}$.

(b) $(*)$ If T is normal, then:

  (i) $\|T^2\| = \|T^*T\| = \|T\|^2$ [2pts]
  (ii) The spectral radius of T is $\|T\|$. [4pts]
  (iii) Every eigenvector of T with eigenvalue $\lambda \in \mathbb{C}$ is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$. *Hint: Consider* $\|(\lambda - T)v\|^2$. [2pts]
  (iv) If $v, w \in \mathcal{H}$ are eigenvectors of T with distinct eigenvalues, then $\langle v, w \rangle = 0$. [2pts]
  (v) If T is also compact, then $\mathcal{H}$ admits an orthonormal basis consisting of eigenvectors of T. [4pts]

(c) If T is *unitary* (meaning $T^*T = TT^* = \mathbb{1}$), then its spectrum is contained in the unit circle $\{|\lambda| = 1\} \subset \mathbb{C}$.
Hint: Show $\|T\| = \|T^{-1}\| = 1$, and use the fact that operators with distance less than 1 from the identity map are invertible.

Problem 2

Assume (X, μ) is a σ-finite measure space, $F : X \to \mathbb{C}$ is a bounded measurable function, and $T : L^2(X) \to L^2(X)$ is the multiplication operator $u \mapsto Fu$.

(a) Show that $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(T)$ if and only if[1]

$$\mu\big(F^{-1}(B_\epsilon(\lambda))\big) > 0 \quad \text{for all} \quad \epsilon > 0, \tag{1}$$

[1] The set of numbers $\lambda \in \mathbb{C}$ satisfying the condition in (1) for a given function $F : X \to \mathbb{C}$ is called the *essential range* of F.

where $B_\epsilon(\lambda) \subset \mathbb{C}$ denotes the open disk of radius ϵ about λ.

(b) Under what condition on F is $\lambda \in \sigma(T)$ an eigenvalue of T? When does it have finite multiplicity?

Problem 3
For a Lebesgue-integrable function $F : \mathbb{T}^n \to \mathbb{C}$, define the operator

$$T : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n) : u \mapsto F * u, \quad \text{where} \quad (F * u)(x) = \int_{\mathbb{T}^n} F(x-y)u(y)\,dy.$$

Young's inequality (or more accurately its analogue for periodic functions) implies that T is bounded, with $\|T\| \leqslant \|F\|_{L^1}$.

(a) ($*$) Prove that if the Fourier coefficients $\{\widehat{F}_k\}_{k\in\mathbb{Z}^n}$ of F satisfy $\lim_{|k|\to\infty} |\widehat{F}_k| = 0$, then T is compact. Show that this holds in particular if $F \in L^2(\mathbb{T}^n)$. [5pts]
Hint: For inspiration, look again at the proof that the inclusions $H^s(\mathbb{T}^n) \hookrightarrow H^t(\mathbb{T}^n)$ for $s > t$ are compact.

(b) Under what assumptions on F is T a self-adjoint operator?

(c) Under what assumptions on F is T a normal operator?

(d) Describe the spectrum $\sigma(T)$, and find an explicit collection of eigenvectors of T that form an orthonormal basis of $L^2(\mathbb{T}^n)$. Assuming the condition in part (a), is every element of $\sigma(T)$ necessarily an eigenvalue?

Problem 4
For a fixed constant $x_0 \in \mathbb{T}^n$, let $T : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ denote the translation operator

$$(Tf)(x) := f(x + x_0).$$

This operator is unitary, and therefore cannot be compact.[2]

(a) Find an explicit spectral representation for T, i.e. a σ-finite measure space (X, μ), unitary isomorphism $U : L^2(\mathbb{T}^n) \to L^2(X)$ and bounded measurable function $F : X \to \mathbb{C}$ such that UTU^{-1} is the multiplication operator $u \mapsto Fu$.
Hint: Use Fourier series.

(b) ($*$) Show that depending on the value of $x_0 \in \mathbb{T}^n$, one of the following must happen: (i) $\sigma(T)$ is a finite set consisting of eigenvalues that each have infinite multiplicity; (ii) $\sigma(T)$ is the entire unit circle in $\mathbb{C}$ and consists of a countably infinite set of eigenvalues, plus an uncountable set of points that are not eigenvalues. [5pts]
Advice: Use the result of Problem 2(a) to identify the spectrum.

(c) Carry out the analogues of parts (a) and (b) for a similar translation operator on $L^2(\mathbb{R}^n)$, and show that if the shift $x_0 \in \mathbb{R}^n$ is nonzero, then the spectrum in this case is always the entire unit circle in $\mathbb{C}$ but contains no eigenvalues.

[2] A Banach space isomorphism is never compact unless the space is finite dimensional. (Why not?)

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Problem 1
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(i) T is normal;
(ii) $T = A + iB$ for two self-adjoint operators $A, B \in \mathscr{L}(\mathcal{H})$ that commute with each other;
(iii) $\|Tx\| = \|T^*x\|$ for every $x \in \mathcal{H}$.

Hint: Consider $\|T(x+y)\|^2$ and $\|T(x+iy)\|^2$ for arbitrary $x, y \in \mathcal{H}$.

(b) $(*)$ If T is normal, then:

$$TT^* = T^*T$$

1.

(a)

$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow i)$$

$i) \Rightarrow ii)$ Suppose T is a normal operator.

define $A = \frac{1}{2}(T + T^*)$

$B = \frac{1}{2}(iT^* - iT)$

Claim $T = A + iB$, and A, B are s.A. and $AB = BA$.

$A, B \in \mathscr{L}(\mathcal{H})$

$$A^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + (T^*)^*)$$
$$= \frac{1}{2}(T^* + T) = A$$

$$B^* = \frac{1}{2}\left(iT^* - iT\right)^* = \frac{1}{2}\left(-iT + iT^*\right) = B$$

$$AB = \frac{1}{4}\left(T + T^*\right)\left(iT^* - iT\right)$$

$$= \frac{1}{4}\left(\cancel{iTT^*} - \underbrace{iTT + iT^*T^*}_{} - \cancel{iT^*T}\right)$$

$$BA = \frac{1}{4}\left(iT^*T^* - iTT\right)$$

$$\Rightarrow \quad AB = BA$$

and $\therefore$ we get ii).

ii) $\Rightarrow$ iii) $\quad T = A + iB$, $AB = BA$, A, B s.a.

Want :- $\|Tx\| = \|T^*x\| \quad \forall x \in \mathcal{H}$.

$$T^* = (A+iB)^* = A^* - iB^* = A - iB.$$

$$\|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle Ax - iBx, Ax - iBx \rangle$$

$$= \langle Ax, Ax \rangle - i\langle Ax, Bx \rangle + i\langle Bx, Ax \rangle + \langle Bx, Bx \rangle$$

$$= \langle Ax, Ax \rangle - i\langle BAx, x \rangle + i\langle ABx, x \rangle + \langle Bx, Bx \rangle$$

$$= \langle Ax, Ax \rangle + \langle Bx, Bx \rangle$$

$$\|Tx\|^2 = \langle Ax, Ax \rangle + \langle Bx, Bx \rangle.$$

$\Rightarrow \quad \|T^*x\| = \|Tx\| \quad \forall x \in \mathcal{H}.$

we get that ii) $\Rightarrow$ iii).

iii) $\Rightarrow$ i)

Want:- $TT^* = T^*T$. we can show this by proving that $\forall x, y \in \mathcal{H}$.

$$\langle (TT^* - T^*T)x, y \rangle = 0$$

$$Re(\langle T^*x, T^*x \rangle) = Re(\langle Tx, Tx \rangle)$$

$$Im(\langle T^*x, T^*x \rangle) = Im(\langle Tx, Ty \rangle)$$

$$\langle T^*x, T^*y \rangle = \langle Tx, Ty \rangle$$

$$2Re(\langle T^*x, T^*y \rangle) = \langle T^*x, T^*y \rangle + \langle T^*y, T^*x \rangle$$

$$= \langle T^*x, T^*x \rangle + \langle T^*x, T^*y \rangle + \langle T^*y, T^*x \rangle + \langle T^*y, T^*y \rangle - \|T^*x\|^2 - \|T^*y\|^2$$

$$= \langle T^*(x+y), T^*(x+y) \rangle - \|T^*x\|^2$$

$$- \|T^* y\|^2$$

$$= 2\,\mathrm{Re}\left(\langle Tx, Ty\rangle\right)$$

similarly

$$2\,\mathrm{Im}\left(\langle T^* x, T^* y\rangle\right) = 2\,\mathrm{Im}\left(\langle Tx, Ty\rangle\right).$$

$$\Rightarrow \quad \langle (TT^* - T^*T)x, y\rangle = 0$$

$$\Rightarrow \quad TT^* = T^*T.$$

$\boxtimes$

(b) $(*)$ If T is normal, then:

(i) $\|T^2\| = \|T^*T\| = \|T\|^2$ [2pts]

(ii) The spectral radius of T is $\|T\|$. [4pts]

(iii) Every eigenvector of T with eigenvalue $\lambda \in \mathbb{C}$ is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$. *Hint: Consider* $\|(\lambda - T)v\|^2$. [2pts]

(iv) If $v, w \in \mathcal{H}$ are eigenvectors of T with distinct eigenvalues, then $\langle v, w\rangle = 0$. [2pts]

(v) If T is also compact, then $\mathcal{H}$ admits an orthonormal basis consisting of eigenvectors of T. [4pts]

i) $\|T^*T\| = \|T\|^2$ always holds for $T \in \mathcal{L}(\mathcal{H})$.

Let $x \in \mathcal{H}$.

$$\|T^2 x\|^2 = \langle T^2 x, T^2 x\rangle = \langle T^* T^* T T x, x\rangle$$

$$\langle TTx, TTx\rangle$$

$$= \langle T^*TT^*Tx, x\rangle = \|T^*Tx\|^2$$

$$\Rightarrow \quad \|T^2\| = \|T^*T\| = \|T\|^2$$

□

ii) $r(T) = \|T\|$.

$$r(T) = \limsup_{n\to\infty} \|T^n\|^{1/n}$$

$\mathcal{L}(H)$ is a Banach algebra, $\|AB\| \le \|A\|\cdot\|B\|$

$$\Rightarrow \forall n, \quad \|T^n\|^{1/n} \le (\|T\|^n)^{1/n} = \|T\|$$

$$\Rightarrow \quad r(T) \le \|T\|. \quad \text{——} \quad 1$$

If we prove that there is a subsequence $\{n_k \mid k\in\mathbb{N}\}$, $\|T^{n_k}\|^{1/n_k} \longrightarrow \|T\|$ as $k\to\infty$, this'll prove that $r(T) \ge \|T\|$.

$$\|T^{2^n}\|^{1/2^n} = \left(\|T\|^{2^n}\right)^{\frac{1}{2^n}} = \|T\|$$

$\Rightarrow$ choose the subsequence $\{\frac{1}{2^n}\}$

$$\Rightarrow \quad r(T) \ge \|T\| \quad \text{——} \quad 2$$

By ① and ②, $r(T) = \|T\|$. ∎

iii) $v \in \mathcal{H}$ $\lambda \in \mathbb{C}$

$$\|(\lambda - T)v\|^2 = \langle (\lambda - T)v, (\lambda - T)v \rangle$$
$$= \langle (\lambda - T)^*(\lambda - T)v, v \rangle$$
$$= \langle (\bar{\lambda} - T^*)(\lambda - T)v, v \rangle$$
$$= \langle (\bar{\lambda}\lambda - \bar{\lambda}T - \lambda T^* + T^*T)v, v \rangle$$
$$= \langle (\lambda\bar{\lambda} - \bar{\lambda}T - \lambda T^* + TT^*)v, v \rangle$$
$$= \langle (\lambda - T)(\bar{\lambda} - T^*)v, v \rangle$$
$$= \|(\bar{\lambda} - T^*)v\|^2$$

If v is an eigenvector of T w/ e.v. λ

$\Rightarrow$ v is an eigenvector of T^* w/ e.v. $\bar{\lambda}$. ∎

iv) Suppose $Tv = \lambda v$ $\lambda \neq \mu$.
$Tw = \mu w$

from part iii) $T^* v = \bar{\lambda} v$

$$\Rightarrow (\lambda - \mu)\langle v, w\rangle = \langle \bar{\lambda} v, w\rangle - \langle v, \mu w\rangle$$
$$= \langle T^* v, w\rangle - \langle v, Tw\rangle$$
$$= \langle v, Tw\rangle - \langle v, Tw\rangle = 0$$

$\therefore \lambda \neq \mu \Rightarrow \langle v, w\rangle = 0$. ▨

v) T is compact.

Suppose $\{\lambda_i\}$ set of all e.v. of T.

$E_i = \ker(T - \lambda_i) \ \forall \ i \in I$. E_i is closed $\forall i$.

pick some o.n. basis $\mathcal{B}_i$ of E_i.

$$\mathcal{B} = \bigcup_{i \in I} \mathcal{B}_i$$ is an orthonormal set of vectors

in $\mathcal{H}$.

Also, every element of $\mathcal{B}$ is in $\ker(T - \lambda_i)$

$\Rightarrow$ $\mathcal{B}$ is an orthonormal set consisting only of e.v.

<u>Claim</u> :- $\mathcal{B}$ is a basis for $\mathcal{H}$. If not

assume $E = \text{span}(\mathcal{B})$ $\therefore$ $E \neq \mathcal{H}$

$\Rightarrow$ we have $E^\perp$ is non-trivial.

$T|_{E^\perp}$ has no e.v. on $E^\perp$

If $Tv = \lambda v \Rightarrow v$ is an eigenvector for T

$\Rightarrow v \in E \Rightarrow v = 0$.

$T|_{E^\perp}$ is again a compact operator w/ no eigenvalues $\Rightarrow$ Riesz - Schauder thm

$$\sigma(T|_{E^\perp}) \subseteq \{0\} \Rightarrow r(T|_{E^\perp}) = 0.$$

$\Rightarrow \|T|_{E^\perp}\| = 0 \Rightarrow T|_{E^\perp} = 0$.

$\Rightarrow E^\perp = \{0\}$ contradiction.

$\therefore E = \text{span}(\mathcal{B}) = \mathcal{H}$

$\Rightarrow \mathcal{H}$ has a basis consisting of e.v.

If T is *unitary* (meaning $T^*T = TT^* = \mathbb{1}$), then its spectrum is contained in the unit circle $\{|\lambda| = 1\} \subset \mathbb{C}$.
Hint: Show $\|T\| = \|T^{-1}\| = 1$, *and use the fact that operators with distance less than 1 from the identity map are invertible.*

c) $T^{-1} = T^*$

$\|Tx\| = \|T^*x\|$ for T normal and every unitary operator is normal.

Let $x \in \mathcal{H}$

$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle x, x \rangle = \|x\|^2$

$\Rightarrow \quad \|T\| = 1 = \|T^{-1}\| = \|T^*\|.$

$\because \quad r(T) = \|T\| = 1. \Rightarrow \sigma(T) \subseteq \{ z \in \mathbb{C} \mid |z| \leq 1 \}$

Suppose $\lambda \in \mathbb{C}$ w/ $|\lambda| < 1$

Want:- $\lambda \notin \sigma(T)$ i.e, $T - \lambda$ is invertible.

$T - \lambda = T(Id - \lambda T^*)$

enough to prove that $Id - \lambda T^*$ is invertible.

In PSET 1, if $\|x\| < 1 \Rightarrow (Id - x)$ is invertible

$\Rightarrow$ we'll prove that $\|\lambda T^*\| < 1$

$\|\lambda T^* x\|^2 = \langle \lambda T^* x, \lambda T^* x \rangle = |\lambda|^2 \langle T^* x, T^* x \rangle$
$= |\lambda|^2 \|x\|^2$

$\|\lambda T^{*}\| = |\lambda| < 1.$

$\therefore \quad \sigma(T) = \{ z \in \mathbb{C} \mid |z| = 1 \}.$

$\boxtimes$

Problem 2

Assume (X, μ) is a σ-finite measure space, $F : X \to \mathbb{C}$ is a bounded measurable function, and $T : L^2(X) \to L^2(X)$ is the multiplication operator $u \mapsto Fu$.

(a) Show that $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(T)$ if and only if[1]

$$\mu\big(F^{-1}(B_\epsilon(\lambda))\big) > 0 \quad \text{for all} \quad \epsilon > 0, \tag{1}$$

[1] The set of numbers $\lambda \in \mathbb{C}$ satisfying the condition in (1) for a given function $F : X \to \mathbb{C}$ is called the *essential range* of F.

where $B_\epsilon(\lambda) \subset \mathbb{C}$ denotes the open disk of radius ϵ about λ.

(b) Under what condition on F is $\lambda \in \sigma(T)$ an eigenvalue of T? When does it have finite multiplicity?

(a) assume that $\exists\ \epsilon > 0$ s.t

$\mu\left(F^{-1}(B_\epsilon(\lambda))\right) = 0$

We'll show that $\lambda \notin \sigma(T) \iff \lambda - T$ is invertible

$\iff \quad \mu\{x \in X \mid |F(x) - \lambda| < \epsilon\} = 0 \quad \text{—— } (1)$

If we prove that $\lambda - T$ is bijective.

Suppose $f \in L^2(X)$

$$\|(\lambda-T)f\|_{L^2}^2 = \|\lambda f - Ff\|_{L^2}^2$$

$$= \int_X |\lambda - F(x)|^2 |f(x)|^2 d\mu$$

$$\geq \int_X \epsilon^2 |f(x)|^2 d\mu \geq \epsilon^2 \|f\|_{L^2}^2$$

In Take-Home midterm, $c = \epsilon^2$

$\Rightarrow$ $\lambda - T$ is injective and has closed range.

If $\lambda - T$ is NOT surjective $\Rightarrow$

λ is in the residual spectrum of T.

$\Rightarrow$ λ is an eigenvalue of T'.

$\Rightarrow$ $\exists$ some $\Lambda \in (L^2(X))^*$ s.t.

$T'\Lambda = \lambda\Lambda \Rightarrow \Lambda(Tu) = \lambda\Lambda(u)$ $\forall u \in L^2(X)$

$\Rightarrow$ By Riesz-representation thm. $\exists f \in L^2(X)$

s.t. $\Lambda = \Lambda_f \Rightarrow \forall u \in L^2(X)$ we have

$$0 = \Lambda_f(Tu - \lambda u) = \int_X \langle f(x), F(x)u(x) - \lambda u(x)\rangle d\mu$$

$$= \int_X (F(x)\langle f(x), u(x)\rangle - \lambda\langle f(x), u(x)\rangle) d\mu$$

$$= \int_X \langle F(x) f(x) - \lambda f(x), \mu(x) \rangle \, d\mu$$

$\Rightarrow \quad f(F-\lambda) = 0 \quad$ a.e.

$\because \quad \lambda \neq 0 \Rightarrow f \neq 0$

$\Rightarrow \quad |F-\lambda| = 0 \quad$ which is a contradiction to (1).

$\Rightarrow \quad \lambda - T$ is bijective $\Rightarrow \lambda - T$ is invertible

$\Rightarrow \quad \lambda \notin \sigma(T)$.

for the other direction, let $\lambda - T$ be invertible.

Want:- $\exists \; \epsilon > 0$ s.t.

$$\mu(\{x \in X \mid |F(x) - \lambda| < \epsilon\}) = 0.$$

$\because \quad \lambda - T$ is injective and has closed range

$\Rightarrow \quad \exists$ some $\epsilon > 0$ s.t

$$\|(\lambda - T) f\|_{L^2} \geq \epsilon \|f\|_{L^2} \quad \forall \; f \in L^2(X).$$

(Take-Home midterm).

$\Rightarrow \quad \forall \; f \in L^2(X)$

$$\int_X |\lambda - F(x)|^2 |f(x)|^2 d\mu \geq \epsilon^2 \int_X |f(x)|^2 d\mu$$

$$A = \{ x \in X \mid |F(x) - \lambda| < \epsilon \}$$

want :- $\mu(A) = 0$.

Suppose not. $\mu(A) > 0$.

$\Rightarrow$ $\exists$ some $B \subseteq A$ of finite non-zero measure, $\because$ X is σ-finite.

Let us define $f : X \to \mathbb{C}$ as

$$f(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}$$

$f \in L^2(X)$.

Then

$$\int_X |\lambda - F(x)|^2 |f(x)|^2 d\mu = \int_B |\lambda - F(x)|^2 d\mu$$

$$< \int_B \epsilon^2 d\mu = \epsilon^2 \int_X |f(x)|^2 d\mu$$

contradiction.

$\Rightarrow$ $\mu(A) = 0$.

$\therefore$ we complete the proof (a). $\square$

(b) $\lambda \in \mathbb{C}$ is an e.v. of T

$\iff \mu\left(\{x \in X \mid F(x) = \lambda\}\right) > 0.$

$\Rightarrow$

If λ is an e.v.

$\Rightarrow \exists\ f \neq 0 \in L^2(X)$ s.t. $Tf = \lambda f$

$\Rightarrow f(x)\,(\underbrace{F(x) - \lambda}_{=0 \text{ a.e.}}) = 0$

$L^2\left(\{x \in X \mid \exists\, n \in \mathbb{N} \text{ s.t. } (F(x))^n = \lambda\}\right)$

is finite-dimensional.

3. $F : \mathbb{T}^n \to \mathbb{C}$ be a Lebesgue integrable function.

(a) $\lim_{|k| \to \infty} |\hat{F}_k| = 0$ is given.

consider $T' = \mathcal{F}T : L^2(\mathbb{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n)$

$\Rightarrow T'(u) = \mathcal{F}(F * u) = \hat{F} \cdot \hat{u}$

Define $T_k' : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$ by

$$T_k'(u)_m = \begin{cases} \hat{F}_m \hat{u}_m & \text{if } |m| \le k \\ 0 & \text{if } |m| > k \end{cases}$$

bounded operators and is of finite rank $\forall\, k \in \mathbb{N} \Rightarrow$ are compact.

Then

$$\|T_k' - T'\|^2 = \sup_{u \in L^2(\mathbb{T}^n) \setminus \{0\}} \frac{\sum_{|m|>k} |\hat{F}_m \hat{u}_m|^2}{\|u\|_{L^2}^2}$$

$\because \lim_{|k| \to \infty} |\hat{F}_k| = 0 \Rightarrow \forall\, \epsilon > 0 \ \exists\, N$ s.t

$$|\hat{F}_k| \le \epsilon \quad \forall\, |k| > N.$$

$$\Rightarrow \sum_{|k|>r} |\hat{F}_m \hat{u}_m|^2 \le \epsilon^2 \sum_{|m|>k} |\hat{u}_m|^2 \le \epsilon^2 \sum_{m \in \mathbb{Z}^n} |\hat{u}_m|^2$$

$$\leq \epsilon^2 \|\hat{u}\|^2_{\ell^2} = \epsilon^2 \|u\|^2_{L^2}$$

$\Rightarrow \quad \|T'_k - T'\| \leq \epsilon$ for $k \in \mathbb{N}$ w/ $k > N$.

$$\Rightarrow \quad \lim_{k\to\infty} T'_k = T'$$

$\Rightarrow \quad T'$ is compact $\Rightarrow \quad T = \mathcal{F}^* T'$ is also compact.

If $f \in L^2(\mathbb{T}^n) \Rightarrow \hat{f} \in \ell^2(\mathbb{Z}^n) \Rightarrow \|\hat{F}\|^2_{\ell^2} < \infty$

$\therefore$ if $\lim_{|k|\to\infty} |\hat{F}_k| = 0 \Rightarrow \sum_{k\in\mathbb{Z}^n} |\hat{F}_k|^2 < \infty$

$\Rightarrow \quad T$ is compact when $f \in L^2(\mathbb{T}^n)$.

(b) T^* is given by $u \mapsto \bar{F}^- * u$

where $F^-(x) = F(-x)$. $\therefore$ T is self-adjoint

$$\Longleftrightarrow \quad F(x) = \bar{F}(-x).$$

(c) Let us calculate $\|Tu\|^2_{L^2}$.

$$\|Tu\|_{L^2}^2 = \langle F*u, F*u \rangle_{L^2}$$

$$= \langle \hat{F}_k \hat{u}_k , \hat{F}_k \hat{u}_k \rangle_{\ell^2}$$

Parseval's

$$= \sum_{k\in\mathbb{Z}^n} |\hat{F}_k|^2 |\hat{u}_k|^2 = \sum_{k\in\mathbb{Z}^n} |\overline{\hat{F}_k}|^2 |\hat{u}_k|^2$$

$$= \langle (\widehat{\overline{F}^-})_k \hat{u}_k , (\widehat{\overline{F}^-})_k \hat{u}_k \rangle_{\ell^2}$$

$$= \langle \overline{F}^- * u, \overline{F}^- * u \rangle_{L^2} = \langle T^* u, T^* u \rangle_{L^2}$$

$$= \|T^* u\|_{L^2}^2$$

$\Rightarrow$ from problem 1(a), we get that T is a normal operator.

d) $\because$ T is normal $\Rightarrow$ $r(T) = \|T\| \le \|F\|_{L^1}$

$$\Rightarrow \sigma(T) \subseteq \{ z \in \mathbb{C} \mid |z| \le \|F\|_{L^1} \}$$

$\{ e^{2\pi i k x} \mid k \in \mathbb{Z} \}$ is an o.n. basis consisting of eigenvectors.

Now, suppose $\lim_{|k|\to\infty} |\hat{F}_k| = 0$.

If $F = 0 \Rightarrow T = 0 \Rightarrow \sigma(T) = \{0\}$

so every element of the spectrum is an eigenvalue.

If $F \neq 0$ then $|\hat{F}_k|$ is an eigenvalue w/ eigenvector $e^{2\pi i k x}$. $\because \lim_{|k|\to\infty} |\hat{F}_k| = 0$ and $\sigma(T)$ is closed $\Rightarrow 0 \in \sigma(T)$. But 0 can't be an eigenvalue as $F * u = 0$ a.e. $\Rightarrow u = 0$ a.e.

$\boxtimes$

4) $x_0 \in \mathbb{T}^n$ and $T : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n)$ given by

$$(Tf)(x) = f(x + x_0).$$

a)

consider $X = \mathbb{Z}^n$ with the counting measure ν. and consider the unitary isomorphism

$$U = \mathcal{F} : L^2(\mathbb{T}^n) \longrightarrow \ell^2(\mathbb{Z}^n).$$

Consider $F : \mathbb{Z}^n \longrightarrow \mathbb{C}$ by $F(k) = e^{2\pi i k x_0}$.

Then $\quad UTU^{-1}(f(k)) = \mathcal{F}T\mathcal{F}^{-1}(f(k))$

$$= F(k)f(k).$$

b) $\because$ F is a bounded measurable function

$\Rightarrow$ we can use problem 2(a) to describe the spectrum.

i) If $x_0 = (x_1, \dots, x_n)$ consists of only rational entries $\Rightarrow$ there is a common denominator, say $q \in \mathbb{N}$. $\Rightarrow$ $k \cdot x_0$ is a multiple of $\frac{1}{q}$.

$$\Rightarrow \quad \{F_k \mid k \in \mathbb{Z}^n\} \subseteq \{e^{2\pi i p/q} \mid p \in \{0, \dots, q-1\}\}$$

$\Rightarrow \sigma(T_F) = \{F_k \mid k \in \mathbb{Z}^n\}$.

$\Rightarrow$ for $\lambda \in \sigma(T_F)$, $\lambda = F_k$ for some $k \in \mathbb{Z}^n$.

$\therefore$ if $u \in l^2(\mathbb{Z}^n)$, $u = 0$ everywhere but $u = 1$ at $k \in \mathbb{Z}^n$

we get that $\quad Fu = F_k u = \lambda u$.

moreover, $\forall\, m \in \mathbb{N}$

$k' = k + (m \cdot q, 0, \dots, 0)$ satisfies

$$f_{k'} = e^{2\pi i k' \cdot x_0} = f_k \, e^{2\pi i m q x_0} = f_k = \lambda.$$

$\Rightarrow$ $\{ v \mid v \text{ is an eigenvector for } \lambda \}$ is infinite dimension.

$\Rightarrow$ $\lambda \in \sigma(T_F)$ is an eigenvector of infinite multiplicity.

ii) If x_0 has atleast one irrational coordinate

In this case $\{ F_k \mid k \in \mathbb{Z}^n \}$ contains every element of the form $e^{2\pi i m x_j}$, $m \in \mathbb{Z}$.

$\Rightarrow$ $\sigma(T_F) = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

The remaining values in the spectrum are the elements of the unit circle that are not eigenvectors and they are not in $\{ F_k \mid k \in \mathbb{Z}^n \}$ $\Rightarrow$ uncountably many.

c) In this case $X = \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{C}$ w/ $F(x) = e^{2\pi i x x_0}$ and $U = \mathcal{F}^{-1}$. Again

$\sigma(T_F) = S^1$.