Session 2

Plan :- 1) Problem set 1

2) Differential under the integral sign (DCT)

3) Banach contraction mapping principle
→ Picard – Lindelöf thm.

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Funktionalanalysis
WiSe 2020–21

Problem Set 1

Due: Thursday, 12.11.2020 (19pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Problem 1
A *Banach algebra* is a Banach space X that is equipped with the additional structure of a product $X \times X \to X : (x, y) \mapsto xy$ satisfying $\|xy\| \leqslant \|x\| \cdot \|y\|$ for all $x, y \in X$.

(a) Suppose X is a Banach space and $\mathscr{L}(X)$ denotes the Banach space of continuous linear operators $X \to X$, endowed with the operator norm. Show that $\mathscr{L}(X)$ with a product structure defined by composition $AB := A \circ B$ is a Banach algebra.

(b) $(*)$ Assume X is a Banach algebra containing an element $\mathbb{1} \in X$ that satisfies $\mathbb{1}x = x\mathbb{1} = x$ for all $x \in X$. Show that for any $x \in X$ with $\|x\| < 1$, the series $\sum_{n=0}^{\infty}(-1)^n x^n$ converges absolutely to an element $y \in X$ satisfying $y(\mathbb{1} + x) = (\mathbb{1} + x)y = \mathbb{1}$. [3pts]

(c) Assume X and Y are Banach spaces and $A_0 \in \mathscr{L}(X, Y)$ is a continuous linear map that admits a continuous inverse $A_0^{-1} \in \mathscr{L}(Y, X)$. Find a constant $c > 0$ such that for every $A \in \mathscr{L}(X, Y)$ with $\|A - A_0\| < c$, A also has an inverse $A^{-1} \in \mathscr{L}(Y, X)$.

Problem 2
For any integer $m \geqslant 0$, let $C^m([0, 1])$ denote the Banach space of m times continuously differentiable functions $x : [0, 1] \to \mathbb{R}$, with the C^m-norm $\|x\|_{C^m} := \sum_{k=0}^{m} \sup_{t\in[0,1]} |x^{(k)}(t)|$. For the subset $X := \{x \in C^2([0, 1]) \mid x(0) = x(1) = 0\}$, prove:

(a) X is a vector space, and endowing it with the C^2-norm makes it a Banach space.
Hint: Closed linear subspaces of Banach spaces are also Banach spaces. (Why?)

(b) For any function $P \in C^0([0, 1])$, the transformation $x \mapsto \ddot{x} + Px$ defines a continuous linear operator $A_P : X \to C^0([0, 1])$, which satisfies $\|A_P - A_0\| \leqslant \|P\|_{C^0}$.[1]

(c) $(*)$ The operator $A_0 \in \mathscr{L}(X, C^0([0, 1]))$ in part (b) has a continuous inverse $A_0^{-1} \in \mathscr{L}(C^0([0, 1]), X)$. [4pts]
Hint: Every $x \in X$ must have $\dot{x}(t_0) = 0$ for some $t_0 \in (0, 1)$. (Why?)

Comment: Problems 1 and 2 together prove the statement from lecture that for all functions $P, f \in C^0([0, 1])$ with $\|P\|_{C^0}$ sufficiently small, there is a unique C^2-function $x : [0, 1] \to \mathbb{R}$ solving the boundary value problem $\ddot{x} + Px = f$ with $x(0) = x(1) = 0$.

Problem 3
Determine which (if any) of the following are closed linear subspaces of the Banach space of bounded continuous functions $f : (0, 1) \to \mathbb{R}$ with the C^0-norm:

(a) The bounded continuously differentiable functions on $(0, 1)$

[1] Here $\dot{x}$ and $\ddot{x}$ denote the first and second derivatives of x respectively.

(b) $(*)$ The uniformly continuous functions on $(0,1)$ [3pts]

Problem 4
For an arbitrary topological vector space X and a seminorm $\|\cdot\|$ on X, consider the following conditions:

(i) $\|\cdot\| : X \to [0,\infty)$ is a continuous function;

(ii) The set $B_1(0) := \{x \in X \mid \|x\| < 1\} \subset X$ is open;

(iii) For every $x_0 \in X$ and $\epsilon > 0$, the set $B_\epsilon(x_0) := \{x \in X \mid \|x - x_0\| < \epsilon\} \subset X$ is open.

(a) Prove that conditions (i), (ii) and (iii) are all equivalent.
Hint: Topological vector spaces have the feature that the affine map $x \mapsto x_0 + \epsilon x$ defines a homeomorphism $X \to X$ for any $x_0 \in X$ and $\epsilon > 0$ (why?). In particular, it maps open sets to open sets.

(b) If additionally X is a locally convex space whose topology is determined by the family of seminorms $\{\|\cdot\|_\alpha\}_{\alpha \in I}$, prove that conditions (i)–(iii) are equivalent to the following: (iv) There exists a nonempty finite subset $I_0 \subset I$ and a constant $C > 0$ such that $\|x\| \leqslant C \sum_{\alpha \in I_0} \|x\|_\alpha$ for all $x \in X$.

(c) Prove that two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on a vector space V are equivalent if and only if they define the same topology.

Problem 5
Assume X is a locally convex space. Prove:

(a) A set $\mathcal{U} \subset X$ is open if and only if for every $x_0 \in \mathcal{U}$, there exists a continuous seminorm $\|\cdot\| : X \to [0,\infty)$ such that $B_1(x_0) := \{x \in X \mid \|x - x_0\| < 1\} \subset \mathcal{U}$.
Hint: Every finite positive linear combination of continuous seminorms is a continuous seminorm.

(b) X is also a topological vector space.

Problem 6 $(*)$
Prove: For two locally convex spaces X and Y, a linear map $A : X \to Y$ is continuous if and only if for every continuous seminorm $\|\cdot\|_Y$ on Y, there exists a continuous seminorm $\|\cdot\|_X$ on X such that $\|Ax\|_Y \leqslant \|x\|_X$ holds for all $x \in X$. [5pts]

Problem 7
Here is an example of a topological vector space whose topology cannot be defined via a metric. Let $C^0_c(\mathbb{R}^n)$ denote the space of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that vanish outside of compact subsets.[2] We endow $C^0_c(\mathbb{R}^n)$ with a locally convex topology defined via the family of seminorms $\{\|f\|_\varphi\}_{\varphi \in I}$ where I denotes the set of all continuous functions $\varphi : \mathbb{R}^n \to [0,\infty)$ and $\|f\|_\varphi := \|\varphi f\|_{C^0}$.

(a) $(*)$ Show that a sequence f_j converges to f_∞ in $C^0_c(\mathbb{R}^n)$ if and only if there exists a compact set $K \subset \mathbb{R}^n$ such that $f_j|_{\mathbb{R}^n \setminus K} \equiv 0$ for every $j \in \mathbb{N} \cup \{\infty\}$ and $f_j \to f$ uniformly on K. [4pts]

(b) To show that $C^0_c(\mathbb{R}^n)$ is not metrizable, one can argue by contradiction and suppose there exists a metric d such that every neighborhood $\mathcal{U} \subset C^0_c(\mathbb{R}^n)$ of 0 contains an open set of the form $B_n := \{f \in C^0_c(\mathbb{R}^n) \mid d(0,f) < 1/n\}$ for $n \in \mathbb{N}$ sufficiently large. Show that in this situation, there must exist functions $\varphi_n \in I$ such that $A_n := \{f \in C^0_c(\mathbb{R}^n) \mid \|f\|_{\varphi_n} < 1\} \subset B_n$ for every n, then derive a contradiction by constructing a neighborhood $\mathcal{U}$ of 0 that does not contain A_n for any $n \in \mathbb{N}$.

[2] We say in this case that the functions $f \in C^0_c(\mathbb{R}^n)$ have *compact support* in $\mathbb{R}^n$.

Problem 1

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(a) Suppose X is a Banach space and $\mathscr{L}(X)$ denotes the Banach space of continuous linear operators $X \to X$, endowed with the operator norm. Show that $\mathscr{L}(X)$ with a product structure defined by composition $AB := A \circ B$ is a Banach algebra.

(b) (∗) Assume X is a Banach algebra containing an element $\mathbb{1} \in X$ that satisfies $\mathbb{1}x = x\mathbb{1} = x$ for all $x \in X$. Show that for any $x \in X$ with $\|x\| < 1$, the series $\sum_{n=0}^{\infty}(-1)^n x^n$ converges absolutely to an element $y \in X$ satisfying $y(\mathbb{1} + x) = (\mathbb{1} + x)y = \mathbb{1}$. [3pts]

(c) Assume X and Y are Banach spaces and $A_0 \in \mathscr{L}(X, Y)$ is a continuous linear map that admits a continuous inverse $A_0^{-1} \in \mathscr{L}(Y, X)$. Find a constant $c > 0$ such that for every $A \in \mathscr{L}(X, Y)$ with $\|A - A_0\| < c$, A also has an inverse $A^{-1} \in \mathscr{L}(Y, X)$.

Problem 2

1. a) $\mathcal{L}(X)$ is a Banach algebra.

Want: If $A, B \in \mathcal{L}(X)$

$$\|A \circ B\| \leq \|A\| \cdot \|B\|$$

If $T \in \mathcal{L}(X) \implies \|Tx\| \leq \|T\| \cdot \|x\|$ —— ①

$$\|T\| = \inf \underbrace{\left\{ c \geq 0 \;\middle|\; \|Tx\| \leq c\|x\|,\ x \in X \right\}}_{\text{non-negative } \mathbb{R}\text{, bounded below}}$$

$\downarrow$ inf is alway attained. $\implies \exists\, c$

$\therefore$ $\|Tx\| \leq \|T\|\,\|x\|$ proves ①.

$x \in X$

$$\|A \circ B(x)\| = \|A(B(x))\| \leq \|A\|\,\|B(x)\| \leq \|A\|\,\|B\|\,\|x\|$$

$\implies$ $\|A \circ B\| \leq \|A\| \cdot \|B\|$ proving $\mathcal{L}(X)$ is Banach algebra.

b) $y = (1+x)^{-1} = \sum (-1)^n x^n$

$\|x\| < 1$ $\quad \sum (-1)^n x^n$ is absolutely convergent

$\Rightarrow \sum \|x^n\|$ converges.

$\because$ X is a Banach algebra, $\|x^n\| = \|\underbrace{x \cdot x \cdots x}_{n\text{-times}}\|$

$\leq \|x\|^n$

$\Rightarrow \sum \|x^n\| \leq \sum \underbrace{\|x\|}_{<1}^n$ — converges

$\Rightarrow \sum (-1)^n x^n$ is convergent in X to some $y \in X$.

$y(1+x) = 1 = (1+x) \cdot y$ — Want.

$$(1+x) \sum_{n=0}^{\infty} (-1)^n x^n = \lim_{n \to \infty} \left(\sum_{k=0}^{n} (-1)^n x^n + \sum_{k=0}^{n} (-1)^n x^{n+1} \right)$$

$(x^0 = 1)$

$$= \lim_{n \to \infty} \left(1 + (-1)^n x^{n+1} \right)$$

$\because \sum (-1)^n x^n$ convergent $\Rightarrow \lim_{n \to \infty} x^n \longrightarrow 0$

$= 1 \in X$

in a similar way : $\sum (-1)^n x^n (1+x) = 1.$

$\|x \cdot y\| \leq \|x\| \cdot \|y\|$

$\|x\| \leq \|x\| \cdot \|1\| \quad \Rightarrow \quad \|x\| (\cancel{1+\|1\|}) \geq 0$ $\quad 1 - \|1\|$

$y = 1$

$\Rightarrow \quad \|x\| \leq \|x\| \cdot \|1\|$

$\Rightarrow \quad \|x\| (1 - \|1\|) \leq 0 \quad \Rightarrow \|x\| (\|1\| - 1) \geq 0$

$\cancel{\|1\| \geq 1}$ $\qquad \|1\| \leq 1$

$\|x^n\| \leq \|x\|^n \qquad \|1\| \leq 1$

c)) Assume X and Y are Banach spaces and $A_0 \in \mathscr{L}(X, Y)$ is a continuous linear map that admits a continuous inverse $A_0^{-1} \in \mathscr{L}(Y, X)$. Find a constant $c > 0$ such that for every $A \in \mathscr{L}(X, Y)$ with $\|A - A_0\| < c$, A also has an inverse $A^{-1} \in \mathscr{L}(Y, X)$.

$$C = \frac{1}{\|A_0^{-1}\|} \quad \text{- Claim.}$$

If $\|A - A_0\| < \dfrac{1}{\|A_0^{-1}\|} \Rightarrow A^{-1}$ makes sense.

$$A - A_0 = (A A_0^{-1} - I_y) A_0$$

$$\|I - A_0^{-1} A\| = \|A_0^{-1}(A_0 - A)\| \leq \|A_0^{-1}\| \, \|A_0 - A\| < 1$$

we have an element in $\mathcal{L}(X)$, $I - A_0^{-1}A : X \to X$

$\| I - A_0^{-1} A \| < 1$

part b) $\Rightarrow$ $I - (I - A_0^{-1}A)$ is invertible

$\Rightarrow$ $A_0^{-1}A$ is invertible

$\Rightarrow$ $A^{-1} = (A_0^{-1}A)^{-1} A_0^{-1}$

$\boxtimes$

(2) (a) X is a closed linear subspace.

$(x_n) \in X$ s.t. $x_n \to x$, $x \in X$ (want)

$x_n \to x$ uniformly. $\Rightarrow$ $x_n \to x$ pointwise

$\therefore$ $\lim\limits_{n\to\infty} x_n(0) = x(0) = \lim\limits_{n\to\infty} x_n(1) = x(1) = 0.$

$\lim\limits_{n\to\infty} x_n' = x'$ (guess)

Suppose $\lim\limits_{n\to\infty} x_n' = g$ want :- $g = x'$

By FTC $x_n(b) - x_n(a) = \int_a^b x_n'(t)\,dt$

$\lim\limits_{n\to\infty} x_n(b) - x_n(a) = x(b) - x(a) = \lim\limits_{n\to\infty} \int_a^b x_n'(t)\,dt$

2) b) $A_p : X \longrightarrow C^0([0,1])$

$$A_p(x) = \ddot{x} + Px$$

Want:- A_p is continuous.

$\exists\ c > 0$ s.t. $\|A_p x\|_{C^0} \leq C \cdot \|x\|_{C^2}$

$$\|\ddot{x} + Px\|_{C^0} \leq \|\ddot{x}\|_{C^0} + \|Px\|_{C^0}$$

$$= \sup|\ddot{x}| + \sup|P| \cdot |x|$$

if $\sup|P| < 1 \longrightarrow A_p$ is continuous.

$$\Rightarrow \|\ddot{x} + Px\|_{C^0} \leq \sup|\ddot{x}| + \sup|x| + \sup|\dot{x}|$$

$$= 1 \cdot \|x\|_{C^2}$$

If $\sup|P| \geq 1$

$$\Rightarrow \|\ddot{x} + Px\|_{C^0} \leq \underbrace{\sup|P|}_{C} \|\ddot{x}\|_{C^2}$$

$\Rightarrow A_p$ is continuous.

$$\|A_p - A_0\| = \sup_{\|x\|=1} \|(A_p - A_0)x\|$$

$$= \sup_{\|x\|=1} \|Px\| \leq \sup_{\|x\|=1} \left(\sup|P|\,|x|\right)$$

$\leq \sup|p| = \|p\|_{C^0}$

$\boxtimes$

c) $\|A \circ \bar{f}\|_{C^2} \leq c \|f\|_{C^0}$, $f \in C^0([0,1])$

$\Uparrow$
X

$\|\ddot{x}\| + \|\dot{x}\| + \|x\|$

MVT

3) a) is not closed $f_n(x) = \sqrt{\frac{1}{n^2} + \left(x - \frac{1}{2}\right)^2}$

b) closed $\to$ uniform converge ✓

$f_n \to |x - \frac{1}{2}|$

(b) (f_n) uniformly continuous

$f_n \longrightarrow f$, want f is uniformly ctns.

$|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)|$

$\leq \underbrace{|f(x) - f_n(x)|}_{< \frac{\epsilon}{3}} + \underbrace{|f_n(x) - f_n(y)|}_{< \epsilon/3} + \underbrace{|f_n(y) - f(y)|}_{< \epsilon/3}$

$< \epsilon$

(4) a) i) $\Rightarrow$ ii), ii) $\Rightarrow$ iii) and iii) $\Rightarrow$ (i)

i) $\Rightarrow$ ii) $\quad \|\cdot\|$ is a continuous semi-norm.

$$B_1(0) = \underbrace{\{x \in X \mid \|x\| < 1\}}_{\|\ \|^{-1}([0,1))}$$

$[0,1)$ $\hookrightarrow$ open in $[0,\infty)$

$\Rightarrow$ $B_1(0)$ is open.

ii) $\Rightarrow$ iii) $\quad$ If $B_1(0)$ is open $\Rightarrow$ $B_{\underline{\epsilon}}(x_0)$

$$\Phi : X \longrightarrow X$$
$$x \longmapsto x_0 + \epsilon x$$

N.B. $\Phi(B_1(0)) = B_\epsilon(x_0)$

Φ is a homeomorphism.

ϕ is a bijection.

$\because$ X is TVS $\Rightarrow$ vector addition and scalar mult. are continuous function.

$$\Phi^{-1} : X \longrightarrow X$$
$$x \longmapsto \frac{1}{\epsilon}(x - x_0)$$

$\Rightarrow$ Φ^{-1} is continuous.

$\Rightarrow$ $B_\epsilon(x_0)$ is open.

iii) $\Rightarrow$ i) If $\forall x_0 \in X$, $\epsilon > 0$, $B_\epsilon(x_0)$ is open

Want $\|\cdot\| : X \to [0,\infty)$ is a cts.

Suppose $U \subset [0,\infty)$ is an open set.

and $x_0 \in X$ s.t. $\|x_0\| \in U$.

Want:- $\| \ \|^{-1}(U)$ is also open in X

Choose some $\epsilon > 0$ s.t. $(\|x_0\| - \epsilon, \|x_0\| + \epsilon) \subset U$

(U is open in $[0,\infty)$, $\|x_0\| \in U$)

claim

$\{x \in X \mid \|x - x_0\| < \epsilon\}$ which is an open set due to iii) $\subset \|\cdot\|^{-1}(U)$.

Let $x \Rightarrow \|x\| = \|x - x_0 + x_0\| \leq \|x - x_0\| + \|x_0\|$

$< \epsilon + \|x_0\|$

$\|x\| \geq \|x_0\| - \|x - x_0\| > \|x_0\| - \epsilon$

$\Rightarrow \|x\| \in (\|x_0\| - \epsilon, \|x_0\| + \epsilon) \subset U$

$\Rightarrow \{x \in X \mid \|x - x_0\| < \epsilon\} \subset \|\cdot\|^{-1}(U)$.

$\Rightarrow$ $\|\cdot\|$ is continuous $\Rightarrow$ a) ✓ $\square$.

b) $(i) \Longleftrightarrow (iv)$

$iv) \Longrightarrow i)$

$\downarrow$

I_0 s.t. $\|x\| \leq C \sum_{\alpha \in I_0} \|x\|_\alpha \quad \forall x.$

Claim: If $\|\cdot\|_1$ is continuous semi-norm and $\|\ \| \leq \|\ \|_1 \Longrightarrow \|\ \|$ is continuous.

Want:- $B = \{ x \in X \mid \|x\| < 1 \}$ is open. (part a))

$x_0 \in B$.

Choose $\epsilon > 0$ s.t. $\|x_0\| + \epsilon < 1$.

$\because \|\cdot\|_1$ is continuous $\Longrightarrow U = \{ x \in X \mid \|x - x_0\|_1 < \epsilon \}$ is open.

(part a)).

But $\|\cdot\| \leq \|\cdot\|_1 \Longrightarrow \forall x \in U$

$$\|x\| \leq \|x_0\| + \|x - x_0\| \leq \|x_0\| + \|x - x_0\|_1 < \|x_0\| + \epsilon < 1$$

$\Longrightarrow \underset{\substack{\text{open}\\ \text{set in } X}}{U} \subset B \Longrightarrow B$ is open $\Longrightarrow \|\cdot\|$ conts.

i) $\Rightarrow$ iv)

$\|\cdot\|$ is continuous $\Rightarrow$ $B = \{x \in X \mid \|x\| < 1\}$ is open.

$\exists$ $I_0 \subset I$, I_0 finite

$\mathcal{U} = \{x \in X \mid \|x\|_\alpha < \epsilon_\alpha \ \forall \alpha \in I_0\} \subset B.$

if $\|x\|_\alpha < \epsilon_\alpha \ \forall \alpha \in I_0 \Rightarrow \|x\| < 1$ — (1)

$\|\lambda x\|_\alpha = 0 < \epsilon_\alpha$

Want: $\|x\| \leq C \sum_{\alpha \in I_0} \|x\|_\alpha$

$$Q(x) = \frac{\|x\|}{\sum_{\alpha \in I_0} \|x\|_\alpha}$$ if $Q(x) \leq C$ $\forall x$ $\Rightarrow$ we're done.

$Q(\lambda x)$

makes sense.

$\|x\| > 0 \Rightarrow \|x\|_\alpha > 0$ for some $\alpha \in I_0$.

If $Q(x)$ is not bounded $\Rightarrow \exists\ x_j \in X$

s.t. $\|x_j\| > 0$ and $Q(x_j) \to \infty$.

. multiply x_j w/ positive scalars s.t.

$$\|\tilde{x}_j\|_\alpha < \epsilon_\alpha \implies \|x\| < 1$$

$$\implies Q(\tilde{x}_j) \not\to \infty.$$

contradiction

$\implies$ $Q(x)$ is bounded $\forall\, x$.

$$\implies \|x\| \le C \sum_{\alpha \in I_0} \|x\|_\alpha$$

$\boxtimes$

$Q(x)$ will also be used in Pr.(6).