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Funktionalanalysis
WiSe 2020–21

Problem Set 5

Due: Thursday, 10.12.2020 (18pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space $(V, \langle\ ,\ \rangle)$ over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by m.*

Problem 1

Show that the space of bounded continuous functions on $\mathbb{R}$ is not dense in $L^\infty(\mathbb{R})$.

Problem 2

Fix $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that if $p > 1$ and $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \langle f, \varphi \rangle \, dm = 0$ for all smooth compactly supported functions $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $f = 0$ almost everywhere.[1]

(b) $(*)$ Assume $1 < p < \infty$, and suppose $T, T^* : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ are two linear operators satisfying the "adjoint" relation

$$\int_{\mathbb{R}^n} \langle Tf, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, T^* g \rangle \, dm \quad \text{for all} \quad f, g \in C_0^\infty(\mathbb{R}^n).$$

Show that T extends to a bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ if and only if T^* extends to a bounded linear operator $T^* : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. [6pts]

Hint: Use the isometric identification of L^p with the dual space of L^q. (In part (a), this makes sense only after restricting to a compact subset.) You will also need to use the density of C_0^∞ in L^p.

Problem 3 $(*)$

Show that for any $f, g \in L^1(\mathbb{R}^n)$ and a compactly supported smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \langle \varphi * f, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, \varphi^- * g \rangle \, dm,$$

where $\varphi^-(x) := \varphi(-x)$. [4pts]
Hint: Here is a useful fact about integrals of vector-valued functions. If $L : V \to W$ is a linear map between finite-dimensional vector spaces and $f : \mathbb{R}^n \to V$ is Lebesgue integrable, then $Lf : \mathbb{R}^n \to W$ is also Lebesgue integrable and $\int_{\mathbb{R}^n} Lf \, dm = L\left(\int_{\mathbb{R}^n} f \, dm\right)$.

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[1] We will see when we study distributions that the result of Problem 2(a) is also true for $p = 1$, but that case is trickier to prove.

Problem 4 $(*)$
For an integer $m \geqslant 0$, let $C_b^m(\mathbb{R}^n)$ denote the Banach space of C^m-functions $\mathbb{R}^n \to V$ whose derivatives up to order m are all bounded, with the usual C^m-norm. Let $C^m(\bar{\mathbb{R}}^n)$ denote the subspace consisting of functions $f \in C_b^m(\mathbb{R}^n)$ whose derivatives of order m are also uniformly continuous.[2] One can show along the lines of Problem Set 1 #3(b) that $C^m(\bar{\mathbb{R}}^m)$ is a closed subspace of $C_b^m(\mathbb{R}^n)$, so it is also a Banach space. Prove that if $f \in C^m(\bar{\mathbb{R}}^n)$ and $\{\rho_j : \mathbb{R}^n \to [0, \infty)\}_{j \in \mathbb{N}}$ is an approximate identity with shrinking support, then

$$\lim_{j \to \infty} \|\rho_j * f - f\|_{C^m} = 0,$$

and conclude that $C^\infty(\mathbb{R}^n) \cap C^m(\bar{\mathbb{R}}^n)$ is dense in $C^m(\bar{\mathbb{R}}^n)$. [8pts]
Hint: A similar (though non-identical) result is proved at the end of §5 *in the lecture notes. We did not cover it in lecture.*

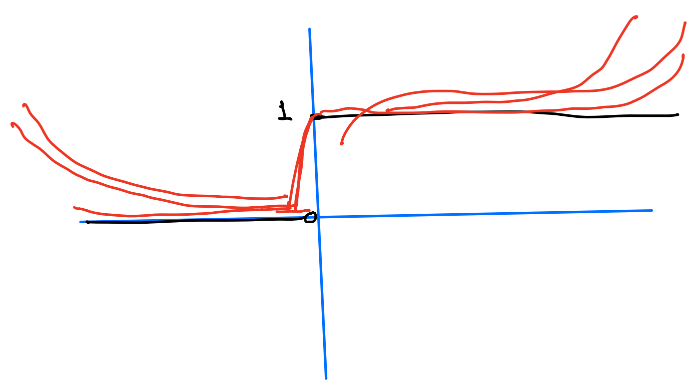
[2]Note that for $f \in C^m(\bar{\mathbb{R}}^n)$, the derivatives of any order $k < m$ are also uniformly continuous, but this is not an extra condition; it follows (via the fundamental theorem of calculus) from the assumption that the derivatives of order $k + 1$ are bounded.

Problem Session 6

Problem 1
Show that the space of bounded continuous functions on $\mathbb{R}$ is not dense in $L^\infty(\mathbb{R})$.

1. Show an example $f \in L^\infty(\mathbb{R})$ which cannot be approximated by any sequence of bounded continuous functions.

$$f(x) = \begin{cases} 0, & x < 0 \\ 1 & x \geq 0 \end{cases} \in L^\infty(\mathbb{R}^n)$$



$\nexists$ any sequence $(f_n) \in C_b(\mathbb{R})$ s.t. $\lim_{n\to\infty} |f_n - f| \to 0$.

Suppose not, then $\forall \epsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $\forall$ $n \geq N$

$|f_n - f| < \epsilon$ Take $\epsilon = \frac{1}{2}$

$\Rightarrow$ $|f_n(x) - f(x)| < 1/2$ $\Rightarrow$ $f_n(x) < \frac{1}{2} + f(x)$

$f_n(x) < \frac{1}{2}$ $\forall\ x<0$

$f_n(x) < \frac{1}{2}+1 = \frac{3}{2}$ $\forall\ x \geq 0$

This cannot happen as f_n are continuous function.

$C_b(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$.

There are many more examples.

Fix $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that if $p > 1$ and $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \langle f, \varphi \rangle\, dm = 0$ for all smooth compactly supported functions $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $f = 0$ almost everywhere.[1]

(b) $(*)$ Assume $1 < p < \infty$, and suppose $T, T^* : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ are two linear operators satisfying the "adjoint" relation

$$\int_{\mathbb{R}^n} \langle Tf, g \rangle\, dm = \int_{\mathbb{R}^n} \langle f, T^* g \rangle\, dm \quad \text{for all} \quad f, g \in C_0^\infty(\mathbb{R}^n).$$

Show that T extends to a bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ if and only if T^* extends to a bounded linear operator $T^* : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. [6pts]

(a) $$\int_{\mathbb{R}^n} \langle f, \varphi \rangle\, dm = 0 \quad \forall\ \varphi \in C_0^\infty(\mathbb{R}^n).$$

Let $\overline{B_R(0)}$ is some closed ball of radius R in $\mathbb{R}^n$.

Claim :- $$\int_{\overline{B_R(0)}} \langle f, g \rangle\, dm = 0 \quad \forall\ g \in L^q_{loc}(\mathbb{R}^n).$$

where $\frac{1}{p} + \frac{1}{q} = 1.$

C_0^∞ is dense in L^q $\Rightarrow$ $\exists$ some $\varphi \in C_0^\infty$ s.t.

$|\varphi - g| < \epsilon$ $\forall\ \epsilon > 0.$

$$\int fg \, dm = \int fg - f\varphi + f\varphi \, dm$$

$$= \underbrace{\int f(g-\varphi)\, dm}_{\to 0} + \underbrace{\int f\varphi \, dm}_{=0 \text{ given}}$$

g was arbitrary $\Rightarrow$ $\int fg \, dm = 0$ —— (1)

L^p and $(L^q)^*$ are isomorphic to each other.

The function on L^q which is given

$\Lambda_f = \int \langle f, \cdot \rangle \, dm$ is 0 by (1).

Every functional on L^q must be of the form Λ_f for some $f \in L^p$ - Riesz Rep.

$\because$ $(L^q)^* \longleftrightarrow L^p$ is an isomorphism

$\Rightarrow$ $\Lambda_f \equiv 0 \iff f = 0$ a.e. on $\overline{B_R(0)}$.

$\Rightarrow$ $\exists$ a measure zero set in $\overline{B_R(0)}$ s.t. $f \neq 0$.

We know that $\mathbb{R}^n$ can be covered by countable union of closed balls.

At most, we have countable measure zero sets where $f \neq 0$. But countable union of measure zero sets is still a measure zero set

$\Rightarrow \quad f = 0$ a.e. on $\mathbb{R}^n$.

$\boxtimes$

b) Suppose $T^*: L^q(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R})$ is a bounded linear operator.

Want: $T: L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is a bounded linear operator.

<u>Lemma</u> X, Y let $V \subset X$ be a dense subspace. If $\Lambda \in \mathcal{L}(V, Y)$ then Λ can be extended to $\tilde{\Lambda} \in \mathcal{L}(X, Y)$ and the operator norm of $\|\tilde{\Lambda}\| = \|\Lambda\|$. [Lec. 3]

Let $f \in C_0^\infty(\mathbb{R}^n)$ and define, $(g \in C_0^\infty(\mathbb{R}^n))$

$$\Lambda(g) = \int_{\mathbb{R}^n} \langle Tf, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, T^* g \rangle \, dm$$

$$: C_0^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}.$$

$$|\Lambda(g)| = \left| \int_{\mathbb{R}^n} \langle f, T^* g \rangle \, dm \right|$$

$$\leq \int_{\mathbb{R}^n} |\langle f, T^* g \rangle| \, dm$$

T^* is a bounded linear op $\Rightarrow$ $\|T^* g\|_{L^q} \leq C \|g\|_{L^q}$

$$\underset{\text{Hölders}}{\leq} \|f\|_{L^p} \|T^* g\|_{L^q}$$

$$\leq C \|f\|_{L^p} \|g\|_{L^q}.$$

T^* can be extended.

Λ has bounded operator norm in L^q

$\Rightarrow$ Λ can be extended to a bounded linear operator from L^q.

$\Rightarrow$ $\Lambda \in (L^q)^*$. $\|\Lambda\| \leq C \|f\|_{L^p}$.

$\Rightarrow$ $\Lambda = \Lambda_h$ for some $h \in L^p$ $\left((L^q)^* \longleftrightarrow L^p \right)$

But from the condition in the problem

$\Lambda_h = Tf$ and $\|Tf\| \leq C \|f\|_{L^p}$

because $(L^q)^* \longleftrightarrow L^p$ is isometric.

$\therefore$ Tf is bounded in the L^p norm $\Rightarrow$ by the lemma, T can be extended to a bounded linear operator : $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$,

Problem 3 (∗)
Show that for any $f, g \in L^1(\mathbb{R}^n)$ and a compactly supported smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \langle \varphi * f, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, \varphi^- * g \rangle \, dm,$$

where $\varphi^-(x) := \varphi(-x)$. [4pts]

$$\int_{\mathbb{R}^n} \langle \varphi * f, g \rangle \, dm = \int_{\mathbb{R}^n} (\varphi * f)(x) \, g(x) \, dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(x-y) f(y) \, dy \right) g(x) \, dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(x-y) g(x) f(y) \, dy \right) dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(x-y) g(x) \, dx \right) f(y) \, dy \quad \text{(Fubini's Thm)}$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(-(y-x)) g(x) \, dx \right) f(y) \, dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi^-(y-x) g(x) \, dx \right) f(y) \, dy$$

$$= \int_{\mathbb{R}^n} (\varphi^- * g)(y) f(y)\, dy$$

$$= \int_{\mathbb{R}^n} \langle \varphi^- * g, f \rangle\, dy.$$

$\boxed{/\!/}$

For an arbitrary vector space V, we just use $\langle \int, \cdot \rangle = \int \langle, \cdot \rangle$. and just follow the same proof.

$\boxed{/\!/}$

Problem 4 $(*)$
For an integer $m \geqslant 0$, let $C_b^m(\mathbb{R}^n)$ denote the Banach space of C^m-functions $\mathbb{R}^n \to V$ whose derivatives up to order m are all bounded, with the usual C^m-norm. Let $C^m(\bar{\mathbb{R}}^n)$ denote the subspace consisting of functions $f \in C_b^m(\mathbb{R}^n)$ whose derivatives of order m are also uniformly continuous.[2] One can show along the lines of Problem Set 1 #3(b) that $C^m(\bar{\mathbb{R}}^m)$ is a closed subspace of $C_b^m(\mathbb{R}^n)$, so it is also a Banach space. Prove that if $f \in C^m(\bar{\mathbb{R}}^n)$ and $\{\rho_j : \mathbb{R}^n \to [0, \infty)\}_{j \in \mathbb{N}}$ is an approximate identity with shrinking support, then

$$\lim_{j \to \infty} \|\rho_j * f - f\|_{C^m} = 0,$$

and conclude that $C^\infty(\mathbb{R}^n) \cap C^m(\bar{\mathbb{R}}^n)$ is dense in $C^m(\bar{\mathbb{R}}^n)$. [8pts]
Hint: A similar (though non-identical) result is proved at the end of §5 in the lecture notes. We did not cover it in lecture.

$\rightarrow$ $\epsilon/3$ trick.

$\rho_j * f$ is smooth. Already seen in the lectures.

$\partial^j(\rho_j * f) = (\partial^j \rho_j) * f$

ρ_j is an approximate identity w/ shrinking support, $r_j \to 0$ and $\text{supp}(\rho_j) \subset B_{r_j}(0)$.

$$f_j = \rho_j * f$$

$$|f_j(x) - f(x)| = \left| \int_{\mathbb{R}^n} f(x-y)\rho(y)\,dy - f(x) \right|$$

$$= \left| \int_{\mathbb{R}^n} \{ f(x-y)\rho_j(y) - f(x)\rho_j(y) + f(x)\rho_j(y) \} dy - f(x) \right|$$

$$= \left| \int_{\mathbb{R}^n} (f(x-y) - f(x))\rho(y)\,dy + f(x)\left(\int_{\mathbb{R}^n} \rho_j\,dm - 1\right) \right|$$

$$\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|\rho_j(y)\,dy + |f(x)|\epsilon_j$$

where ϵ_j is appearing b/c ρ_j is an approximate identity $\Rightarrow \int_{\mathbb{R}^n} \rho_j\,dm \to 1$ as $j \to \infty$

$$\Rightarrow \left| \int_{\mathbb{R}^n} \rho_j\,dm - 1 \right| < \epsilon_j \quad \forall \epsilon_j > 0.$$

$\because$ f is bounded and ρ_j have support contained in B_{r_j}

$\partial^{|\alpha|} f(x-y) - \partial^{|\alpha|} f(x)$

$$\Rightarrow \left|f_j(x) - f(x)\right| \le \int_{B_{\lambda_j}} \underbrace{|f(x-y) - f(x)|}_{< \epsilon'} \rho_j \, dm + \underbrace{M}_{\text{bound on } f} \epsilon_j$$

as f is uniformly continuous

$|f(x) - f(y)| \le 2M$ where M is the bound on f.

$$\le \epsilon' \int_{B_{\lambda_j}} \rho_j \, dm + M \epsilon_j$$

as $j \to \infty \Rightarrow \int \rho_j \, dm \to 1$

$$\le \epsilon' + M \epsilon_j$$

$$< \epsilon \qquad \text{for } j \text{ large enough.}$$

$\therefore \quad |f_j(x) - f(x)| < \epsilon$

$\Rightarrow \quad f_j$ converge uniformly to f.

Similarly use the fact that f have all of their derivatives of order $\le m$ bounded and are uniformly continuous.

$$\Rightarrow \quad f_j = \rho_j * f \xrightarrow[j \to \infty]{} f \quad \text{in } C^m(\overline{\mathbb{R}^n}).$$

$\boxtimes$