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Funktionalanalysis
WiSe 2020–21

Problem Set 7

Due: Thursday, 7.01.2021 (22pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space* $(V, \langle\ ,\ \rangle)$ *over* $\mathbb{C}$.

Problem 1
Assume $f : \mathbb{T}^n \to \mathbb{C}$ is of class L^2.

(a) What condition on the Fourier coefficients $\{\widehat{f}_k \in \mathbb{C}\}_{k\in\mathbb{Z}^n}$ is equivalent to the condition that f is a *real*-valued function?

(b) Show that if f is real-valued, then it can be presented as an L^2-convergent series[1]

$$f(x) = \sum_{k\in\mathbb{Z}^n} \left[a_k \cos(2\pi k \cdot x) + b_k \sin(2\pi k \cdot x)\right]$$

with uniquely determined real coefficients $a_k, b_k \in \mathbb{R}$ that satisfy $a_{-k} = a_k$ and $b_{-k} = -b_k$ for all $k \in \mathbb{Z}^n$ and are square summable, i.e. the functions $k \mapsto a_k$ and $k \mapsto b_k$ belong to $\ell^2(\mathbb{Z}^n)$. Write down explicit formulas for a_k and b_k as integrals.

(c) Under what conditions on a real-valued function f does the trigonometric series in part (b) contain only cosine terms or only sine terms?

(d) Show that for $n = 1$, the real-valued functions $\varphi_0(x) := 1$, $\varphi_k(x) := \sqrt{2}\cos(2\pi k x)$ and $\psi_k(x) := \sqrt{2}\sin(2\pi k x)$ for $k \in \mathbb{N}$ form an orthonormal basis of the space of real-valued L^2-functions on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.

Problem 2
This is just Problem 1, but for Fourier transforms instead of Fourier series. Assume for simplicity that $f : \mathbb{R}^n \to \mathbb{C}$ is in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, since this will force all integrals to converge.

(a) What condition on the Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ is equivalent to the condition that f is real-valued?

(b) Show that if f is real-valued, then

$$f(x) = \int_{\mathbb{R}^n} \left[u(p)\cos(2\pi p \cdot x) + v(p)\sin(2\pi p \cdot x)\right] dp$$

for uniquely determined real-valued functions $u, v \in \mathscr{S}(\mathbb{R}^n)$ such that u is even and v is odd. Write down formulas for u and v as integrals. Under what conditions on f does one obtain $u \equiv 0$ or $v \equiv 0$?

[1] In the context of real-valued functions, this trigonometric series is also called the "Fourier series" of f.

Problem 3
Each of the following real-valued functions on the interval $[-1/2, 1/2)$ has a unique extension to a (not necessarily continuous) function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Compute explicitly the Fourier expansions $\sum_{k\in\mathbb{Z}} e^{2\pi ikx}\widehat{f}_k$ of each function f, and rewrite them in the form $\sum_{k=0}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx)$ with real coefficients $a_k, b_k \in \mathbb{R}$. In each case, either prove that the series converges to $f(x)$ for every $x \in \mathbb{R}$ or find a specific point $x \in \mathbb{R}$ where it does not converge to $f(x)$.[2]

(a) $(*)$ The sawtooth wave: $f(x) = x$ [4pts]

(b) The square wave: $f(x) = \begin{cases} 1 & \text{for } 0 \leqslant x < 1/2 \\ -1 & \text{for } -1/2 \leqslant x < 0 \end{cases}$

(c) $(*)$ The triangle wave: $f(x) = |x|$ [4pts]

Problem 4 $(*)$
Prove that the space $\mathscr{S}(\mathbb{Z}^n)$ of rapidly decreasing functions on the lattice $\mathbb{Z}^n$ is dense in $\ell^p(\mathbb{Z}^n)$ for every $p \in [1, \infty)$, but not for $p = \infty$. [4pts]

Problem 5 $(*)$
Prove the claim (stated in lecture) that the following two conditions on a pair of functions $f, g \in L^2(\mathbb{R}^n)$ are equivalent:

(i) g is equal almost everywhere to the Fourier transform of f;

(ii) There exists a sequence $R_j \to \infty$ such that $g(p) = \lim_{j\to\infty} \int_{B_{R_j}(0)} e^{-2\pi ip\cdot x} f(x)\, dx$ for almost every $p \in \mathbb{R}^n$. [6pts]
Hint: We are not assuming $f \in L^1(\mathbb{R}^n)$, so the integral $\int_{\mathbb{R}^n} e^{-2\pi ip\cdot x} f(x)\, dx$ might not be well defined. However, the product of f with the characteristic function of $B_R(0)$ is in $L^1(\mathbb{R}^n)$ for every $R > 0$.

Problem 6
In this problem, we consider pairs of functions f and g for which pointwise products $f(x)g(x)$ are well defined, e.g. f can be vector valued and g scalar valued, or vice versa. Use Fubini's theorem to prove the following relations between Fourier transforms/series and convolutions:

(a) $(*)$ For $f, g \in L^1(\mathbb{R}^n)$, the Fourier transform of $f * g$ is given by $\widehat{f * g}(p) = \widehat{f}(p)\widehat{g}(p)$ for all $p \in \mathbb{R}^n$. [4pts]

(b) For $f, g \in L^1(\mathbb{T}^n)$, the Fourier series of $f * g$ has coefficients $\widehat{f * g}_k = \widehat{f}_k\widehat{g}_k$ for $k \in \mathbb{Z}^n$.[3]

(c) For two continuous fully periodic functions f, g whose Fourier coefficients satisfy $\widehat{f}, \widehat{g} \in \ell^1(\mathbb{Z}^n)$, the Fourier series of fg has coefficients $\widehat{fg}_k = \sum_{j\in\mathbb{Z}^n} \widehat{f}_{k-j}\widehat{g}_j$ for $k \in \mathbb{Z}^n$.[4]

(d) Use a density argument to extend the relation in part (c) to the case where f satisfies the same hypothesis but g is an arbitrary function in $L^2(\mathbb{T}^n)$.

[2] All three functions are in $L^2(\mathbb{T}^1)$, so their Fourier series will converge in L^2 no matter what, but possibly not pointwise.

[3] The convolution of two functions on $\mathbb{T}^n$ is defined via the obvious formula $(f * g)(x) := \int_{\mathbb{T}^n} f(x - y)g(y)\, dy$. The proof of Young's inequality can be adapted almost verbatim to the fully periodic setting in order to show that $f * g \in L^1(\mathbb{T}^n)$ whenever $f, g \in L^1(\mathbb{T}^n)$.

[4] The right hand side of this relation could also be written as $(\widehat{f} * \widehat{g})_k$ after defining the convolution of two functions on $\mathbb{Z}^n$ in the obvious way as an integral with respect to the counting measure. The proof of Young's inequality also adapts to this setting, so that $f * g \in \ell^1(\mathbb{Z}^n)$ for $f, g \in \ell^1(\mathbb{Z}^n)$.

Problem session 8

Problem 6 $\longrightarrow$ Continuous functions that are nowhere differentiable $\longrightarrow$ Problem 5

Problem 6

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(a) (∗) For $f, g \in L^1(\mathbb{R}^n)$, the Fourier transform of $f * g$ is given by $\widehat{f * g}(p) = \hat{f}(p)\hat{g}(p)$ for all $p \in \mathbb{R}^n$. [4pts]

(b) For $f, g \in L^1(\mathbb{T}^n)$, the Fourier series of $f * g$ has coefficients $\widehat{f * g}_k = \hat{f}_k \hat{g}_k$ for $k \in \mathbb{Z}^n$.[3]

(c) For two continuous fully periodic functions f, g whose Fourier coefficients satisfy $\hat{f}, \hat{g} \in \ell^1(\mathbb{Z}^n)$, the Fourier series of fg has coefficients $\widehat{fg}_k = \sum_{j \in \mathbb{Z}^n} \hat{f}_{k-j}\hat{g}_j$ for $k \in \mathbb{Z}^n$.[4]

(d) Use a density argument to extend the relation in part (c) to the case where f satisfies the same hypothesis but g is an arbitrary function in $L^2(\mathbb{T}^n)$.

(a) By defn $\hat{f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} f(x)\, dx$

$$\widehat{f*g}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} (f*g)(x)\, dx$$

$$= \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} \left(\int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \right) dx$$

$$\underset{\text{Fubini's Thm}}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} f(y)\, g(x-y)\, dy\, dx$$

Change of variable, $u = x - y \Rightarrow du = dx$

$$= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-2\pi i p\cdot(u+y)} \cdot g(u)\, du \right) dy$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i p\cdot y} \left(\int_{\mathbb{R}^n} e^{-2\pi i p\cdot u} g(u)\, du \right) dy$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-2\pi i p\cdot y}\, dy\ \hat{g}(p)$$

$$= \hat{f}(p)\hat{g}(p)$$

$\boxed{}$

(b) By Young's inequality for convolution of functions over $\mathbb{T}^n$

$$\|f*g\|_{L^1} \le \|f\|_1 \|g\|_1 \Rightarrow \text{if } f, g \in L^1(\mathbb{T}^n)$$

$$\Rightarrow \quad f*g \in L^1(\mathbb{T}^n)$$

$$\widehat{f*g}_k = \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} (f*g)(x)\, dx$$

$$= \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} \left(\int_{\mathbb{T}^n} f(y)\, g(x-y)\, dy \right) dx$$

Change of variable $u = x - y \Rightarrow du = dx$

as in part a) $\widehat{f*g}_k = \hat{f}_k \hat{g}_k$. $\square$

(c) if $\hat{f}, \hat{g} \in \ell^1(\mathbb{Z}^n) \Rightarrow \widehat{f*g} \in \ell^1(\mathbb{Z}^n)$

$$(f*g)(x) = \sum f(y)\, g(x-y)$$

$\square$

(d) Assume $\hat{f} \in \ell^1(\mathbb{Z}^n)$ and $g \in L^2(\mathbb{T}^n)$.

$\Rightarrow f \in C^0(\mathbb{T}^n)$ and $\hat{g} \in \ell^2(\mathbb{Z}^n)$

$\mathcal{F}^*$ Choose a sequence $\hat{g}_j$ of compactly supported functions on $\mathbb{Z}^n$ $\xrightarrow[\ell^2]{} \hat{g}$.

$$\mathcal{F}^* : \mathcal{S}(\mathbb{Z}^n) \longrightarrow C^\infty(\mathbb{T}^n)$$

to the sequence of functions $\hat{g}_j$ gives a sequence of function $g_j \in C^\infty(\mathbb{T}^n)$.

$$\langle \hat{f}, \hat{g} \rangle_{\ell^2} = \langle f, g \rangle_{L^2}$$

$$g_j \xrightarrow[L^2]{} g.$$

$\because$ $\hat{g}_j \in \ell^1(\mathbb{Z}^n) \Rightarrow$ by part c) $\Rightarrow$

$$\widehat{fg_j} = \hat{f} * \hat{g}_j \quad \text{as } j \to \infty$$

Young's inequality $\Rightarrow$ $\hat{f} * \hat{g}_j \longrightarrow \hat{f} * \hat{g}$ in ℓ^2

$$fg_j \xrightarrow[L^2]{} fg \quad \Rightarrow \quad \widehat{fg_j} \longrightarrow \widehat{fg} \text{ in } \ell^2$$

$\square$

Nowhere differentiable functions

Fix constants $a, b > 1$

$f : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{a^k} e^{2\pi i b^k x}$$

$$a > 1 \Rightarrow \sum^{\infty} \frac{1}{a^k} < \infty$$

$k=0$

$\Rightarrow$ partial sums converge uniformly to a continuous function $\Rightarrow f$ is continuous.

Suppose $b \in \mathbb{N} \Rightarrow f$ is periodic

$$f'(x) = 2\pi i \sum \frac{b^k}{a^k} e^{2\pi i b^k x}$$

if $b < a \Rightarrow \frac{b}{a} < 1$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{b^k}{a^k} < \infty \Rightarrow f'(x) \text{ converges}$$

$\Rightarrow f$ is differentiable.

What can be said if $b \geq a$?

Thm :- If $b \geq a > 1$ then f is not differentiable at any point.

Proof: Proof by contradiction. Suppose $\exists\, x_0 \in \mathbb{R}$ where f is differentiable.

$$F(h) = D_h f(x_0) = \frac{f(x_0+h) - f(x_0)}{h}, \quad h \neq 0$$

$\Rightarrow \quad f'(x) = \lim_{h \to 0} F(h)$ exists.

F bounded continuous function on $\mathbb{R}$.

we'll show that this leads to the conclusion that $b < a$, $\lim_{k \to \infty} \left(\frac{b}{a}\right)^k = 0$.

$$(\psi * f)(x) = e^{2\pi i k \cdot x} \hat{f}_k$$

Choose a smooth function

$\hat{\psi} : \mathbb{R} \longrightarrow [0,1]$ w/ $\hat{\psi}(1) = 1$

and $\hat{\psi}$ has compact support in $\left(\frac{1}{b}, b\right)$.

Define
$$\hat{\psi}_k(p) = \hat{\psi}\left(\frac{p}{b^k}\right)$$

$$\frac{1}{b} < \frac{p}{b^k} < b \Rightarrow b^{k-1} < p < b^{k+1}$$

$\Rightarrow \quad \operatorname{supp}(\hat{\psi}_k) \subset (b^{k-1}, b^{k+1})$.

$$\Rightarrow \quad \hat{\psi}_k(b^n) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{k\} \end{cases}$$

$$\because \quad \hat{\psi}_k \in C_0^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$$

$\Rightarrow$ $\hat{\psi}_k$ are Fourier-transforms of some function $\psi_k \in \mathcal{S}(\mathbb{R})$

$$\left(\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \right)$$

Apply the Fourier inversion formula.

$$\psi_k(x) = \int_{-\infty}^{\infty} e^{2\pi i p x} \hat{\psi}\left(\frac{p}{b^k}\right) dp \qquad \left(\frac{p}{b^k} = t \Rightarrow dp = b^k dt \right)$$

$$= b^k \int_{-\infty}^{\infty} e^{2\pi i b^k p x} \hat{\psi}(p)\, dp$$

$$= b^k \psi(b^k x)$$

$$0 \notin \operatorname{supp}(\hat{\psi}_k)$$

$$\Rightarrow \quad 0 = \hat{\psi}_k(0) = \int_{-\infty}^{\infty} \psi_k(x)\, dx$$

and differentiating

$$0 = \hat{\psi}_K'(0) = -2\pi i \int_{-\infty}^{\infty} x\, \psi_K(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x_0)\, \psi_K(x)\, dx = 0 \quad \text{———} \quad 1$$

$$e^{2\pi i K\cdot x}\, \hat{\psi}_K(b) = (f * \psi_K)(x_0)$$

$$= \frac{1}{a^K} e^{2\pi i b^K x_0} \qquad = \int_{-\infty}^{\infty} f(x_0 - x)\, \psi_K(x)\, dx$$

$$= \int_{-\infty}^{\infty} [f(x_0 - x) - f(x_0)]\, \psi_K(x)\, dx$$

$$= -\int_{-\infty}^{\infty} x\, F(-x)\, \psi_K(x)\, dx$$

$$= -b^K \int_{-\infty}^{\infty} x\, F(-x)\, \psi(b^K x)\, dx \qquad \left(b^K x \rightsquigarrow t \Rightarrow dx \to \frac{dt}{b^K} \right)$$

$$= -\int_{-\infty}^{\infty} \frac{x}{b^K} F\left(-\frac{x}{b^K}\right) \psi(x)\, dx$$

$$\Rightarrow \left(\frac{b}{a}\right)^k e^{2\pi i b^k x_0} = -\int_{-\infty}^{\infty} F\left(\frac{-x}{b^k}\right) x\,\varphi(x)\,dx$$

———— (2)

$\underbrace{\qquad\qquad}$ is bounded $\forall\ k \geq 0$

$\leq C\,|x|\,\varphi$

as we assume f is diff $\Rightarrow F$ is bounded.

$$\xrightarrow[k\to\infty]{} F(0)\, x\, \varphi(x)$$

$$= f'(x_0)\, x\, \varphi(x)$$

The sequence of integrand on RHS satisfy the hypothesis of the DCT $\Rightarrow$

$$\lim_{k\to\infty} \left(\frac{b}{a}\right)^k e^{2\pi i b^k x_0} = -f'(x_0) \int_{-\infty}^{\infty} x\,\varphi(x)\,dx$$

$$= 0$$

$$\Rightarrow \lim_{k\to\infty} \left(\frac{b}{a}\right)^k = 0 \Rightarrow \frac{b}{a} < 1$$

$\Rightarrow \quad b \leq a.$

$\boxed{}$

1. (a) $\hat{f}_{-k} = \overline{\hat{f}_k}$

(b) $$a_k = \int_{\mathbb{T}^n} f(x) \cos(2\pi k \cdot x)\, dx$$

$$b_k = \int_{\mathbb{T}^n} f(x) \sin(2\pi k \cdot x)\, dx$$

(c) f even function

f odd function.

2. (a) $\hat{f}(-p) = \overline{\hat{f}(p)}$

(b) $$u(p) = \int_{\mathbb{R}^n} f(x) \cos(2\pi p \cdot x)\, dx$$

$$v(p) = \int_{\mathbb{R}^n} f(x) \sin(2\pi p \cdot x)\, dx.$$

3. (a) $f(x) = \frac{i}{2\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^k}{k} e^{2\pi i k x}$

$= \frac{i}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(2\pi k x)$

(b) $f(x) = -\frac{2i}{\pi} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \frac{1}{k} e^{2\pi i k x}$

(c) $f(x) = \frac{1}{4} - \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \frac{1}{k^2} e^{2\pi i k x}$

Problem 5 (∗)
Prove the claim (stated in lecture) that the following two conditions on a pair of functions $f, g \in L^2(\mathbb{R}^n)$ are equivalent:

(i) g is equal almost everywhere to the Fourier transform of f;

(ii) There exists a sequence $R_j \to \infty$ such that $g(p) = \lim_{j\to\infty} \int_{B_{R_j}(0)} e^{-2\pi i p\cdot x} f(x)\, dx$ for almost every $p \in \mathbb{R}^n$. [6pts]
Hint: We are not assuming $f \in L^1(\mathbb{R}^n)$, so the integral $\int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} f(x)\, dx$ might not be well defined. However, the product of f with the characteristic function of $B_R(0)$ is in $L^1(\mathbb{R}^n)$ for every $R > 0$.

5) i) $\Rightarrow$ ii)

$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$

is the unique continuous extension of the L^2-bounded operator $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

$$L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

Choose a sequence $R_j \to \infty$ and consider

$$f_j = \chi_{B_{R_j}(0)} f \ : \mathbb{R}^n \to \mathbb{R}$$

$f_j \in L^2$ is supported on $B_{R_j}(0)$ whose measure is finite.

$$\Rightarrow \quad f_j \in L^1 \quad \Rightarrow \quad f_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R})$$

$$f_j \xrightarrow[L^2]{} f$$

$\Rightarrow$ we can look at the Fourier transform of f_j

$$\Rightarrow \quad \hat{f}_j(p) = \int_{B_{R_j}(0)} e^{-2\pi i p \cdot x} f(x)\, dx$$

$$\Rightarrow \quad \hat{f}_j \xrightarrow[L^2]{} \hat{f}$$

$\Rightarrow$ almost everywhere of a subsequence of $\hat{f}_j$.

$\Rightarrow$ $\hat{f}_j(p) \longrightarrow \hat{f}(p)$ a.e. p.

$\Rightarrow$ $g(p) = \lim_{j \to \infty} \hat{f}_j(p)$

$\because$ $\hat{f}_j$ converges point to $\hat{f}$

$\Rightarrow$ $g = \hat{f}$ a.e. on $\mathbb{R}^n$

$\Rightarrow$ ii) $\Rightarrow$ i)

$$\mathcal{F}(f) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx$$

$h \in L^p$

$\varphi_j \in$

$h * \varphi_j$