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Funktionalanalysis
WiSe 2020–21

Problem Set 8

Due: Thursday, 14.01.2021 (21pts total)

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Problem 1
Fix $s \geqslant 0$ and a multi-index α of order $m := |\alpha| \in \mathbb{N}$.

(a) Use the Fourier transform and Fourier inverse operators $\mathscr{F}, \mathscr{F}^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ to write down an explicit formula for the unique extension of $\partial^\alpha : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ to a bounded linear operator $\partial^\alpha : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$.

(b) (∗) Show that a sequence $f_j \in H^m(\mathbb{R}^n)$ converges in the H^m-norm to $f \in H^m(\mathbb{R}^n)$ if and only if $\partial^\beta f_j \to \partial^\beta f$ in the L^2-norm for all multi-indices β of order $|\beta| \leqslant m$. [3pts]

(c) (∗) Show that for any scalar-valued function $f \in L^1(\mathbb{R}^n)$, the convolution with f defines a bounded linear operator $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n) : g \mapsto f * g$, and if $g \in H^{s+m}(\mathbb{R}^n)$, then $\partial^\alpha(f * g) = f * \partial^\alpha g$. [4pts]
Hint: Problem Set 7 #6 proves the formula $\widehat{f * g} = \hat{f}\hat{g}$ for $f, g \in L^1(\mathbb{R}^n)$. For the present problem, you may assume this formula also holds when $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$; this case was omitted from the lecture due to lack of time, but is proved via an easy density argument as Theorem 8.18 in the lecture notes.

(d) Show that for any $f \in H^m(\mathbb{R}^n)$ and any approximate identity $\rho_j : \mathbb{R}^n \to [0, \infty)$ with shrinking support, the functions $f_j := \rho_j * f$ are in $H^m(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and converge in the H^m-norm to f as $j \to \infty$.

Problem 2
Suppose $\rho \in \mathscr{S}(\mathbb{R}^n)$ satisfies $\rho \geqslant 0$ and $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$, and define $\rho_j(x) := j^n \rho(jx)$ for $j \in \mathbb{N}$.

density of C_0^∞ in L^p-spaces.

(a) (∗) Show that for any $s \geqslant 0$ and $f \in H^s(\mathbb{R}^n)$, the sequence $\rho_j * f \in C^\infty(\mathbb{R}^n)$ satisfies

$$\|\rho_j * f\|_{H^s} \leqslant \|f\|_{H^s} \qquad \text{and} \qquad \rho_j * f \xrightarrow{H^s} f \text{ as } j \to \infty.$$

Note that the convergence does not follow from Problem 1(d) since we are not assuming $s \in \mathbb{N}$. [4pts]

(b) Show that the same result holds if $\rho_j \in \mathscr{S}(\mathbb{R}^n)$ is instead defined as $\mathscr{F}^*\psi_j$ for a sequence of smooth functions $\psi_j : \mathbb{R}^n \to [0, 1]$ with compact support in the increasingly large ball $B_{j+1}(0) \subset \mathbb{R}^n$ and $\psi_j|_{B_j(0)} \equiv 1$.

Problem 3
Prove that for every $m \in \mathbb{N}$, functions $f \in C^m(\mathbb{T}^n)$ are also in $H^m(\mathbb{T}^n)$ and the inclusion $C^m(\mathbb{T}^n) \hookrightarrow H^m(\mathbb{T}^n)$ is continuous.

$\|\rho_j * f\|_{L^p} \leq \|f\|_{L^p}$ 1 $\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}$

§ 2.

Problem 4
Fix constants $a, b > 1$ with $b \in \mathbb{N}$ and consider the periodic function $f(x) := \sum_{k=0}^{\infty} \frac{1}{a^k} e^{2\pi i b^k x}$, which is continuous since the series converges absolutely and uniformly. Prove:

(a) $(*)$ $f \in H^s(S^1)$ if and only if $s < \log_b a$. [4pts]

(b) $f \in C^{0,\alpha}(S^1)$ for every $\alpha \in (0,1)$ with $\alpha \leqslant \log_b a$.
Hint: Use Lemma 9.27 from the lecture notes. Note that the partial sums are continuously differentiable: estimate their $C^{0,1}$-norms.

Remark: f is a variant of the function famously introduced by Weierstrass in 1872, which is of class C^1 if $b < a$, but nowhere differentiable if $b \geqslant a$. Part (a) establishes a weak version of the latter statement by proving $f \notin H^1(S^1)$, which implies $f \notin C^1(S^1)$ via Problem 3. Notice that while the Sobolev embedding theorem provides a continuous inclusion $H^s(S^1) \hookrightarrow C^{0,\alpha}(S^1)$ whenever $\alpha \leqslant s - 1/2$, f turns out to be in a wider range of Hölder spaces than is guaranteed by that theorem. (That is just a coincidence—there is no interesting phenomenon behind it that I am aware of.)

Problem 5
Assume Ω is a compact subset of either $\mathbb{R}^n$ or $\mathbb{T}^n$, and $0 < \alpha < \beta \leqslant 1$.

(a) Prove via the Arzelà-Ascoli theorem that the inclusion $C^{0,\beta}(\Omega) \hookrightarrow C^0(\Omega)$ is compact.

(b) Show that if $f_k \in C^{0,\beta}(\Omega)$ is a uniformly $C^{0,\beta}$-bounded sequence that is C^0-convergent to $f \in C^0(\Omega)$, then f is also in $C^{0,\beta}(\Omega)$.
Caution: Do not try to prove that f_k is also $C^{0,\beta}$-convergent to f—that is not generally true.

(c) Show that the inclusion $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ is also compact.
Hint: Given $f_k \to f$ as in part (b), use the relation

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} = \left(\frac{|g(x) - g(y)|}{|x - y|^\beta} \right)^{\alpha/\beta} \cdot |g(x) - g(y)|^{1 - \frac{\alpha}{\beta}}.$$

for the functions $g := f - f_k$.

Problem 6
The linear inhomogeneous Cauchy-Riemann equation for functions $f : \mathbb{C} \to \mathbb{C}$ of a complex variable $z = x + iy$ is a first-order PDE taking the form

$$\bar{\partial} f := \partial_x f + i \partial_y f = g.$$

Use the coordinates (x, y) to identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider functions that are fully periodic on $\mathbb{R}^2$; these are equivalent to complex-valued functions on the torus $\mathbb{T}^2$. Prove:

(a) $(*)$ If $f \in H^1(\mathbb{T}^2)$ and $g = \bar{\partial} f \in H^m(\mathbb{T}^2)$ for some $m \in \mathbb{N}$, then $f \in H^{m+1}(\mathbb{T}^2)$. [6pts]

(b) If $f \in C^1(\mathbb{T}^2)$ and $g = \bar{\partial} f$ is smooth, then f is smooth.

Problem Session 9

Problem 3
Prove that for every $m \in \mathbb{N}$, functions $f \in C^m(\mathbb{T}^n)$ are also in $H^m(\mathbb{T}^n)$ and the inclusion $C^m(\mathbb{T}^n) \hookrightarrow H^m(\mathbb{T}^n)$ is continuous.

$f \in C^m(\mathbb{T}^n) \Rightarrow f$ is m-times continuously differentiable on $\mathbb{T}^n$ (compact) $\Rightarrow \partial^\alpha f$ $|\alpha| \leq m$ is bounded $\Rightarrow \partial^\alpha f \in L^2(\mathbb{T}^n)$ for $|\alpha| \leq m$, $\mathbb{T}^n$ is of finite measure

$\Rightarrow$ $f \in C^m(\mathbb{T}^n) \Rightarrow f \in H^m(\mathbb{T}^n)$.

want :- $\|f\|_{H^m} \leq C \|f\|_{C^m}$

$$\|f\|_m = \sum_{k \in \mathbb{Z}^n} (1+|k|^2)^m |\hat{f}|^2 = \sum \frac{(1+|k|^2)^m}{(2\pi i k)^{2\alpha}} (2\pi i k)^{2\alpha} |\hat{f}|^2$$

$$= \sum \frac{(1+|k|^2)^m}{(2\pi i k)^{2\alpha}} |\widehat{\partial^\alpha f}|^2$$

$$= \sum \frac{(1+|k|^2)^m}{(2\pi i k)^{2\alpha}} |\partial^\alpha f|^2$$

$$= C \|f\|_{C^m}$$

$\Rightarrow$ $C^m(\mathbb{T}^n) \hookrightarrow H^m(\mathbb{T}^n)$ is continuous. $\boxtimes$

Problem 4

Fix constants $a, b > 1$ with $b \in \mathbb{N}$ and consider the periodic function $f(x) := \sum_{k=0}^{\infty} \frac{1}{a^k} e^{2\pi i b^k x}$, which is continuous since the series converges absolutely and uniformly. Prove:

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(b) $f \in C^{0,\alpha}(S^1)$ for every $\alpha \in (0,1)$ with $\alpha \leqslant \log_b a$.
Hint: Use Lemma 9.27 from the lecture notes. Note that the partial sums are continuously differentiable: estimate their $C^{0,1}$*-norms.*

(a)
$$\|f\|_{H^s(S^1)} = \sum (1+|b^k|^2)^s \left(\frac{1}{a^k}\right)^2$$
$$\leq 2 \sum b^{2ks} \left(\frac{1}{a^k}\right)^2$$
$$\leq c \sum \frac{b^{2ks}}{a^{2k}} = c \sum \left(\frac{b^s}{a}\right)^{2k}$$

$< \infty$ want for $f \in H^s(S^1)$

$b^s < a \implies$ Taking $\log_b$, we get

$f \in H^s(S^1) \iff s < \log_b a.$ ∎

Lemma 9.27 Suppose f_k is a sequence of continuous functions on $\Omega \subset \mathbb{R}^n$ converging uniformly to f and there are constants $a > 1$, $b \geq 1$, $C > 0$

and $\beta \in (0,1]$ s.t.

$$\|f - f_K\|_{C^0} \leq \frac{C}{a^K} \quad \text{and} \quad |f_K|_{C^{0,\beta}} \leq Cb^K$$

Then $f \in C^{0,\alpha}(\Omega)$ for $\alpha = \dfrac{\beta}{1+\log_a b}$.

4(b) $f(x) = \sum \frac{1}{a^K} e^{2\pi i b^K x}$

$$f_K(x) = \sum_{n=0}^{K} \frac{1}{a^n} e^{2\pi i b^n x}$$

$$\|f - f_K\|_{C^0} = \left\| \sum_{n=K+1}^{\infty} \frac{1}{a^n} e^{2\pi i b^n x} \right\|$$

$$\leq \sum_{n=K+1}^{\infty} \left| \frac{1}{a^n} \right| = \frac{\frac{1}{a^{K+1}}}{1 - \frac{1}{a}}$$

$$= \frac{a}{a^{K+1}(a-1)} \leq \frac{C}{a^K}$$

$\beta = 1$

$$\|f_K\|_{C^{0,1}} = \left\| \sum_{n=0}^{K} \frac{1}{a^n} e^{2\pi i b^n x} \right\|_{C^{0,1}}$$

find bounds on the derivative of $f_k(x)$

which are finitely many terms.

$$f_k'(x) = 2\pi i \sum_{n=0}^{k} \frac{b^n}{a^n} e^{2\pi i b^n x}$$

$$\|f_k\|_{C^{0,1}} \leq C\left(\frac{b}{a}\right)^k$$

Using the lemma

$$f \in C^{0,\alpha}, \alpha \in (0,1), \alpha = \frac{\beta}{1+\log_a b}$$

$$\alpha = \frac{1}{1+\log_a\left(\frac{b}{a}\right)}$$

$$= \frac{1}{\cancel{1} + \log_a b - \cancel{\log_a a}}$$

$$= \frac{1}{\log_a b} = \log_b a$$

$$\therefore \quad f \in C^{0,\alpha}, \quad \alpha \leq \log_b a$$

<u>Proof of the Lemma</u> :- Want to estimate

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq C \quad \text{for} \quad \alpha = \frac{\beta}{1+\log_a b}$$

$\sim a$

We'll prove for $0 < |x-y| \leq C$

$$|f(x)-f(y)| = |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)|$$

$$\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f(y) - f_k(y)|$$

$$\leq \frac{2C}{a^k} + C b^k |x-y|^\beta$$

Choose $k \in \mathbb{N}$ big enough so that

$$\frac{1}{(ab)^{k+1}} \leq |x-y|^\beta \leq \frac{1}{(ab)^k}$$

$$\Rightarrow |f(x)-f(y)| \leq \frac{2C}{a^k} + Cb^k \cdot \frac{1}{(ab)^k}$$

$$\leq \frac{3aC}{a^{k+1}}$$

$$\Rightarrow \frac{|f(x)-f(y)|}{|x-y|^\alpha} = \frac{|f(x)-f(y)|}{|x-y|^{\frac{\beta}{1+\log_b a}}}$$

$$\leq \frac{3aC}{a^{k+1} |x-y|^{\beta/1+\log_b a}}$$

$$= \frac{3aC}{a^{k+1}|x-y|^{\beta}} |x-y|^{1+\log_b a}$$

$$\leq \frac{3aC}{a^{k+1}} |x-y|^{1+\log_b a} \cdot (ab)^{k+1}$$

$$= 3aC\, |x-y|^{1+\log_b a}\, b^{k+1}$$

$$a^{1+\log_a b} = ab \Rightarrow a = (ab)^{\frac{1}{1+\log_a b}}$$

$$\parallel$$

$$a,\ \underbrace{a^{\log_a b}}_{b}$$

$$\frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \leq C$$

where $\alpha = \dfrac{\beta}{1+\log_b a}$ $\boxtimes$

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for the functions $g := f - f_k$.

(a) $C^{0,\beta}(\Omega) \hookrightarrow C^0(\Omega)$ compact

$(f_K) \in C^{0,\beta}$, bounded

- uniform boundedness
- equicontinuous

f_K is uniformly bounded.

$|f_K(x) - f_K(y)| \leq \epsilon$ whenever $|x-y| < \delta$

$\because$ $f_K \in C^{0,\beta}$

$\Rightarrow$ $|f_K(x) - f_K(y)| \leq C|x-y|^\beta$

choose $\epsilon = \dfrac{\delta^{1/\beta}}{C}$

$\Rightarrow$ f_K are equicontinuous

$\Rightarrow$ Arzelá-Ascoli (f_K) has a convergent subsequence in C^0

$\Rightarrow$ $C^{0,\beta} \hookrightarrow C^0$ is compact.

b) $f \in C^{0,\beta}(\Omega)$

Want:- $\dfrac{|f(x) - f(y)|}{|x-y|^\beta} \leq C$

$$|f(x)-f(y)| = |f(x) - f_K(x) + f_K(x) - f_K(y) + f_K(y) - f(y)|$$

$$\leq \underbrace{|f(x) - f_K(x)|}_{\leq C_1 |x-y|^\beta} + \underbrace{|f_K(x) - f_K(y)|}_{\leq C |x-y|^\beta} + \underbrace{|f_K(y) - f(y)|}_{C_1 |x-y|^\beta}$$

$$|f(x) - f(y)| \leq C |x-y|^\beta$$

$\boxed{\nabla_{//}}$

Problem 6

The linear inhomogeneous Cauchy-Riemann equation for functions $f : \mathbb{C} \to \mathbb{C}$ of a complex variable $z = x + iy$ is a first-order PDE taking the form

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(b) If $f \in C^1(\mathbb{T}^2)$ and $g = \bar{\partial} f$ is smooth, then f is smooth.

$\Delta f = g$

If $g \in H^m$

$\Rightarrow f \in H^{m+2}$

$\bar{\partial} f = \partial_x f + i\partial_y f$ is a 1st order operator.

(a) $\bar{\partial} f = g$

$$\Rightarrow \partial_x f + i\partial_y f = g$$

$$\Rightarrow \widehat{\partial_x f + i\partial_y f} = \hat{g}$$

$$= 2\pi i k_1 \hat{f}_k + i\, 2\pi i\, k_2 \hat{f}_k = \hat{g}_k$$

$$= 2\pi i k_1 \hat{f}_k - 2\pi k_2 \hat{f}_k = \hat{g}_k$$

$$|k|^2 = k_1^2 + k_2^2$$

We'll look at H^s-norm of f, $s \leq m+1$

for $s \leq m+1 \Rightarrow 2(s-1) \leq 2m$

$$\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{2s} |\hat{f}_k|^2 \leq C \sum_k \frac{|k|^{2s} |\hat{g}_k|^2}{|k|^2}$$

$$= C \sum |k|^{2s-2} |\hat{g}_k|^2$$

$$= C \sum |k|^{2(s-1)} |\hat{g}_k|^2$$

$$\leq C \sum |k|^{2m} |\hat{g}_k|^2$$

$$= \|g\|_{H^m} < \infty$$

$\Rightarrow f \in H^{m+1}(\mathbb{T}^2)$ provided $g \in H^m(\mathbb{T}^m)$

(b) $\bigcap_s H^s(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$ If $g \in C^\infty \Rightarrow g \in H^s$ $\forall s$

by part a) $f \in H^{s+1} \Rightarrow f \in H^s \ \forall s$

$\Rightarrow \quad f \in C^\infty(\mathbb{T}^n)$.

If the operator is elliptic, the solution will be as regular as the problem allows

$$Lf = g \quad , \quad L \text{ is order } k$$

If $\quad g \in W^{m,p} \Rightarrow f \in W^{m+k,p}$.

- $\Delta f = 0$. $\qquad W^{m,p} \hookrightarrow C^{k,\alpha}$