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Funktionalanalysis
WiSe 2020–21

Problem Set 6

Due: Thursday, 17.12.2020 (22pts total)

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space* $(V, \langle\ ,\ \rangle)$ *over a field* $\mathbb{K}$ *which is either* $\mathbb{R}$ *or* $\mathbb{C}$. *The Lebesgue measure on* $\mathbb{R}^n$ *is denoted by* m.

Problem 1

Assume $f : I \to V$ is a function defined on an interval $I \subset \mathbb{R}$.

(a) (∗) Show that f is Lipschitz-continuous with Lipschitz constant $C > 0$[1] if and only if there exist constants $a \in I$ and $v_0 \in V$ and a function $g \in L^\infty(I)$ with $\|g\|_{L^\infty} \leqslant C$ such that $f(x) = v_0 + \int_a^x g(t)\,dt$ for all $x \in I$. [5pts]
Hint: Start by proving directly that Lipschitz-continuity implies absolute continuity.

(b) Find an explicit example of a function $f : [0,1] \to \mathbb{R}$ that is absolutely but not Lipschitz-continuous.

(c) (∗) One says that $f : I \to V$ has a *jump discontinuity* at $x_0 \in I$ if $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ both exist but are not equal. Assume $f \in L^1_{\text{loc}}(I)$ and set $F(x) := \int_a^x f(t)\,dt$ for some constant $a \in I$. Show that if f has a jump discontinuity at x_0, then F is not differentiable at x_0. [3pts]

(d) Let $\varphi = \chi_{[0,\infty)} : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of $[0,\infty)$. Given an enumeration of the rational numbers $q_1, q_2, q_3, \ldots \in \mathbb{Q}$, show that $f(x) := \sum_{n=1}^\infty \frac{1}{2^n}\varphi(x - q_n)$ defines a real-valued function $f \in L^\infty(\mathbb{R})$ such that for every $n \in \mathbb{N}$, $\lim_{x\to q_n^+} f(x) = f(q_n)$ and $\lim_{x\to q_n^-} f(x) = f(q_n) - \frac{1}{2^n}$.
Hint: For any $n, N \in \mathbb{N}$, *there exists a neighborhood* $J \subset I$ *of* q_n *such that the function* $\varphi_m(x) := \frac{1}{2^m}\varphi(x - q_m)$ *is constant on* J *for all* $m \leqslant N$ *with the exception of* $m = n$. *(Why?)*

(e) Write down an example of a Lipschitz-continuous function $F : \mathbb{R} \to \mathbb{R}$ that fails to be differentiable on a dense subset of $\mathbb{R}$. (By part (a) and the Lebesgue differentiation theorem, it will still be differentiable almost everywhere.)

Problem 2

Determine the set of Lebesgue points for each of the following functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$:

(a) (∗) The characteristic function of $\mathbb{Q}^n \subset \mathbb{R}^n$ [3pts]

(b) The characteristic function of $\mathbb{R}^n \backslash \mathbb{Q}^n \subset \mathbb{R}^n$

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[1] Recall: $C > 0$ is a Lipschitz constant for $f : I \to V$ if $|f(x) - f(y)| \leqslant C|x - y|$ holds for all $x, y \in I$.

Problem 3
Let $\mathbb{D}^n \subset \mathbb{R}^n$ denote the unit ball, and consider a function of the form $f(x) := \frac{1}{|x|^\alpha}$ on $\mathbb{R}^n \backslash \{0\}$ for some constant $\alpha > 0$.

(a) For which values of $\alpha > 0$ does f belong to $L^1_{\text{weak}}(\mathbb{D}^n)$, and for which of these is it also in $L^1(\mathbb{D}^n)$?

(b) For which values of $\alpha > 0$ does f belong to $L^1_{\text{weak}}(\mathbb{R}^n \backslash \mathbb{D}^n)$, and for which of these is it also in $L^1(\mathbb{R}^n \backslash \mathbb{D}^n)$?

Problem 4 (*)
Assume (X, μ) is a measure space and λ is another measure that is finite and absolutely continuous with respect to μ. Give a direct proof (without citing the Radon-Nikodým theorem) of the following result: for every $\epsilon > 0$, there exists $\delta > 0$ such that every measurable set $A \subset X$ satisfying $\mu(A) < \delta$ also satisfies $\lambda(A) < \epsilon$. [5pts]

Problem 5
Our proof of the Radon-Nikodým theorem in lecture established the following slightly stronger result. Assume λ and μ are two σ-finite measures defined on the same σ-algebra on a set X, and consider the space $L^1(X, \lambda + \mu)$ of measurable functions $g : X \to \mathbb{R}$ that satisfy $\int_X |g|\, d(\lambda + \mu) = \int_X |g|\, d\lambda + \int_X |g|\, d\mu < \infty$. The map $g \mapsto \int_X g\, d\lambda$ then defines a bounded linear functional $L^1(X, \lambda+\mu) \to \mathbb{R}$, so by the Riesz representation theorem, there exists a unique $h \in L^\infty(X, \lambda + \mu)$ such that

$$\int_X g\, d\lambda = \int_X hg\, d(\lambda + \mu) \quad \text{for all} \quad g \in L^1(X, \lambda + \mu).$$

Theorem (proved in lecture): $h : X \to \mathbb{R}$ satisfies $0 \leqslant h < 1$ outside of a subset $E \subset X$ with $\mu(E) = 0$, and the resulting function $f := \frac{h}{1-h} : X \backslash E \to [0, \infty)$ satisfies

$$\int_A f\, d\mu \leqslant \lambda(A) \quad \text{for all measurable sets } A \subset X, \tag{1}$$

with equality for all mesurable sets $A \subset X$ whenever $\lambda \ll \mu$.
In the following, we consider pairs of measures λ and μ on certain σ-algebras on sets $X \subset \mathbb{R}^n$, where each of λ and μ is either the Lebesgue measure m, the counting measure ν, or the Dirac measure δ.[2] In each case, find the function $f : X \backslash E \to [0, \infty)$ explicitly, determine the collection of measurable subsets $A \subset X$ on which (1) becomes an equality, and determine whether $\lambda \ll \mu$.

(a) (*) $X = \mathbb{R}^n$, $\mu = m$ and $\lambda = \delta$ on the σ-algebra of Lebesgue-measurable sets [3pts]

(b) $X = \mathbb{R}^n$, $\mu = \delta$ and $\lambda = m$ on the σ-algebra of Lebesgue-measurable sets

(c) (*) $X = \mathbb{Z}^n$, $\mu = \nu$ and $\lambda = \delta$ on the σ-algebra of all subsets [3pts]

(d) $X = \mathbb{Z}^n$, $\mu = \delta$ and $\lambda = \nu$ on the σ-algebra of all subsets

One last brainteaser:

(e) For $X = \mathbb{R}^n$ with $\mu = \nu$ and $\lambda = m$ defined on the σ-algebra of Lebesgue-measurable sets, show that $\lambda \ll \mu$ but the function f is identically zero, so (1) cannot typically be an equality. What went wrong?

[2] Recall: $\nu(A)$ is the number of elements in A, while $\delta(A)$ is defined to be 1 if $0 \in A$ and 0 otherwise.