Connections (continued)

On $E \xrightarrow{\pi} M$:

- horizontal subbundle $HE \subseteq TE$ (defn. 1) $\quad$ ($\Leftrightarrow$ linear proj. $K : TE \to E$ s.t. $HE = \ker K$)
- horizontal lift maps $\mathrm{Hor}_v : T_{\pi(v)} M \xrightarrow{\cong} H_v E \subseteq T_v E$
- parallel transport maps $P_\gamma^t : E_{\gamma(0)} \xrightarrow{\cong} E_{\gamma(t)}$ ($\forall$ paths γ)
- covariant derivative: $s \in \Gamma(E)$, $\underset{= \gamma(0)}{p \in M}$, $\underset{= \dot\gamma(0)}{X \in T_p M}$,

$$\nabla_X s = \frac{d}{dt} (P_\gamma^t)^{-1} (s(\gamma(t)))$$

Note: $Ts(X) = \mathrm{Hor}_{s(p)}(X) + \mathrm{Vert}_{s(p)}(\nabla_X s)$

$\Rightarrow \nabla_X s = K(Ts(X))$.

ex: Trivial bundle $E = M \times \mathbb{F}^m \rightsquigarrow$ trivial connection:

$$T_{(p,v)} E = \underset{=: H_{(p,v)}E}{T_p M} \oplus \underset{= \mathbb{F}^m}{T_v \mathbb{F}^m} =: V_{(p,v)} E \quad , \quad K(X, w) = w$$

$P_\gamma^t : \mathbb{F}^m \xrightarrow{\mathrm{Id}} \mathbb{F}^m$, $\quad \Gamma(E) = C^\infty(M, \mathbb{F}^m)$

$$\nabla_X s = ds(X)$$

$\exists$ an operator $\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{Hom}(TM, E))$

$$\nabla s(p) : T_pM \longrightarrow E_p : X \longmapsto \nabla_X s$$

<u>EX 1</u>: $\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{Hom}(TM,E))$ is linear.

<u>EX 2</u>: $\nabla(fs) = df \cdot s + f \nabla s$

(i.e. for $p \in M$, $X \in T_pM$,

$$\nabla_X(fs) = df(X)\, s(p) + f(p)\, \nabla_X s \;)$$

$\forall \; f \in C^\infty(M, \mathbb{F})$, $\; s \in \Gamma(E)$.

<u>defn. 3</u>: a <u>connection</u> on $E \xrightarrow{\pi} M$ is a linear operator $\nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{Hom}(TM,E))$ satisfying
$\nabla(fs) = df \cdot s + f\nabla s \quad \forall \; f \in C^\infty(M,\mathbb{F}), \; s \in \Gamma(E)$.

<u>claim</u>: defn. 3 $\Rightarrow$ defn. 1.

<u>prop</u>: $\forall$ pairs of ops. $\nabla, \hat{\nabla}$ as in defn. 3, can write
$\hat{\nabla} s = \nabla s + As$ for a bndl map $A : E \longrightarrow \mathrm{Hom}(TM, E)$.

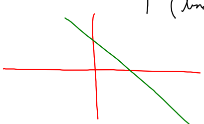
<u>pf</u>: Defn. $\hat{A} := \hat{\nabla} - \nabla : \Gamma(E) \longrightarrow \Gamma(\mathrm{Hom}(TM,E))$.
Then $\hat{A}$ is C^∞-linear: $\forall \; f \in C^\infty(M, \mathbb{R})$, $s \in \Gamma(E)$,
$\hat{A}(fs) = f \cdot \hat{A}s$.

$$\text{''} \quad \hat{\nabla}(fs) - \nabla(fs) = df \cdot s + f\hat{\nabla}s - df \cdot s - f\nabla s$$
$$= f(\hat{\nabla}s - \nabla s) = f\hat{A}s.$$

$\Rightarrow \exists$ bndl map $A : E \longrightarrow \mathrm{Hom}(TM,E)$ s.t. $\forall \; s \in \Gamma(E)$,

$\hat{A}s(p) = A(p)s(p)$. $\quad \square$

<u>rk</u>: $\Rightarrow$ {connections (defn. 3) on $E \xrightarrow{\pi} M$} is an <u>affine</u> space over $\Gamma(\mathrm{Hom}(E, \mathrm{Hom}(TM,E)))$
$= \Gamma($bndl of <u>bilinear</u> maps $E \oplus TM \to E)$.



Local coords:

$\Phi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{F}^m$ local triv

$\rightsquigarrow$ $\exists$ trivial conn. on $E|_{U_\alpha}$, call it ∇^0

$\Rightarrow$ For $s \in \Gamma(E)$ over U_α, can write

$\nabla_X s$ $=: \nabla s(p) X = \nabla^0 s(p) X + \Gamma_\alpha(X, s(p))$

for a smooth bilinear bndl map $\Gamma_\alpha : (TM \oplus E)|_{U_\alpha} \to E|_{U_\alpha}$.

Choose chart $(x^1, \ldots, x^n)$ over U_α; write

$e_1, \ldots, e_m \in \Gamma(E|_{U_\alpha})$ for the local frame equiv. to Φ_α

Then $s = s^a e_a \in \Gamma(E|_{U_\alpha})$ for cpnt. fns $s^a : U_\alpha \to \mathbb{F}$

$(a = 1, \ldots, m)$.

Write $\Gamma_\alpha(\partial_i, e_b) = \Gamma^a_{ib} e_a$ defns fns.

$\Gamma^a_{ib} : U_\alpha \longrightarrow \mathbb{F}$ "Christoffel symbols"

Write $\nabla_{\frac{\partial}{\partial x^i}} =: \nabla_i$, now

$$\nabla_X s = \nabla_{X^i \partial_i} s = X^i \nabla_i (s^b e_b) = X^i (\partial_i s^b \cdot e_b + s^b \nabla_i e_b)$$

$$\nabla_i e_b = \underbrace{\nabla^0_i e_b}_{=0} + \Gamma_\alpha(\partial_i, e_b) = \Gamma^a_{ib} e_a$$

$$\Rightarrow \quad \nabla_X s = X^i (\partial_i s^b \cdot e_b + s^b \Gamma^a_{ib} e_a)$$
$$= X^i (\partial_i s^a + s^b \Gamma^a_{ib}) e_a$$

$$\boxed{(\nabla_i s)^a = \partial_i s^a + \Gamma^a_{ib} s^b}$$

Γ_α deps. on choice of local triv. over U_α

Γ^a_{ib} are <u>not</u> cpnts. of any globally def'd tensor field!

Alternative perspective: Connection 1-forms

Recall: local triv. Φ_α associates to any $s \in \Gamma(E)$ its local representative $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$

$$\Phi_\alpha(s(p)) = (p, s_\alpha(p)).$$

$\Rightarrow$ can write $\boxed{(\nabla_X s)_\alpha(p) = \underbrace{d s_\alpha(X)}_{\substack{\text{trivial conn.} \\ \text{w.r.t. } \Phi_\alpha}} + A_\alpha(X)\, s_\alpha(p)}$

for $X \in T_pM$.

for an $\mathbb{F}^{m\times m}$-valued 1-form $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathbb{F}^{m\times m})$

A_α = "conn. 1-form" w.r.t. Φ_α.

EX: In local coords, $A_\alpha(\partial_i)^a{}_b = \Gamma^a_{ib}$.

EX: For another triv. Φ_β related by transition fns.

$g_{\beta\alpha} =: g : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to GL(m, \mathbb{F})$,

$$\boxed{A_\alpha(X) = g^{-1} A_\beta(X)\, g + g^{-1} dg(X)}$$

"gauge transformation"

defn: Spse $\pi : E \to M$ has a G-structure ($G \subseteq GL(m, \mathbb{F})$ a Lie subgroup). A connection is G-compatible if $\exists$ G-compat. local trivs. over subsets $\mathcal{U}_\alpha \subseteq M$ & paths γ in $\mathcal{U}_\alpha$, par. transp. takes the form $P^t_\gamma = g(t) : \underset{E_{\gamma(0)}}{\mathbb{F}^m} \to \underset{E_{\gamma(t)}}{\mathbb{F}^m}$

for some fn. $g(t) \in G$.

ex: $G = O(m)$ or $U(m)$ means E has a bndl metric, conn. is G-compat. $\Leftrightarrow$ P.T. maps preserve inner products on fibers.

thm: a conn. is G-compatible iff all conn. 1-forms A_α w.r.t. G-compat triv Φ_α take values in $\mathfrak{g}$, i.e. $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$. ($\mathfrak{g} := T_{\mathrm{Id}} G$)

pf: $\Rightarrow$: Spse γ a path in $\mathcal{U}_\alpha$, $s \in \Gamma(E)$ parallel along γ, then G-compatible $\Rightarrow$

$s_\alpha(\gamma(t)) = g(t)\, s_\alpha(\gamma(0))$ for a fn $g(t) \in G$.

Then $(\nabla_{\dot\gamma(0)} s)_\alpha = 0 = \partial_t s_\alpha(\gamma(t))\big|_{t=0} + A_\alpha(\dot\gamma(0))\, s_\alpha(\gamma(0))$

$$\Rightarrow \quad A_\alpha(\dot\gamma(0))\, s_\alpha(\gamma(0)) = \underbrace{-\dot g(0)}_{\in\, \mathfrak{g}}\, s_\alpha(\gamma(0))$$

$\Leftarrow$: P.T. comes from an ODE, in local triv., can always write in terms of a fn.

$g(t) \in GL(m, \mathbb{F})$ w/ $g(0) = \mathrm{Id}$.

Show: $\dot g(t)$ is always in $TG \subseteq T(GL(m,\mathbb{F}))$,

i.e. $g(t)$ is a flow line of a vec. fld. on $GL(m,\mathbb{F})$ that is tangent to G along $G \Rightarrow g(t) \in G \;\; \forall\, t$.