(Σ, g) = Riemannian 2-mfd, oriented

Recall: $SO(2) \cong U(1) \Rightarrow T\Sigma$ is a cpx line bdl w/ $U(1)$-str.

∇ + local triv. on $\mathcal{U}_\alpha \subseteq \Sigma \rightsquigarrow$ conn. 1-form $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{u}(1))$, $\mathfrak{u}(1) = i\mathbb{R}$

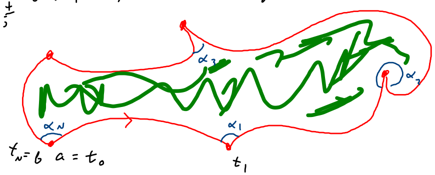
$\rightsquigarrow$ global curvature 2-form $F \in \Omega^2(\Sigma, \mathfrak{u}(1))$, $F = dA_\alpha$ on $\mathcal{U}_\alpha$

If ∇ = L-C. conn; $F = -i K_G \cdot d\mathrm{vol}$

$\Rightarrow$ on any cpt region $P \subseteq \mathcal{U}_\alpha$, $\int_P K_G \, d\mathrm{vol} = i \int_{\partial P} A_\alpha$

Write $A_\alpha = i\lambda$ for $\lambda \in \Omega^1(\mathcal{U}_\alpha)$, $\int_P K_G \, d\mathrm{vol} = -\int_{\partial P} \lambda$.

Defn: A smooth polygon P in $\mathbb{R}^2$ is a cpt region bdd by a simple closed curve $\gamma : [a,b] \to \mathbb{R}^2$, smooth & embedded except at fin. many pts $a < t_1 < \dots < t_{N-1} < b$ w/ $\lim_{t \to t_j^\pm} \dot\gamma(t) \neq 0$ exist but maybe $\neq$.



$\Rightarrow \exists$ angles $\alpha_j \in [0, 2\pi]$ at each vertex ($t_1, \dots, t_{N-1}$, also $t_0 := a$, $t_N := b$).

Write $\dot\gamma(t) = r(t) e^{i\phi(t)} \in \mathbb{C} = \mathbb{R}^2$ for $r(t) > 0$, $\phi(t) \in \mathbb{R}$ smooth on $(a,b) \setminus \{t_1, \dots, t_{N-1}\}$.

Write $\Delta\phi_j := \lim_{t \to t_j^+} \phi(t) - \lim_{t \to t_j^-} \phi(t) \in [-\pi, \pi]$, so $\Delta\phi_j = \pi - \alpha_j$ for $j = 1, \dots, N-1$. Now ϕ is unique mod 2π.

Write $\Delta\phi_N := \phi(a) - \phi(b) + 2\pi k$ for some $k \in \mathbb{Z}$ s.t. $\in [-\pi, \pi]$ & $\Delta\phi_N = \pi - \alpha_N$.

Lemma: $\int_a^b \dot\phi(t)\, dt + \sum_{j=1}^N \Delta\phi_j = 2\pi$. $\boxtimes$

Defn: A smooth polygon $P \subseteq \Sigma$ is a cpct subset s.t. some nbhd of P admits a chart sending P to a smooth polyg. in $\mathbb{R}^2$.

Defn. a frame X for $T\Sigma$ over P as $g\,\partial_1$ (using chart) for a fn. $g : P \longrightarrow (0,\infty)$ s.t. $|X| \equiv 1$.

Parametrize ∂P by a path $\gamma : [0,T] \longrightarrow \Sigma$ w/ $|\dot\gamma| \equiv 1$ except at nonsmooth pts. $t_0 := 0 < t_1 < \ldots < t_{N-1} < T =: t_N$.

$\alpha_j :=$ angle $\in [0, 2\pi]$ at vertex t_j, sim. α_N at $\gamma(0) = \gamma(T)$.

Edges of ∂P are the smooth curves $\gamma([t_{j-1}, t_j])$.

Each edge $\ell \subseteq \partial P$ inherits from Σ a bndry orientation, $\rightsquigarrow$ volume form $d\mathrm{vol}_{\partial P} \in \Omega^1(\ell)$.

Now $\dot\gamma(t) = e^{i\theta(t)} X(\gamma(t))$ for some $\theta : [a,b] \setminus \{t_1, \ldots, t_{N-1}\} \longrightarrow \mathbb{R}$,

$\Delta\theta_j := \lim_{t \to t_j^+} \theta(t) - \lim_{t \to t_j^-} \theta(t) \in [-\pi, \pi]$, $\Delta\theta_j = \pi - \alpha_j$

$\Delta\theta_N$ sim.

Lemma: $\boxed{\int_0^T \dot\theta(t)\,dt + \sum_{j=1}^N \Delta\theta_j = 2\pi.}$

Pf: Suff. to assume $P \subseteq \mathbb{R}^2$ but w/ a nonstandard metric g.

If $g =$ Eucl., follows from previous lemma.

Sum is always $\in 2\pi\mathbb{Z}$.

Space of Riem. metrics is convex $\Rightarrow$ can deform g to Eucl. metric.

$\Rightarrow$ smooth deformation of $2\pi k$ through $2\pi\mathbb{Z}$ to 2π $\Rightarrow k = 1$. ∎

Recall $A_\alpha = i\lambda$ defn $\lambda \in \Omega^1(\text{nbhd}(P))$.

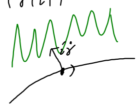
Pick an edge $\ell_j := \gamma([t_{j-1}, t_j]) \subseteq \partial P$.

goal: $\int_{\ell_j} \lambda = ?$

$$\nabla_t \dot\gamma(t) = \left(\partial_t e^{i\theta(t)} + A_\alpha(\dot\gamma(t))\, e^{i\theta(t)}\right) X(\gamma(t))$$

$$= \left(\dot\theta(t) + \lambda(\dot\gamma(t))\right) i e^{i\theta(t)} X(\gamma(t))$$

$$= \left(\dot\theta(t) + \lambda(\dot\gamma(t))\right) i \dot\gamma(t)$$



defn: For a 1-dim. submfd $\ell \subseteq \Sigma$ w/ chosen normal vec. fld $\nu \in \Gamma(T\Sigma|_\ell)$. The geodesic curvature $\kappa_\ell : \ell \to \mathbb{R}$ is ! fn. s.t. for any parametrization $\gamma : (a,b) \to \ell$ w/ $|\dot\gamma| = 1$,

$$\boxed{\nabla_t \dot\gamma = \kappa_\ell \cdot \nu(\gamma)} \qquad (\Rightarrow \kappa_\ell = 0 \text{ iff } \ell \text{ is a geodesic})$$

$\Rightarrow$ lemma: $\dot\theta(t) + \lambda(\dot\gamma(t)) = \kappa_{\ell_j}(\gamma(t))$. ☒

Now $\int_{\partial P} \lambda = \sum_{j=1}^N \int_{\ell_j} \lambda = \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \lambda(\dot\gamma(t))\, dt$

$= -\int_P K_0\, d\mathrm{vol}_\Sigma$

$$= \sum_j \underbrace{\int_{t_{j-1}}^{t_j} \kappa_{\ell_j}(\gamma(t))\, dt}_{\int_{\ell_j} \kappa_{\ell_j}\, d\mathrm{vol}_{\partial P}} - \underbrace{\int_0^T \dot\theta(t)}_{\sum_j \Delta\theta_j - 2\pi}$$

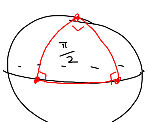
$$= \sum_j \int_{\gamma_j} \kappa_{\ell_j}\, d\mathrm{vol}_{\partial P} + \sum_j (\pi - \alpha_j) - 2\pi = \sum_j \int_{\ell_j} \kappa_{\ell_j}\, d\mathrm{vol}_{\partial P} + (N-2)\pi - \sum_j \alpha_j$$

$\Rightarrow$ thm (Gauss-Bonnet I):

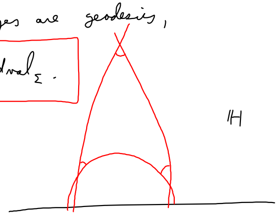
$$\boxed{\sum_{j=1}^N \alpha_j = (N-2)\pi + \int_P K_0\, d\mathrm{vol}_\Sigma + \sum_{j=1}^N \int_{\ell_j} \kappa_{\ell_j}\, d\mathrm{vol}_{\partial P}.}$$ ☒

cor: For a polygon whose edges are geodesics,

$$\boxed{\sum_j \alpha_j = (N-2)\pi + \int_P K_0\, d\mathrm{vol}_\Sigma.}$$



S^2



$\mathbb{H}$

defn: a *polygonal triangulation* of Σ is a set of sm. polygons $\{P_\alpha \subseteq \Sigma\}_{\alpha \in I}$ s.t. $\bigcup_\alpha P_\alpha = \Sigma$ &:

(1) Each edge ℓ either is contained in $\partial\Sigma$ or $\ell \cap \partial\Sigma \subseteq \partial\ell$, & if latter case, ℓ is an edge of exactly 2 of the "faces" P_α

(2) for $\alpha \neq \beta$, $P_\alpha \cap P_\beta$ is empty or a union of common edges

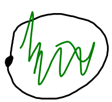
(3) Every vertex is a vertex of at most fin.-many faces.

defn: Given a finite triang of Σ w/ v vertices, e edges, f faces, the *Euler characteristic* of Σ is

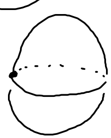
$$\chi(\Sigma) := v - e + f.$$

We'll show: $\chi(\Sigma)$ indep. of choice of triang.

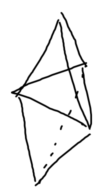
ex:



$\chi(\mathbb{D}^2) = 1 - 1 + 1 = 1.$

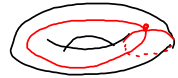


$\chi(S^2) = 1 - 1 + 2 = 2.$



$\chi(S^2) = 5 - 9 + 6 = 2.$

$\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0.$

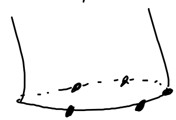


Assume Σ has a finite polyg. triang. ($\Rightarrow \Sigma$ cpct),

w/ $v = v_0 + v_\partial$ vertices (v_∂ = # on bdry)

$e = e_0 + e_\partial$ edges

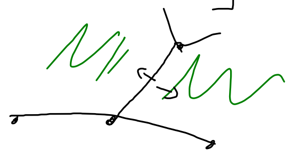
f = faces



$e_\partial = v_\partial$.

$$\int_\Sigma K_G \, dvol_\Sigma = \sum_{\text{face}} \int_{\text{face}} K_G \, dvol$$

$$= \sum_{\text{faces}} \left[-\sum_{\text{edges}} \int \kappa_g \, dvol_{\partial r} + \sum_j \alpha_j + (2-N)\pi \right]$$



<u>Sum contains</u>:

(1) $-\int_{\partial\Sigma} \kappa_{\partial\Sigma} \, dvol_{\partial\Sigma}$

(2) 2π for each interior vertex,
π for " bdry vertex $\Rightarrow 2\pi v_0 + \pi v_\partial$
$= 2\pi v - \pi v_\partial$

(3) $\sum_{\text{faces}} (2-N)\pi = 2\pi f - 2\pi e_0 - \pi e_\partial = 2\pi(f-e) + \pi e_\partial$

$\Rightarrow$ <u>thm (Gauss-Bonnet 2)</u>: $\boxed{\int_\Sigma K_G \, dvol_\Sigma + \int_{\partial\Sigma} \kappa_{\partial\Sigma} \, dvol_{\partial\Sigma} = 2\pi \chi(\Sigma)}$

<u>cor</u>: $\chi(\Sigma)$ indep. of triangulation.

<u>cor</u>: For any metric on a closed surface, $\int_\Sigma K_G \, dvol_\Sigma$ is always the same mult. of 2π.

<u>cor</u>: $\forall$ metrics on S^2, $K_G > 0$ somewhere.