Chern-Weil theory:

goal: Connection ∇ on bndl $E \xrightarrow{\pi} M$,
curvature $\rightsquigarrow$ topol. invt of VBs

special case: $\mathrm{rk}(E) = 1$ over $\mathbb{F} := \mathbb{C}$; choose bndl metric $\langle \, , \rangle$
$\rightsquigarrow$ str. grp. $U(1)$. (Recall: $\mathfrak{u}(1) = i\mathbb{R}$)

Lemma: For any 2 $U(1)$-compat. conns. $\nabla, \hat{\nabla}$ on $E \xrightarrow{\pi} M$
w/ curvature 2-forms $F, \hat{F} \in \Omega^2(M, \mathfrak{u}(1))$, $\exists \, \lambda \in \Omega^1(M)$
s.t. $\hat{F} = F + i\, d\lambda$.

pf: $\exists$ bilinear bndl map $B: TM \oplus E \to E$ s.t.
$$\hat{\nabla}_X s = \nabla_X s + B(X, s) =: \nabla_X s + \beta(X) s \quad \text{for}$$
$\beta \in \Omega^1(M, \mathrm{End}(E))$. Since $\dim E_p = 1$, $\mathrm{End}(E_p) = \mathbb{C}$
$\Rightarrow$ actually $\beta \in \Omega^1(M, \mathbb{C})$.
$\Rightarrow$ In a local triv. $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}$,
$$(\nabla_X s)_\alpha = \partial_X s_\alpha + A_\alpha(X) s, \qquad (\hat{\nabla}_X s)_\alpha = \partial_X s_\alpha + \hat{A}_\alpha(X) s$$
$\Rightarrow \hat{A}_\alpha(X) - A_\alpha(X) = \beta(X)$. Since $A_\alpha, \hat{A}_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{u}(1))$,
β also valued in $\mathfrak{u}(1) = i\mathbb{R}$, so $\beta = i\lambda$ for some $\lambda \in \Omega^1(M)$.
$\Rightarrow \hat{F} - F = d\beta = i\, d\lambda$. $\square$

defn: The 1st Chern class of the cpx. line bndl $E \to M$ is
$$c_1(E) := \left[-\frac{1}{2\pi i} F \right] \in H^2_{dR}(M)$$
for any choice of bndl metric $\langle \, , \rangle$ & any choice of $U(1)$-compat. conn. ∇ w/ curvature 2-form F.

<u>thm</u>: (1) $c_1(E)$ is indep of choice of bndl metric.

(2) If $\exists$ bndl iso.
$$\begin{tikzcd} E \arrow[r] \arrow[dr] & E' \arrow[d] \\ & M \end{tikzcd}$$
then $c_1(E) = c_1(E')$.

(3) E trivial $\Rightarrow c_1(E) = 0$.

(4) For any $E \to M$ & $f : N \to M$, $c_1(f^*E) = f^* c_1(E)$
$$\cap$$
$$H^2_{dR}(N).$$

<u>Pf</u>: (2) $\Psi : E \to E'$ bndl iso. preserving bndl metrics.

Any $U(1)$-compat. conn. ∇ & loc. triv. Φ_α on E,

can push forward through Ψ to defn. conn. & triv. on E'

s.t. conn. 1-forms are the <u>same</u>. $\Rightarrow$ curv. 2-forms also same.

(3) $\exists$ global triv. $\Rightarrow \exists$ global conn. 1-form $A_\alpha \in \Omega^1(M, \mathfrak{u}(1))$

$\Rightarrow F = dA_\alpha$ is exact.

(1) $\langle\, ,\rangle_0$ & $\langle\, ,\rangle_1$ bndl metrics on E.

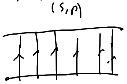
$\rightsquigarrow$ family of bndl metrics $\langle\, ,\rangle_s := s\langle\, ,\rangle_1 + (1-s)\langle\, ,\rangle_0$

for $s \in [0,1]$.

Let $\hat{E} := pr^* E \to [0,1] \times M$ for proj. $[0,1] \times M \xrightarrow{pr} M$

i.e. $\hat{E}_{(s,p)} = E_p$. defn. bndl metric on $\hat{E}$ s.t.

$\langle\, ,\rangle_{(s,p)} := \langle\, ,\rangle_s$ at p. Choose a metric conn. on $\hat{E}$,



P.T. along paths $s \mapsto (s,p)$ defn.

iso. $(E_p, \langle\, ,\rangle_0) \to (E_p, \langle\, ,\rangle_1)$.

(1) follows from (2).

(4) Given ∇ on $E \to M$ & loc. triv. $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}$,

$\exists$ pullback conn. on $f^*E \to N$, & pullback triv

$f^*\Phi_\alpha : f^*E\big|_{f^{-1}(\mathcal{U}_\alpha)} \to f^{-1}(\mathcal{U}_\alpha) \times \mathbb{C}$ s.t. conn. 1-form becomes

$f^* A_\alpha \in \Omega^1(f^{-1}(\mathcal{U}_\alpha), \mathfrak{u}(1))$. $\Rightarrow f^*F$ is the curvature 2-form for pullback conn.

$\boxtimes$

<u>defn</u>: For Σ a closed oriented surface & $E \to \Sigma$ a cpx line bndl, the <u>1st Chern number</u> is

$$\int_\Sigma c_1(E) := \int_\Sigma \omega \quad \text{for a 2-form } \omega \text{ representing } c_1(E) \in H^2_{dR}(M).$$

<u>rk</u>: If $\underset{= d\alpha}{\omega' - \omega}$ exact, then $\int_\Sigma \omega' - \int_\Sigma \omega = \int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha = 0.$

<u>rk</u>: For $E \to M$, $c_1(E)$ is determined by the number

$\int_\Sigma f^* \omega$ for all possible closed oriented surfaces Σ, maps $f : \Sigma \to M$, $\omega \in \Omega^2(M)$ representing $c_1(E)$.

<u>note</u>: $[f^*\omega] = f^*[\omega] = f^* c_1(E) = c_1(f^* E)$.

Assume $\Sigma = \Sigma_\alpha \cup \Sigma_\beta$ for cpct subsurfaces w/ matching bndry

$\partial \Sigma_\beta = \partial \Sigma_\alpha = \coprod_{j=1}^N C_j$, $C_j \cong S^1$. Orient C_j as $\partial \Sigma_\beta$.
($\partial \Sigma_\alpha$ is oriented oppositely)

Also: $\Sigma_\alpha \subseteq U_\alpha \overset{\text{open}}{\subseteq} \Sigma$, $\Sigma_\beta \subseteq U_\beta \overset{\text{open}}{\subseteq} \Sigma$ s.t. $\exists$ loc. trivs. Φ_α & Φ_β on these nbhds,

denote $g := g_{\beta\alpha} : U_\alpha \cap U_\beta \to U(1)$.

$$\int_\Sigma c_1(E) = -\frac{1}{2\pi i} \int_\Sigma F = -\frac{1}{2\pi i}\left(\int_{\Sigma_\alpha} dA_\alpha + \int_{\Sigma_\beta} dA_\beta \right)$$

$$= -\frac{1}{2\pi i}\left(\int_{\partial\Sigma_\alpha} A_\alpha + \int_{\partial\Sigma_\beta} A_\beta \right) = -\frac{1}{2\pi i} \sum_{j=1}^N \int_{C_j} (A_\beta - A_\alpha).$$

<u>PSET 10 #1</u>: $A_\alpha(X) = g^{-1} A_\beta(X)\, g + g^{-1} dg(X) = A_\beta(X) + g^{-1} dg(X)$

$\Rightarrow \int_\Sigma c_1(E) = \frac{1}{2\pi i} \sum_{j=1}^N \int_{C_j} g^{-1} dg$.

$\exists!$ smooth map $\theta : U_\alpha \cap U_\beta \to \mathbb{R}/2\pi\mathbb{Z}$ s.t. $g = e^{i\theta}$,

$g^{-1} dg = e^{-i\theta} d(e^{i\theta}) = e^{-i\theta}\, i e^{i\theta}\, d\theta = i\, d\theta$, $d\theta \in \Omega^1(U_\alpha \cap U_\beta)$

$$\Rightarrow \boxed{\int_\Sigma c_1(E) = \frac{1}{2\pi} \sum_{j=1}^N \int_{C_j} d\theta}$$

EX: $\frac{1}{2\pi}\int_{C_j} d\theta = \text{wind}_{C_j}(g) :=$ the ! $k \in \mathbb{Z}$ s.t.

if $\gamma : [0,1] \to C$ is an orient.-pres. parametrization
(i.e. $\gamma(0) = \gamma(1)$ s.t. $(0,1) \xrightarrow{\cong} C \setminus \{pt.\}$),
$g(\gamma(t)) = r(t) e^{i\phi(t)}$ for some smooth $r(t) > 0$,
$\phi(t) \in \mathbb{R}$, $\phi(1) = \phi(0) + 2\pi k$.

$\Rightarrow$ prop: $\boxed{\int_\Sigma c_1(E) = \sum_{j=1}^N \text{wind}_{C_j}(g) \in \mathbb{Z}.}$

Q: "How many" zeroes should a section $s \in \Gamma(E)$ have?

Spse $s^{-1}(0) := \{ p \in \Sigma \mid s(p) = 0 \in E_p \} \subseteq \Sigma$ is finite.

defn: For an isolated zero $p \in s^{-1}(0)$, the index / order of p
is $\text{ind}(s;p) := \text{wind}_{\partial \mathcal{D}}(s_\alpha) \in \mathbb{Z}$

for $\mathcal{D} \subseteq \Sigma$ a $\underbrace{\text{small disk}}$ encircling p, s_α = local rep. of
s w.r.t. a triv. def'd on $\mathcal{D}$. small enough $\mathcal{D} \cap s^{-1}(0) = \{p\}$.

EX: $\text{ind}(s;p)$ is indep. of choices.
Hint: winding #'s are homotopy-inv't!

defn: The algebraic count of zeroes of $s \in \Gamma(E)$ is

$$\# s^{-1}(0) := \sum_{p \in s^{-1}(0)} \text{ind}(s;p) \in \mathbb{Z}.$$

thm: $\# s^{-1}(0) = \int_\Sigma c_1(E)$.

pf: Choose small disks $\mathcal{D}_p \subseteq \Sigma$ around each $p \in s^{-1}(0)$,
s.t. $\mathcal{D}_p \cap \mathcal{D}_q = \emptyset$ $\forall p \neq q$, set $\Sigma_\beta := \coprod_{p \in s^{-1}(0)} \mathcal{D}_p \subseteq \mathcal{U}_\beta$
s.t. $(\mathcal{U}_\beta, \Phi_\beta)$ is a local triv. $\Sigma_\alpha := \overline{\Sigma \setminus \Sigma_\beta}$.
Let $\mathcal{U}_\alpha := \Sigma \setminus s^{-1}(0)$, defn. Φ_α corresponding to frame
$\frac{s}{|s|}$. Now $s_\alpha : \mathcal{U}_\alpha \to \mathbb{C}$ takes values in $(0,\infty) \subseteq \mathbb{R}$.
Then $s_\beta = g s_\alpha \Rightarrow \text{wind}_{\partial \mathcal{D}_p}(s_\beta) = \text{wind}_{\partial \mathcal{D}_p}(g)$. ☒

Case $E = T\Sigma$, ∇ = L.-C. conn.

$$\int_\Sigma c_1(E) = -\frac{1}{2\pi i}\int_\Sigma F = \frac{1}{2\pi}\int_\Sigma iF = \frac{1}{2\pi}\int_\Sigma K_G \, d\text{vol}_\Sigma = \chi(\Sigma).$$

$\Rightarrow$ Poincaré-Hopf thm: $\forall$ vec. flds. $X \in \mathfrak{X}(\Sigma)$ w/ at most
fin-many zeroes on a closed oriented surface Σ,

$$\boxed{\# X^{-1}(0) = \chi(\Sigma)}.$$ ☒

cor: S^2 admits no nowhere-zero vec. flds. ☒