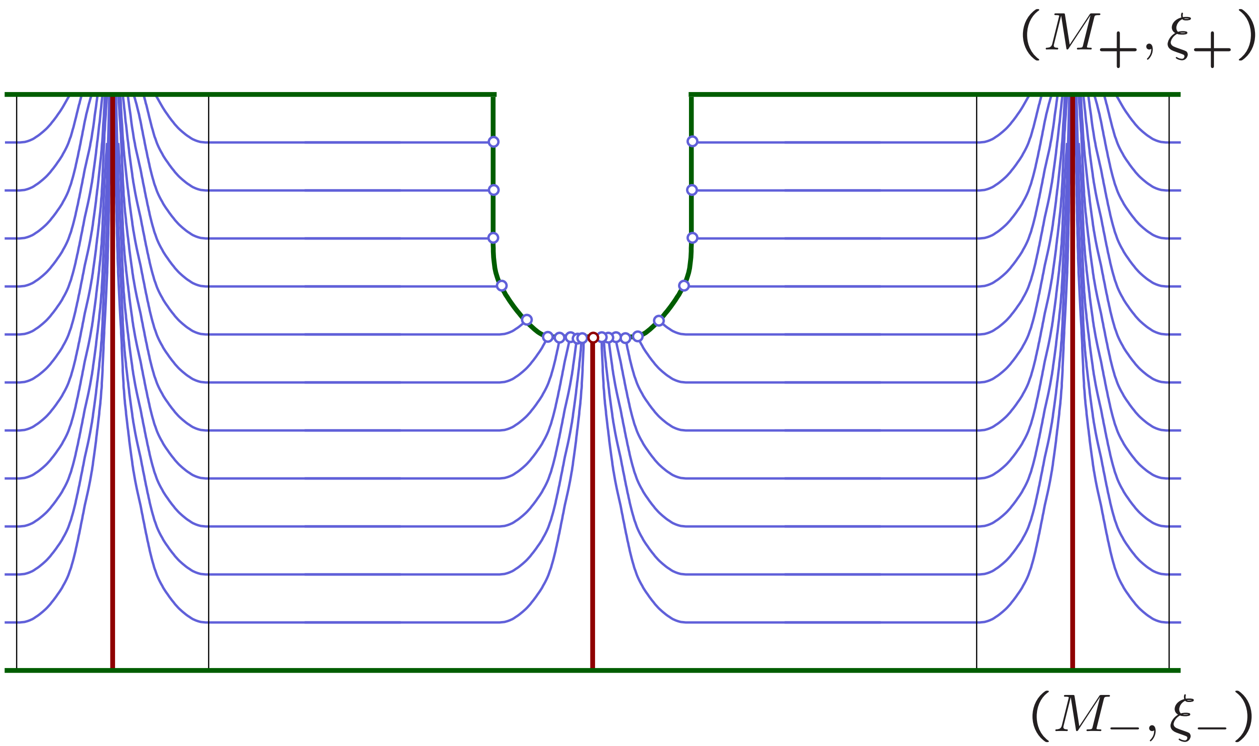
On Symplectic Cobordisms Between Contact Manifolds



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Slides available at:

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Prologue

The following famous quotation is due to George Orwell:

> *All animals are equal, but some animals are more equal than others.*

The following is not:

> *Most contact manifolds are non-fillable, but some are more non-fillable than others.*

Outline

- Part 1: On Symplectic Fillings
- Part 2: On Symplectic Cobordisms
- Part 3: A Hierarchy of Obstructions
- Part 4: Open Books and Fiber Sums
- Part 5: Non-Exact Cobordisms
  (or *some low-tech proofs of results that used to seem hard*)

Part 1
On Symplectic Fillings

Definitions

(W, ω) compact, symplectic, $\partial W = M$.
Assume η is a Liouville vector field, i.e.

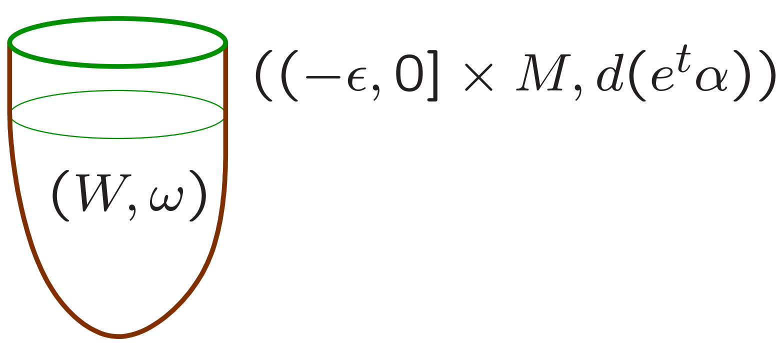
$$\mathcal{L}_\eta \omega = \omega,$$

defined near ∂W and pointing transversely outward. Then

$$\lambda := \iota_\eta \omega$$

satisfies $d\lambda = \omega$ and is a positive contact form on M, defining a contact structure $\xi = \ker \lambda$.

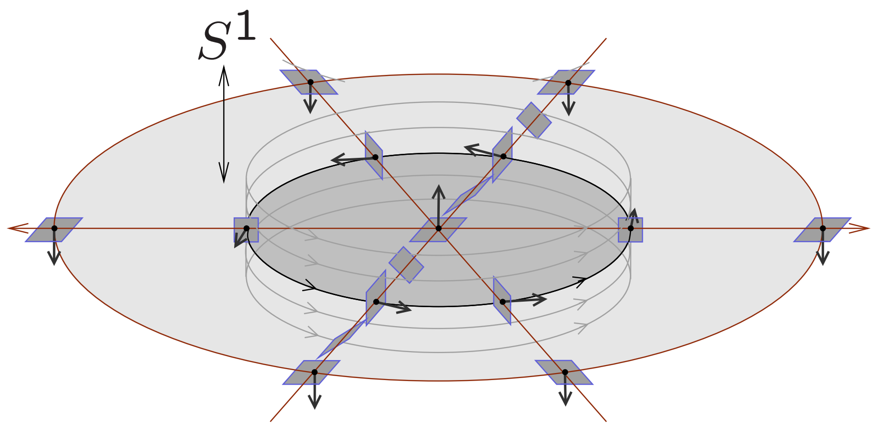
(W, ω) is a **strong** (symplectic) **filling** of (M, ξ).



(W, ω) is an **exact filling** of (M, ξ) $\Longleftrightarrow$
η (or equivalently λ) exists globally.

Gromov '85, Eliashberg '89

(M, ξ) overtwisted $\Rightarrow$ not fillable.



Proof requires technology:
e.g. *holomorphic curves, Seiberg-Witten, Heegaard Floer...*

A modern proof: overtwisted $\Rightarrow$
the *ECH contact invariant* vanishes.

Recall **Embedded Contact Homology**:

Assume $\dim M = 3$ and choose:
- Contact form α for ξ
- Compatible J on $\mathbb{R} \times M$

Choices $\rightsquigarrow$

- Chain complex $C_*(M, \alpha)$ generated by sets of Reeb orbits
- Differential $\partial : C_*(M, \alpha) \to C_*(M, \alpha)$ counting embedded J-holomorphic curves in $\mathbb{R} \times M$.

$$\mathrm{ECH}_*(M, \alpha, J) := H_*(C_*(M, \alpha), \partial)$$

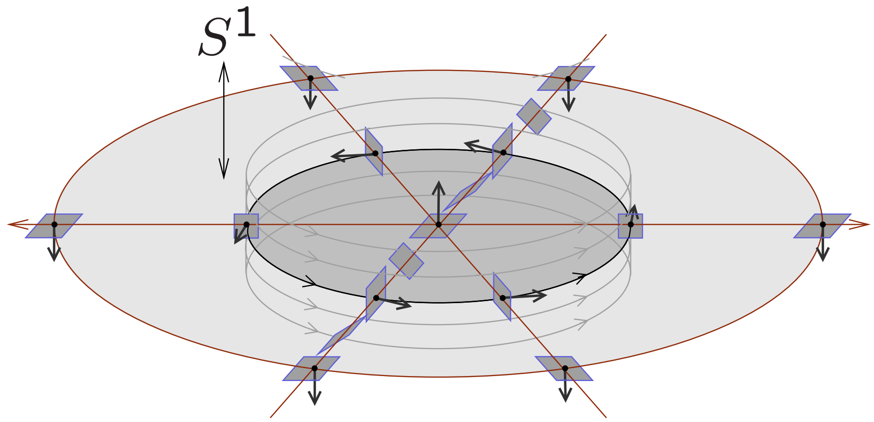
matches the Seiberg-Witten Floer homology of M (Taubes '08).

ECH contact invariant := "homology class of the *empty* orbit set"

$$c_{\mathrm{ech}}(\xi) = [\emptyset] \in \mathrm{ECH}_*(M, \alpha, J).$$

Taubes '08 + Kronheimer-Mrowka '97: $c_{\mathrm{ech}}(\xi)$ is an invariant of (M, ξ), and is nonzero whenever (M, ξ) is strongly fillable.

(M,ξ) overtwisted $\Rightarrow$ contains a *"Lutz tube"* (Eliashberg classification '89)



$\Rightarrow$ an orbit γ spanned by a unique embedded rigid J-holomorphic **plane**. Thus

$$\partial(\gamma) = \emptyset,$$

so $c_{\text{ech}}(\xi) = [\emptyset] = 0$, $\Rightarrow$ not fillable. $\square$

Remark 1

Same argument proves trivial contact homology: $HC_*(M,\xi) = \{1\}$.

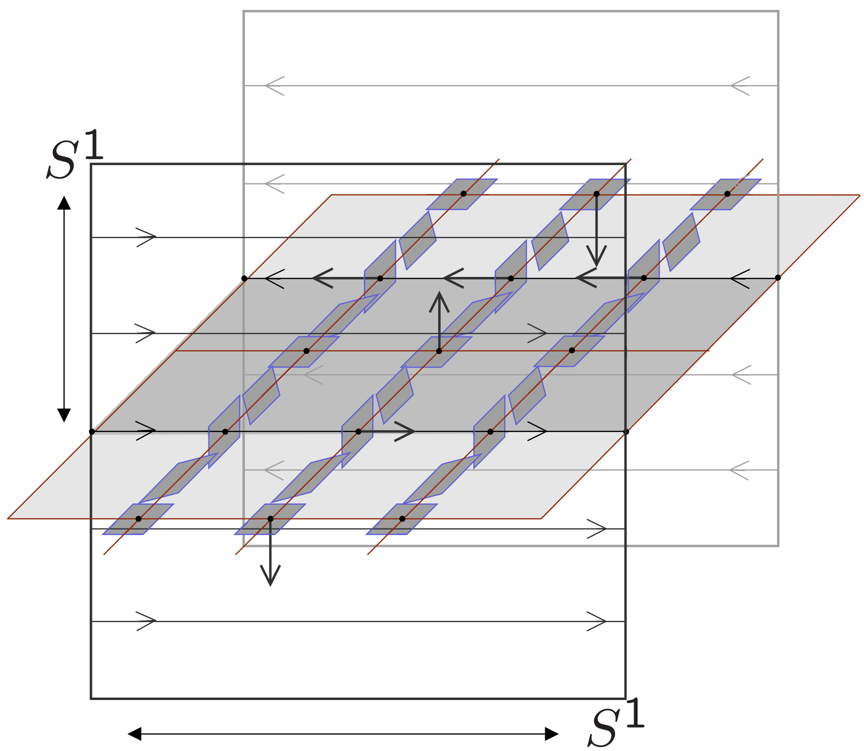
Remark 2

Conjecturally, $c_{\text{ech}}(\xi)$ is equivalent to the Ozsváth-Szabó contact invariant in Heegaard Floer homology.

D. Gay '06:
(M, ξ) has Giroux torsion $\geq 1 \Rightarrow$ not fillable.

Recall:
(M, ξ) has *Giroux torsion* N if it contains $[0, 1] \times T^2 \ni (s, \phi, \theta)$ with contact structure

$$\xi_N := \ker [\cos(2\pi N s)\, d\theta + \sin(2\pi N s)\, d\phi] \,.$$



Proof by ECH: count holomorphic **cylinders**
$\Rightarrow \partial(\gamma_1 \gamma_2) = \emptyset \Rightarrow c_{\text{ech}}(\xi) = 0$. $\square$

(Corresponding Heegaard result by
Ghiggini, Honda, Van Horn-Morris '07.)

Part 2
On Symplectic Cobordisms

Definitions

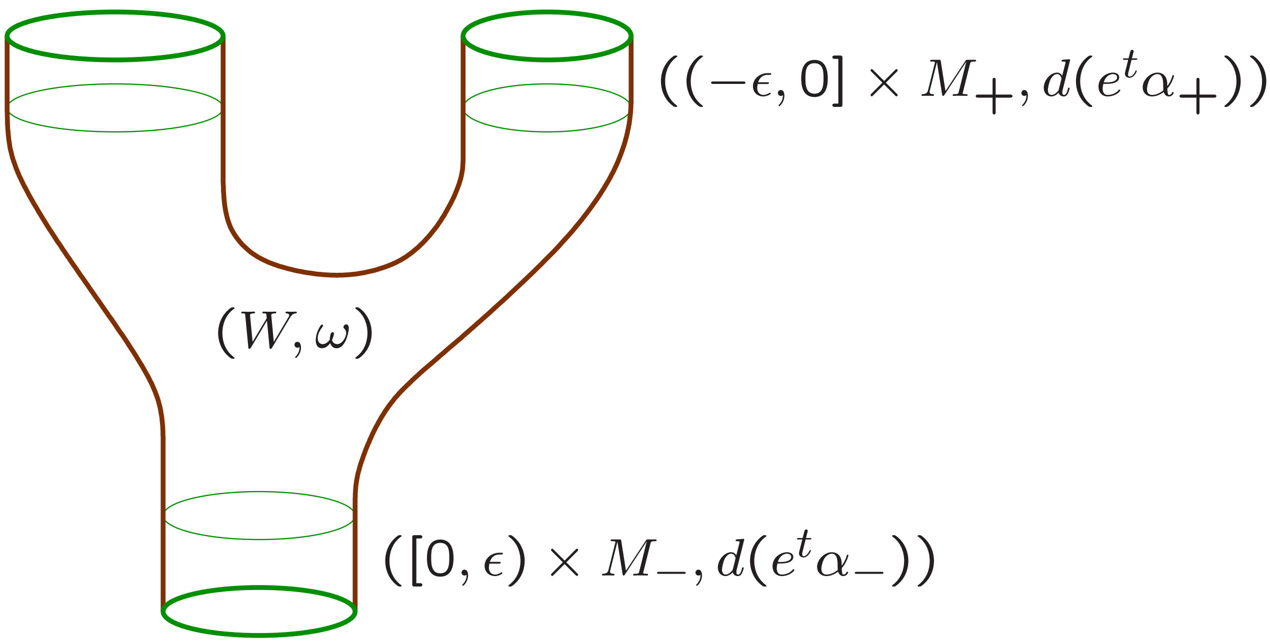
(W, ω) compact, symplectic,

$$\partial W = M_+ \sqcup (-M_-),$$

with Liouville vector field η near ∂W pointing outward at M_+ and inward at M_-.

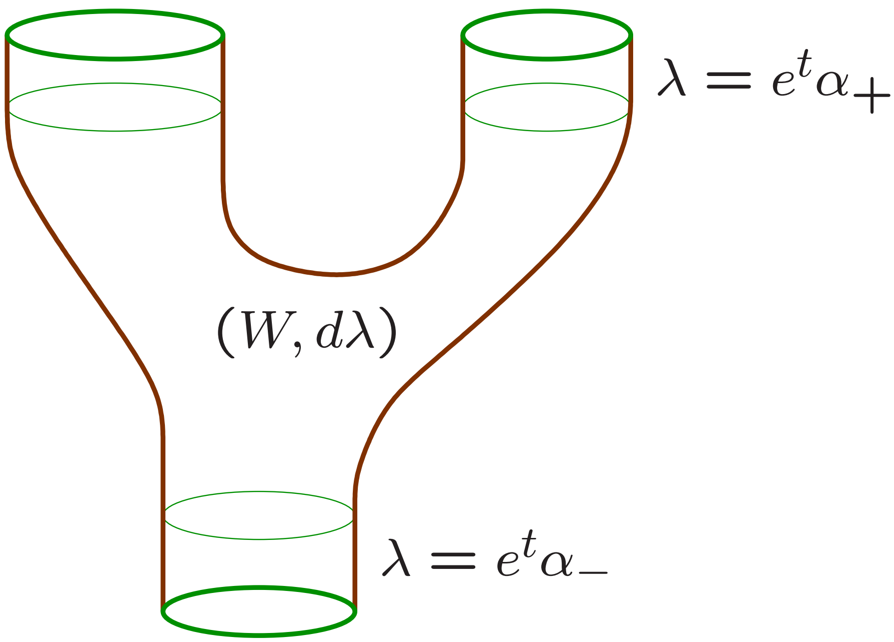
Call this a **symplectic cobordism** from (M_-, ξ_-) to (M_+, ξ_+), and write

$$(M_-, \xi_-) \preccurlyeq (M_+, \xi_+).$$



If η exists globally, call (W, ω) an **exact cobordism** and write

$$(M_-, \xi_-) \prec (M_+, \xi_+).$$



Observe $M_- \prec M_+$ implies $M_- \preccurlyeq M_+$.

Each is a preorder (reflexive and transitive) on the contact category.

Some facts about cobordisms

Abbreviate $M = (M, \xi)$.
Let M_{ot} *denote anything overtwisted.*

- $\emptyset \preccurlyeq M \Leftrightarrow$ fillable ; $\emptyset \prec M \Leftrightarrow$ exactly fillable
- No M satisfies $M \prec \emptyset$. (Stokes theorem)
- All M satisfy $M \preccurlyeq \emptyset$. (Etnyre-Honda '02)
- If $M_- \preccurlyeq M_+$ and M_- is fillable, then M_+ is also fillable. For example,

  $$M \preccurlyeq M_{\text{ot}} \quad \Rightarrow \quad M \text{ not fillable}.$$

- $M_{\text{ot}} \prec M$ for **all** M. (Etnyre-Honda '02)

Are overtwisted contact manifolds more non-fillable than some others?

Is there a non-fillable M such that

$$M \not\preccurlyeq M_{\text{ot}}$$

for all overtwisted M_{ot}?

Yes:
$M \not\prec M_{\text{ot}} \Rightarrow$ by adapting a holomorphic disk argument due to Hofer, M always has a contractible Reeb orbit.

There are non-fillable examples without contractible orbits, e.g. (T^3, ξ_N) for $N \geq 2$
($\Rightarrow$ Giroux torsion $N-1$).

We'll show:
these *do* admit non-exact cobordisms to some M_{ot} (a result of Gay '06 for $N \geq 3$).

Exercise for bored listeners:
There are symplectic cobordisms from (T^3, ξ_{std}) to (S^3, ξ_{std}), but they are never exact.

Part 3
A Hierarchy of Obstructions

Theorem (joint with J. Latschev)
For closed contact manifolds (M, ξ) in all dimensions, one can use Symplectic Field Theory to define the *algebraic torsion*

$$\mathsf{AT}(M, \xi) = \inf \left\{ k \geq 0 \;\middle|\; [\hbar^k] = 0 \in H_*^{\mathsf{SFT}}(M, \xi) \right\} \in \mathbb{N} \cup \{0, \infty\},$$

which has the following properties:

1. $\mathsf{AT}(M, \xi) < \infty \Rightarrow$ not strongly fillable.
2. $HC_*(M, \xi) = \{1\} \Leftrightarrow \mathsf{AT}(M, \xi) = 0$
3. positive Giroux torsion $\Rightarrow \mathsf{AT}(M, \xi) \leq 1$.
4. For every integer $k \geq 0$, there are examples (M_k, ξ_k) with $\mathsf{AT}(M_k, \xi_k) = k$.
5. $(M_-, \xi_-) \prec (M_+, \xi_+) \Rightarrow$
   $\mathsf{AT}(M_-, \xi_-) \leq \mathsf{AT}(M_+, \xi_+)$.

Morally:
"Larger $\mathsf{AT}(M, \xi) \cong$ *closer to fillability*."

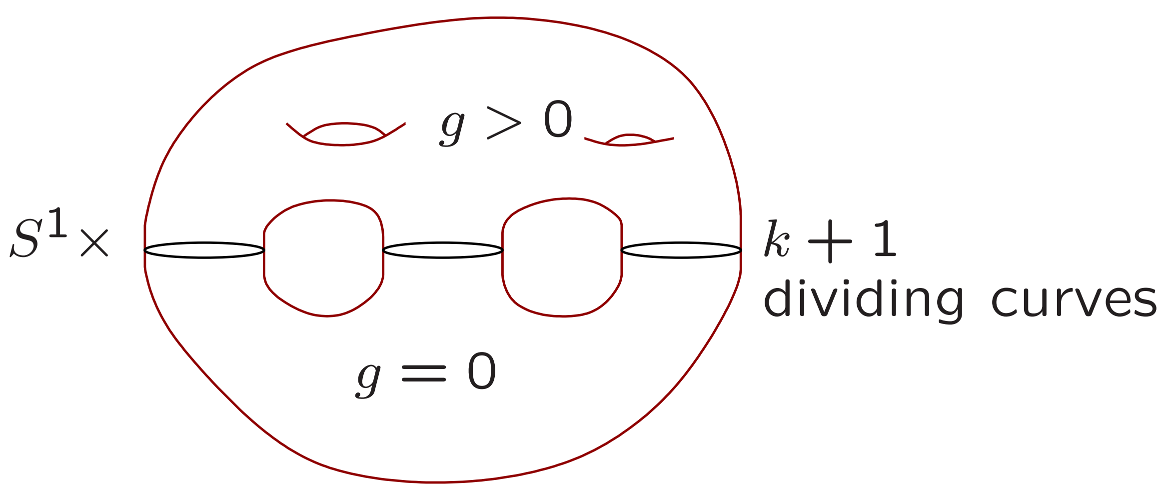
Remark 1
As we'll see, all examples I know for which $\mathrm{AT}(M) < \infty$ satisfy:

1. ECH contact invariant $= 0$
2. $M \preccurlyeq M_{\mathrm{ot}}$

Hence by Etnyre-Honda, they are (non-exactly!) cobordant to everything.

Remark 2
An analogue of $\mathrm{AT}(M, \xi)$ can be defined via ECH. Heegaard???

The examples (M_k, ξ_k)



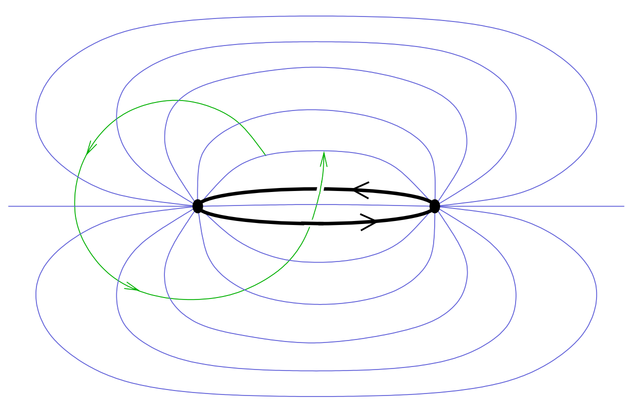
Part 4
Open Books and Fiber Sums

Initial Goal:
Find more general contact subdomains (M_0, ξ_0) (possibly with boundary) such that

$$(M_0, \xi_0) \hookrightarrow (M, \xi) \quad \Rightarrow \quad c_{\text{ech}}(\xi) = 0.$$

Observation:
Informally, there is a correspondence (Hofer-Wysocki-Zehnder, Abbas, W.)

pages of supporting open books
$\longleftrightarrow$
embedded J-holomorphic curves



$$\pi : M \setminus B \to S^1$$

Two operations on open books (and contact structures)

1. *Blow up a binding component* $\gamma \subset B$:
   Replace γ with $\widehat{\gamma} := (\nu\gamma \setminus \gamma)/\mathbb{R}_+ \cong T^2$.
   $\rightsquigarrow$ natural basis $\{\lambda, \mu\} \in H_1(\widehat{\gamma})$.

2. *Binding sum of* $\gamma_1, \gamma_2 \subset B$:
   Blow up both and attach such that
   $\lambda \mapsto \lambda$, $\mu \mapsto -\mu$.

   $\cong$ *contact fiber sum* along γ_1, γ_2
   (Gromov, Geiges)

   $\gamma_1 \cup \gamma_2$ replaced by one "*interface*" torus.

Definitions

Blown up summed open book :=
result of blowing up and/or summing some binding components of an open book.

$\rightsquigarrow$ compact mfd. M (maybe with boundary), and fibration

$$\pi : M \setminus (B \cup \mathcal{I}) \to S^1$$

Here:

- B (the *"binding"*) = a link
- $\mathcal{I}$ (the *"interface"*) = a disjoint union of 2-tori with homology bases $(\lambda, \pm\mu)$
- ∂M = 2-tori with homology bases (λ, μ)

pages := connected components of fibers.
π is *irreducible* $\Leftrightarrow$ fibers connected.

Planar := irreducible with genus 0 pages.

Any blown up summed open book decomposes into *irreducible subdomains*

$$M = M_1 \cup \ldots \cup M_n$$

glued along interface tori.

Definition

The decomposition *supports* a contact structure ξ on M if there is a Reeb vector field X such that:

1. X is positively transverse to all pages
2. X is positively tangent to all boundaries of pages
3. Characteristic foliation at $\mathcal{I} \cup \partial M$ is parallel to $\pm\mu$

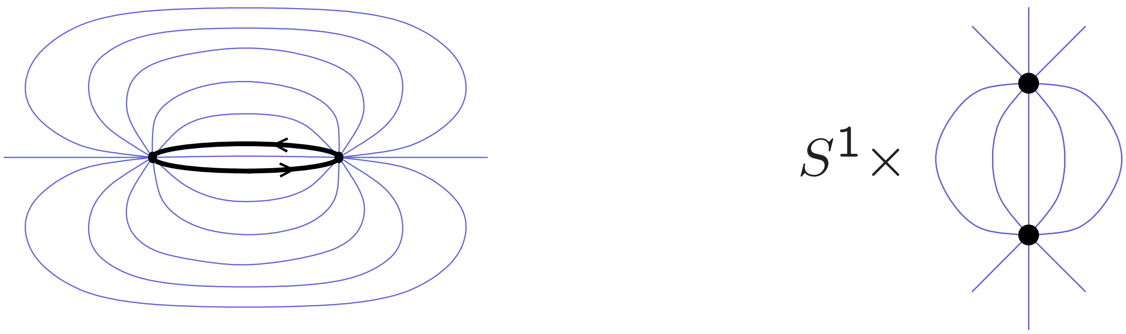
Proposition

Unless $B \cup \mathcal{I} \cup \partial M = \emptyset$, a supported contact structure exists.

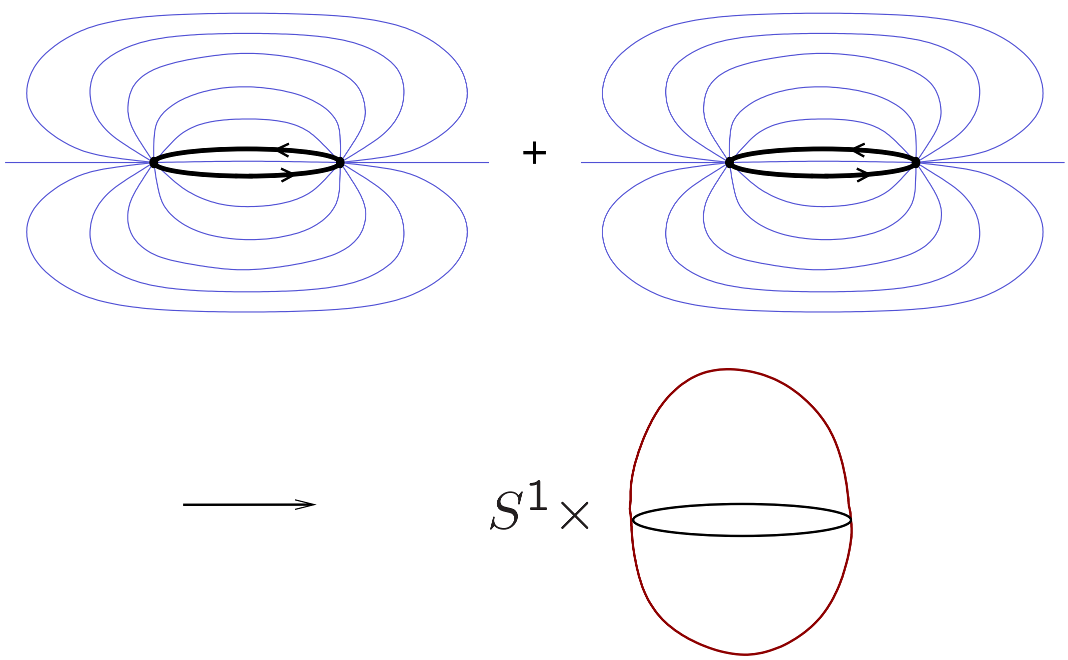
(Otherwise $\pi : M \to S^1$ has closed fibers.)

Examples

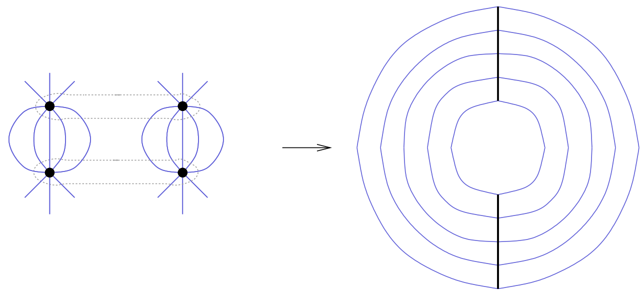
Consider simple open books on the tight S^3 and $S^1 \times S^2$:



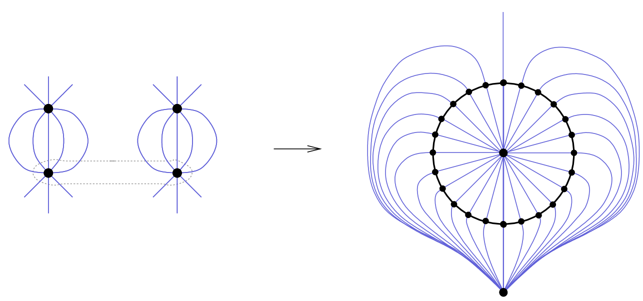
(1) Two copies of S^3 with disk pages binding sum $\rightsquigarrow$ tight $S^1 \times S^2$



(2) Two copies of tight $S^1 \times S^2$
two binding sums $\rightsquigarrow (T^3, \xi_1)$



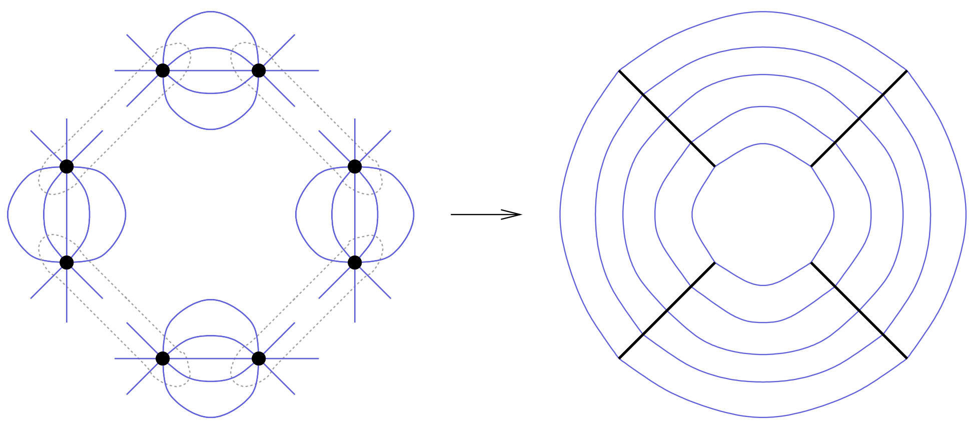
(3) Two copies of $S^1 \times S^2$
one binding sum $\rightsquigarrow$ overtwisted $S^1 \times S^2$



Definition
A blown up summed open book is *symmetric* if it has exactly two irreducible subdomains, all its pages are diffeomorphic, and it has no binding or boundary.

Examples
(1) and (2) are symmetric, (3) is not.

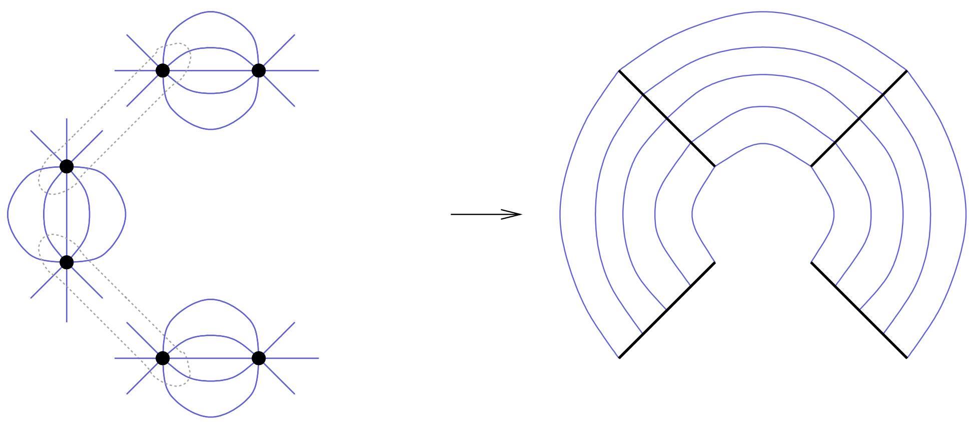
(4) Four copies of $S^1 \times S^2$
four binding sums in a ring $\rightsquigarrow (T^3, \xi_2)$

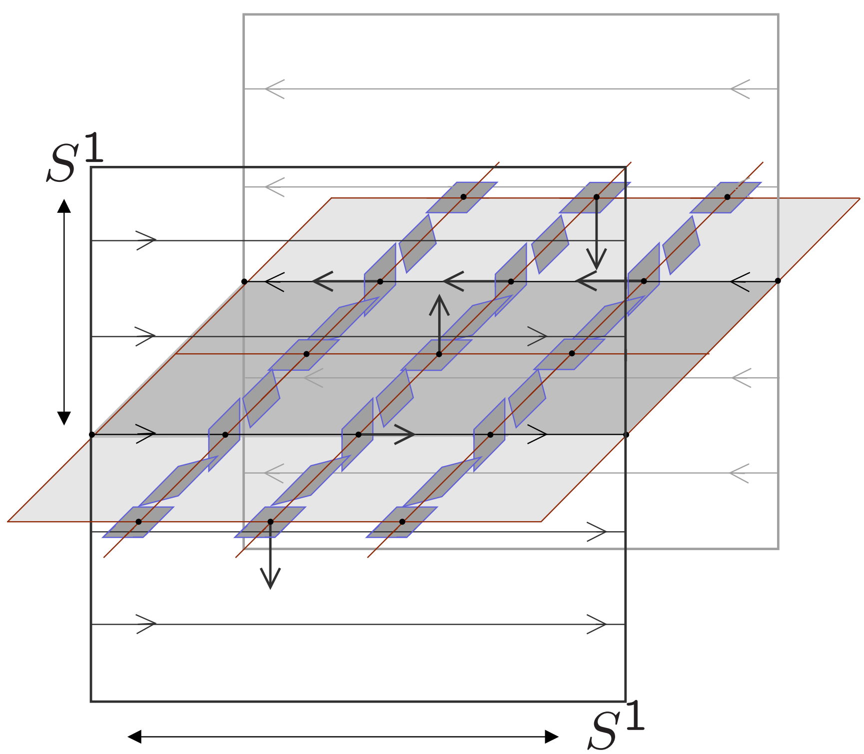


(5) One copy of $S^1 \times S^2$, sum one binding component to the other
$\rightsquigarrow$ Stein fillable torus bundle $T^3/\mathbb{Z}_2$

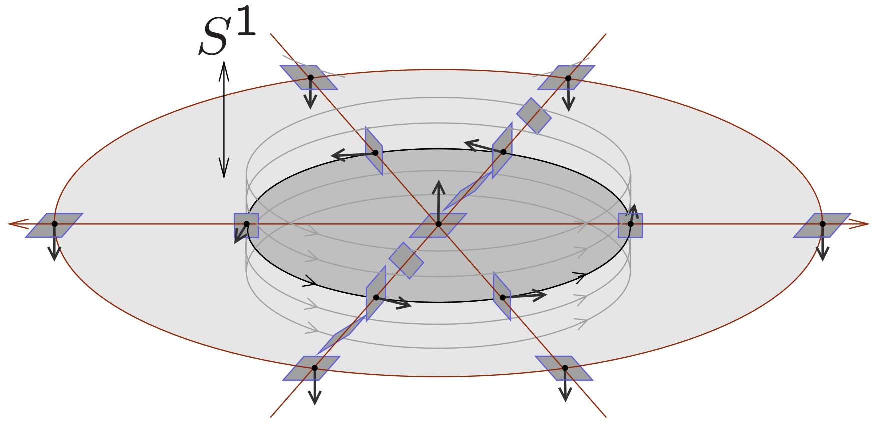
(sorry, I can't draw this)

(6) Three copies of $S^1 \times S^2$,
two binding sums and two blow-ups
$\rightsquigarrow ([0,3/2] \times T^2, \xi_1)$, i.e.
Giroux torsion domain





(7) S^3 summed to $S^1 \times S^2$, remaining binding blown up $\rightsquigarrow$ *Lutz tube*



Definition

For $k \geq 0$, a compact contact domain (M_0, ξ_0) with supporting blown up summed open book is a *planar k-torsion domain* if:

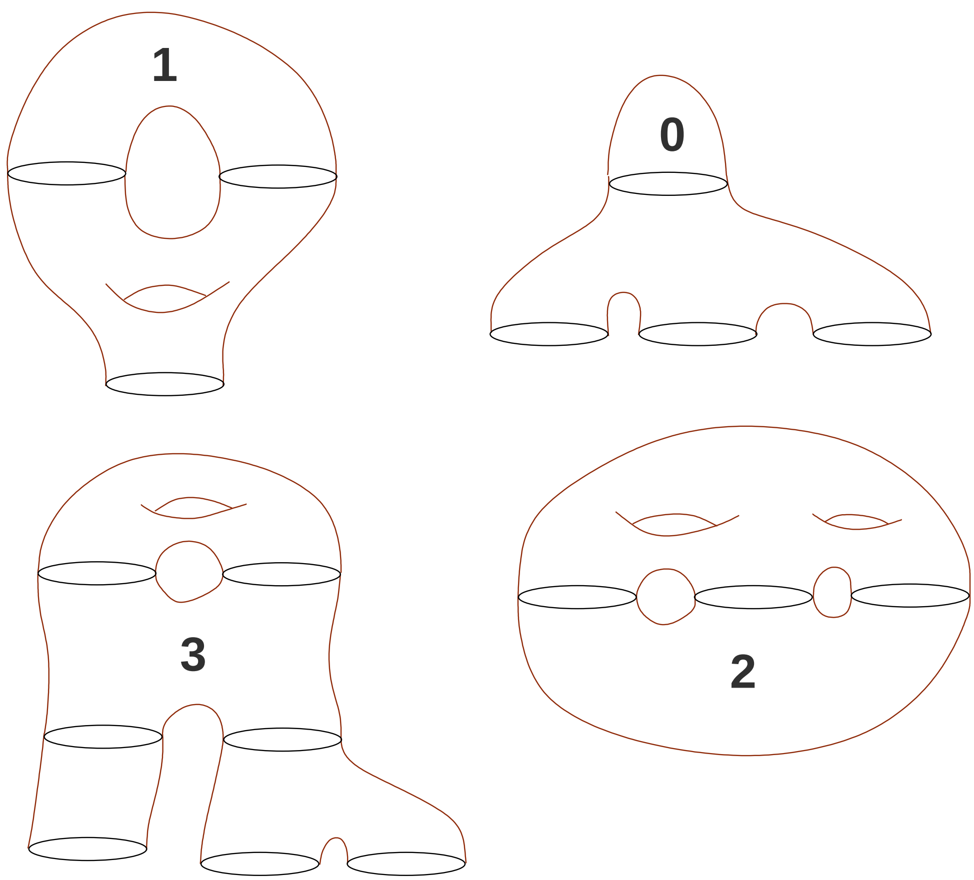
1. It is not symmetric.
2. The interior contains a planar irreducible subdomain

$$M_0^P \subset \text{int}\, M_0,$$

   the *planar piece*, whose pages have $k+1$ boundary components. We call $M_0 \setminus M_0^P$ the *padding*.

A closed contact 3-manifold has *planar k-torsion* if it admits a contact embedding of a planar k-torsion domain.

Some planar torsion domains of the form $S^1 \times \Sigma$



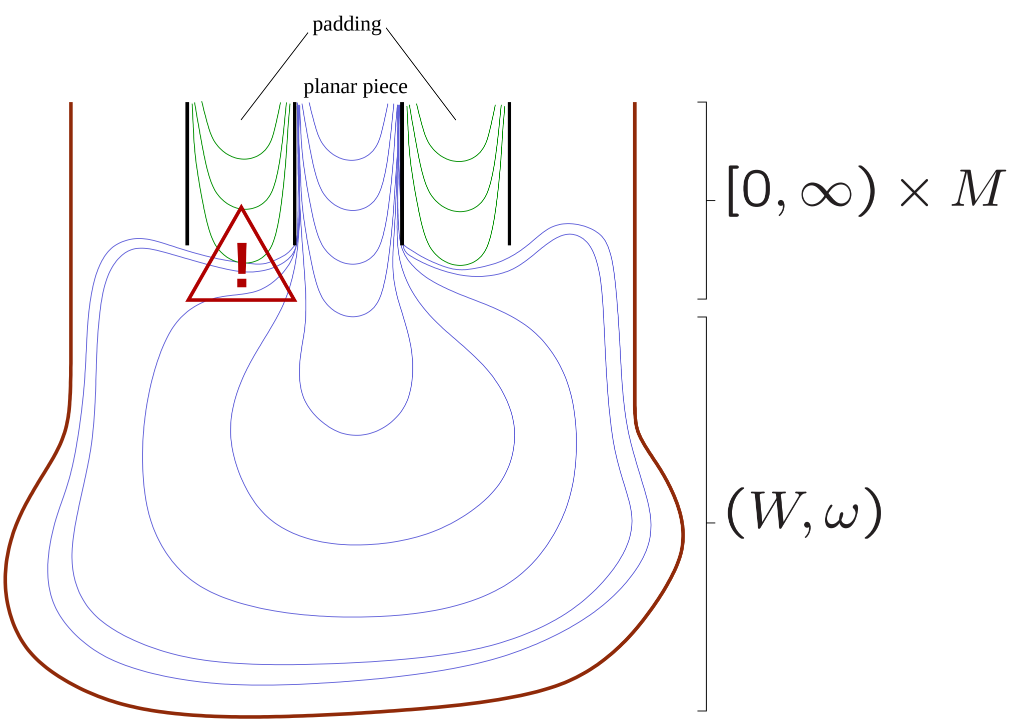
Theorem
If (M, ξ) has planar k-torsion then it is not strongly fillable. Moreover,

1. $c_{\text{ech}}(\xi) = 0$ and $\text{AT}(M, \xi) \leq k$

2. Overtwisted $\Leftrightarrow$ planar 0-torsion

3. Giroux torsion $\Rightarrow$ planar 1-torsion

4. The examples (M_k, ξ_k) for $k \geq 2$ have planar k-torsion but no Giroux torsion.



Part 5
Non-Exact Cobordisms

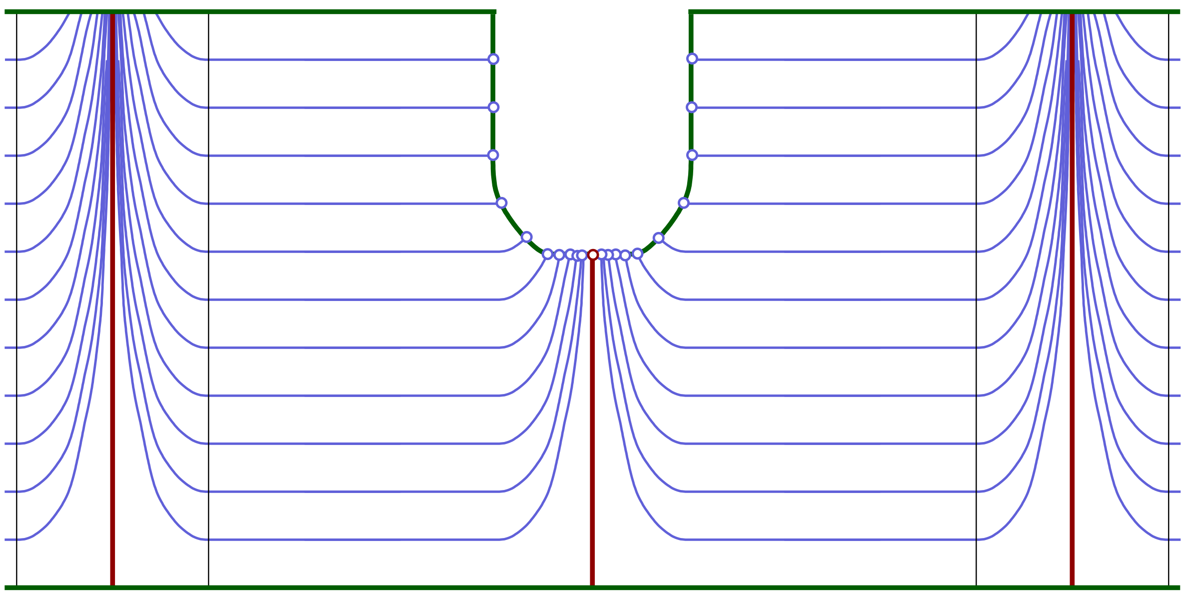
Eliashberg '04 *(symplectic capping)*:
symplectically attaching 2-handles to binding
$\rightsquigarrow$ 0-surgery removes the binding

Gay-Stipsicz '09: doing this at *some*
(not all!) binding components $\rightsquigarrow$
symplectic cobordism between two open books

Blown up version
can attach a round 1-handle

$$S^1 \times [0,1] \times \mathbb{D}$$

to remove an interface torus and cap off pages.



Theorem

If (M_-, ξ_-) has planar k-torsion for $k \geq 1$, then $(M_-, \xi_-) \preccurlyeq (M_+, \xi_+)$ for some contact manifold (M_+, ξ_+) with planar $(k-1)$-torsion.

Moreover, this induces a U-equivariant map

$$\mathrm{ECH}_*(M_+, \xi_+) \to \mathrm{ECH}_*(M_-, \xi_-)$$

taking $c_{\mathrm{ech}}(\xi_+)$ to $c_{\mathrm{ech}}(\xi_-)$.

(Last part is known for Heegaard in simple open book case; J. Baldwin '09)

Corollary
M with k-torsion is cobordant to something overtwisted, and hence to everything.

($\Rightarrow$ not fillable and $c_{\text{ech}}(\xi) = 0$.)

Final Remark

Using such cobordisms, the proof that M_{ot} is not fillable can be reduced to the following:

Lemma
Suppose (W, ω) is a compact symplectic manifold with all boundary components either convex or Levi-flat, and it contains an embedded symplectic sphere of self-intersection 0. Then all boundary components of W are symplectic sphere-bundles.

Proof uses *closed* holomorphic curves; it's still technology, but it's *simpler* technology. Just read McDuff "Rational and Ruled..." 1990, and think about it.