29.07.2014

Minicourse 2

Lecture 2 (R. Siefring)

Recall: we were considering the mapping torus of a twist $S^1 \times \mathbb{R}^2$, $\tilde{J}$, and $\tilde{J}$-holomorphic cylinders

$$(s,t) \in \mathbb{R}^+ \times S^1 \longmapsto (s,t,h(s,t)) \in \mathbb{R} \times S^1 \times \mathbb{R}^2.$$

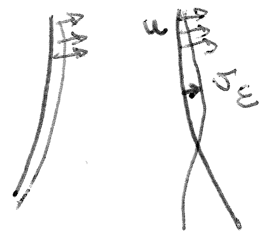
$\tilde{J}$-holomorphicity $\Rightarrow$ $h(s,t) = \sum_{\substack{k \in \mathbb{Z} \\ k < \alpha}} a_k e^{(k-\alpha)s} e^{ikt}.$

$$u(s,t) = e^{-(1+\alpha)s} e^{-it}.$$

$$v_\varepsilon(s,t) = e^{-(1+\alpha)s}(1+\varepsilon) e^{-it} + e^{-(2+\alpha)s} e^{-2it}$$

$$u - v_\varepsilon(s,t) = -\left[\varepsilon e^{-(1+\alpha)s} e^{-it} + e^{-(2+\alpha)s} e^{-2it}\right]$$

We saw 1-intersection when $\varepsilon \neq 0$.
0-intersections when $\varepsilon = 0$.



But int. no. of u and $v_\varepsilon + \varepsilon'$ was 2 regardless of the value of ε

Claim: # of intersections created when separating the ends is $-\underset{s \text{ large}}{\text{wind}}\left[u(s,\cdot) - v_\varepsilon(s,\cdot)\right] \geq 1$

Asymptotic behaviour

$\mathbb{R} \times S^1 \times \mathbb{R}^2$

T-per. orbit at $S^1 \times \{0\}$.

maps of the form $(s,t) \in \mathbb{R}^+ \times S^1 \longmapsto (Ts, t, u(s,t))$ are $\tilde{J}$-hol precisely when:

$$\partial_s u - Au = \delta(s,t,u,\nabla u)$$ ← nonlinear perturbation → 0 as $s \to \infty$

$$A = -J(\nabla_t - T\nabla X).$$

Thm (HWZ, Mora)

$$u(s,t) = e^{\lambda s}(e(t) + r(s,t))$$

$\lambda < 0$; $\lambda \in \sigma(A)$.

$e \in \ker(A - \lambda \cdot 1) \setminus \{0\}$

$r \to 0$ exponentially

If $u(s,t)$ and $v(s,t)$ are different curves, but the leading e-vector in the asymptotic formula then we need more information.

Thm (Siefring '08 → Analogue of M & W result)

$$u(s,t) - v(s,t) = e^{\lambda s}(e(t) + r(s,t)).$$

for λ, e, r satisfying the same conditions from before. (but different e, λ, r)

Corollary: • Finiteness of the intersection no.

• Given a family of $\tilde{J}$-hol half-cylinders, asymptotic to covers of γ, then one can find smooth $\mathbb{R} \times S^1 \times \mathbb{R}^2$ coordinates near $\mathbb{R} \times \gamma$ s.t. all curves are of the form

$$(s,t) \to \left(Tks, kt, \sum_{\ell=0}^{N} e^{\lambda_\ell s} e_\ell(t)\right)$$

$\lambda_\ell < 0$ is an e-val of A_k

$e_\ell \in \ker(A_k - \lambda_\ell) \setminus \{0\}$

Theorem (HWZ "Properties II")

Given an integer K, the span of the set of e-vect's with winding $= K$ is 2-dimensional.

Moreover, winding is monotonic in eigenvalue.

An e-vector with largest possible negative e-value has winding $= \left\lfloor \frac{\mu_{CZ}(\gamma)}{2} \right\rfloor$

$(s,t) \longmapsto (s,t,U(s,t))$ $\quad U - V = e^{\lambda s}(e(t) + r(s,t))$

$\longmapsto (s,t,V(s,t))$

$$\underset{s \text{ large}}{\text{wind}} \left[U(s,t) - V(s,t)\right] = \text{wind } e(t) \leq \left\lfloor \frac{\mu_{CZ}(\gamma)}{2} \right\rfloor$$

Can argue that if we perturb one of the cylinders at the end by a constant vector, then:

$$-\underset{s \text{ large}}{\text{wind}^{\Phi}}\, U - V \geq -\left\lfloor \frac{\mu_{CZ}^{\Phi}(\gamma)}{2} \right\rfloor \qquad \Phi \text{ is a triv. of } \gamma^*\xi$$

but

$$\underbrace{-\text{wind}^{\Phi}(U-V) + \left\lfloor \frac{\mu_{CZ}^{\Phi}(\gamma)}{2} \right\rfloor}_{\text{this is independent of } \Phi} \geq 0$$

Given 2 half-cylinders asymptotic to covers of the same orbit γ, with covering numbers k, ℓ, denote the number of intersections created at the end when perturbing in the direction of a trivialization Φ by $i^{\Phi}(u,v)$.

Hutchings '02 : $i^{\Phi}(u,v) \geq -\max\left\{ k \left\lfloor \mu^{\Phi}(\gamma^{\ell})/2 \right\rfloor, \ell \left\lfloor \mu^{\Phi}(\gamma^{k})/2 \right\rfloor \right\}$.

Also, the difference of the 2 sides is independent of the trivialization Φ

$$\delta_\infty(u,v) = i^{\Phi}(u,v) - \max\{k\lfloor \mu^{\Phi}(\gamma^\ell)/2\rfloor, \ell\lfloor \mu^{\Phi}(\gamma^k)/2\rfloor\}$$

↳ call δ_∞ the <u>asymptotic intersection index</u>.

Denote the number of intersections created when shifting a cylinder u off itself by Φ, by $i^{\Phi}_\infty(u)$. Then, a similar formula holds

$$i^{\Phi}_\infty(u) \geq -(k-1)\lfloor \mu^{\Phi}(\gamma^k)/2\rfloor - \gcd(k, \lfloor\mu^{\Phi}(\gamma^k)/2\rfloor - 1)$$

Define the asymptotic self-intersection index

$\delta_\infty(u) = \text{LHS} - \text{RHS} \geq 0$ and indep. of Φ

<u>Defining a global invariant</u>

Given to $\tilde{J}$-hol curves u, v & a triv of ξ along the asymptotic orbits, we can define a relative intersection no. by perturbing one near the ends in a direction determined by the trivialization, and we denoted by:

$Q^{\Phi}(u,v)$.

<u>Define</u>: $[u]*[v] = Q^{\Phi}(u,v) + \sum \max\{k\lfloor\mu^{\Phi}(\gamma^\ell)/2\rfloor, \ell\lfloor\mu^{\Phi}(\gamma^k)/2\rfloor\}$

everytime u and v have an end asymptotic to γ^k, γ^ℓ respectively, with the same sign

Theorem If u, v are $\tilde{J}$-hol with non identical images then

$$[u] * [v] = \underset{\substack{\uparrow \\ \text{asymp intersections}}}{\delta_\infty(u,v)} + \underset{\substack{\uparrow \\ \text{actual intersections}}}{\delta(u,v)} \geq 0$$

$= 0$ precisely when $\delta(u,v) = \delta_\infty(u,v) = 0$.

Theorem If u is simple then:

$$[u] * [u] - \tfrac{1}{2}\mu_{CZ}(u) + \chi(\Sigma) + \tfrac{1}{2}\Gamma_{odd} - \bar{\sigma}(u)$$

$$= 2\big(\underset{\geq 0}{\delta(u)} + \underset{\geq 0}{\delta_\infty(u)}\big).$$

$$\bar{\sigma}(u) = \sum_{z \in \Gamma} \gcd\left(k, \lfloor \mu^{\Phi}(\gamma^k)/2 \rfloor\right)$$