Summary of Siefring intersection theory

1. Preparations

M^3 with stable Hamiltonian structure (α, ω),

$\xi := \ker \alpha \subset TM, \qquad \omega(X, \cdot) \equiv 0, \quad \alpha(X) \equiv 1.$

Assume all periodic orbits nondegenerate.

Fix trivialisation Φ of ξ along all closed orbits γ, $\rightsquigarrow$ Conley-Zehnder index $\mu_{\mathrm{CZ}}^{\Phi}(\gamma) \in \mathbb{Z}$.

γ with period $T > 0 \rightsquigarrow$ asymptotic operator

$$\mathbf{A}_\gamma := -J(\nabla_t - T\nabla X) : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi)$$

($\nabla :=$ any symmetric connection on M).

$$\alpha_-^{\Phi}(\gamma) := \max\{\mathrm{wind}^{\Phi}(e) \mid \mathbf{A}_\gamma e = \lambda e, \lambda < 0\}$$
$$\alpha_+^{\Phi}(\gamma) := \min\{\mathrm{wind}^{\Phi}(e) \mid \mathbf{A}_\gamma e = \lambda e, \lambda > 0\}$$
$$p(\gamma) := \alpha_+^{\Phi}(\gamma) - \alpha_-^{\Phi}(\gamma) \in \{0, 1\}.$$

Hofer-Wysocki-Zehnder *"Properties II"*:

$$\boxed{\mu_{\mathrm{CZ}}^{\Phi}(\gamma) = 2\alpha_-^{\Phi}(\gamma) + p(\gamma) = 2\alpha_+^{\Phi}(\gamma) - p(\gamma)}.$$

$\Rightarrow$ e.g. $\alpha_-^{\Phi}(\gamma) = \lfloor \mu_{\mathrm{CZ}}^{\Phi}(\gamma)/2 \rfloor$.

2. The intersection pairing

$(\widehat{W}^4, \omega)$ symplectic cobordism with stable Hamiltonian cylindrical ends. For $i = 1, 2$, consider punctured Riemann surfaces

$$\dot{\Sigma}_i = \Sigma_i \setminus (\Gamma_i^+ \cup \Gamma_i^-),$$

and smooth asymptotically cylindrical maps

$$u_i : \dot{\Sigma} \to \widehat{W}$$

asymptotic at $z \in \Gamma_i^\pm$ to orbits $\gamma_z^{k_z}$. Let

$$[u_1]*[u_2] := q^\Phi(u_1, u_2) - \sum_{(z,\zeta)\in\Gamma_1^\pm\times\Gamma_2^\pm} \Omega_\pm^\Phi(\gamma_z^{k_z}, \gamma_\zeta^{k_\zeta}).$$

Here, $q^\Phi(u_1, u_2) \in \mathbb{Z}$ is the relative intersection number $u_1 \cdot u_2^\Phi$, where $u_2^\Phi :=$ a perturbation of u_2, pushed in direction Φ near ∞.

$\Omega_\pm^\Phi(\gamma_1^k, \gamma_2^\ell) \in \mathbb{Z}$ is an a priori winding bound for certain asymptotic eigenvectors:

- $\Omega_\pm^\Phi(\gamma_1^k, \gamma_2^\ell) := 0$ if $\gamma_1 \neq \gamma_2$;
- $\Omega_\pm^\Phi(\gamma^k, \gamma^\ell) := \min\left\{\mp k\alpha_\mp^\Phi(\gamma^\ell), \mp\ell\alpha_\mp^\Phi(\gamma^k)\right\}$.

Properties of $*$:

1. $[u_1] * [u_2]$ depends only on asymptotic orbits and relative homology classes; in particular, it is homotopy invariant.

2. If u_i are J-holomorphic and $u_1(\dot{\Sigma}_1) \neq u_2(\dot{\Sigma}_2)$, then

$$[u_1] * [u_2] = \delta(u_1, u_2) + \delta_\infty(u_1, u_2),$$

where

- $\delta(u_1, u_2) \geq 0$ is the algebraic count of actual intersections;
- $\delta_\infty(u_1, u_2) \geq 0$ is the count of hidden intersections at ∞.

Corollary: $[u_1] * [u_2] \geq 0$ and it bounds the number of geometric intersections from above. In particular,

$$[u_1]*[u_2] = 0 \quad \Rightarrow \quad u_1(\dot{\Sigma}_1) \cap u_2(\dot{\Sigma}_2) = \emptyset.$$

Converse is false! (But "generically" true.)

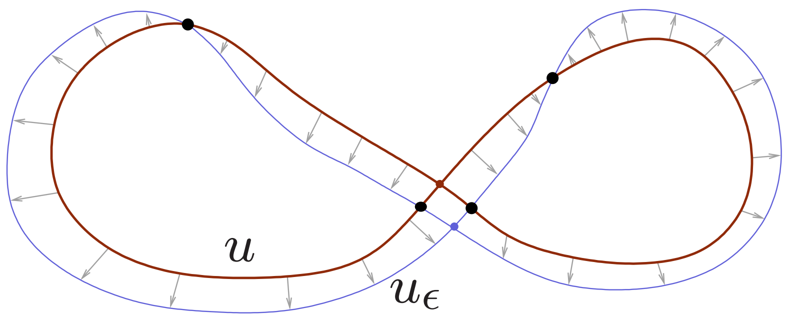
3. The adjunction formula

Closed case: for $u : \Sigma \to W$ simple,

$$[u] \cdot [u] = 2\delta(u) + c_N(u),$$

where

- $\delta(u) \geq 0$ is the algebraic count of double points and critical points;
- $c_N(u) := c_1(u^*TW) - \chi(\Sigma)$ is the normal Chern number of u.



Exercise: $c_N(u)$ is related to the Fredholm index of u by

$$2c_N(u) = \text{ind}(u) - 2 + 2g.$$

Consider a simple punctured curve

$$u : \dot{\Sigma} = \Sigma \setminus (\Gamma^+ \cup \Gamma^-) \to \widehat{W}.$$

Now $\delta(u)$ cannot be homotopy invariant, and $c_1(u^*TW)$ carries no information (it is always 0).

Ingredient A: the normal Chern number

$$c_N(u) := c_1^\Phi(u^*T\widehat{W}) - \chi(\dot{\Sigma}) + \sum_{z \in \Gamma^\pm} \pm \alpha_\mp^\Phi(\gamma_z^{k_z}),$$

where $c_1^\Phi(u^*T\widehat{W})$ is the relative first Chern number with respect to Φ.

Interpretation:
If u is immersed, $c_N(u)$ is the algebraic count of zeroes of a section of its normal bundle, including "hidden zeroes at infinity".

Exercise (using HWZ *Properties II*):

$$\boxed{2c_N(u) = \mathrm{ind}(u) - 2 + 2g + \#\Gamma_{\mathrm{even}}},$$

where $\Gamma_{\mathrm{even}} := \left\{ z \in \Gamma \mid \mu_{\mathrm{CZ}}^\Phi(\gamma_z^{k_z}) \in 2\mathbb{Z} \right\}$.

Ingredient B: spectral covering number

$$\bar{\sigma}(u) := \sum_{z \in \Gamma^\pm} \bar{\sigma}_\mp(\gamma_z^{k_z}),$$

where for an orbit γ, $\bar{\sigma}_\pm(\gamma)$ is the covering multiplicity of any nontrivial eigenvector e of $\mathbf{A}_\gamma$ with $\text{wind}^\Phi(e) = \alpha_\pm^\Phi(\gamma)$. One can show:

$$\bar{\sigma}_\pm(\gamma) = \text{gcd}(\text{cov}(\gamma), \alpha_\pm^\Phi(\gamma)).$$

Remark:
$\bar{\sigma}(u) - \#\Gamma \geq 0$, and it vanishes whenever all the orbits are simple.
$\Rightarrow$ *we can usually ignore this term!*

Ingredient C: double points at infinity

$$\delta_\infty(u) := \frac{1}{2}\Bigg[i_\infty^\Phi(u) - \sum_{z,\zeta \in \Gamma^\pm,\, z \neq \zeta} \Omega_\pm^\Phi(\gamma_z^{k_z}, \gamma_\zeta^{k_\zeta}) - \sum_{z \in \Gamma^\pm} \Omega_\pm^\Phi(\gamma_z^{k_z}) \Bigg] \geq 0.$$

Here, $i_\infty^\Phi(u)$ is the algebraic count of intersections near infinity of u with a small(!) perturbation u^Φ.

$\Omega^{\Phi}_{\pm}(\gamma^k) \in \mathbb{Z}$ is another a priori winding bound for certain asymptotic eigenvectors:

$$\Omega^{\Phi}_{\pm}(\gamma^k) := \mp(k-1)\alpha^{\Phi}_{\mp}(\gamma^k) + \left[\bar{\sigma}_{\mp}(\gamma^k) - 1\right].$$

The adunction formula:

$$\boxed{[u] * [u] = 2\left[\delta(u) + \delta_{\infty}(u)\right] + c_N(u) + \left[\bar{\sigma}(u) - \#\Gamma\right]}.$$

Corollary: $\delta(u) + \delta_{\infty}(u)$ is homotopy invariant and bounds the number of geometric double points from above. In particular,

$$\delta(u) + \delta_{\infty}(u) = 0 \quad \Rightarrow \quad u \text{ is embedded.}$$

Again, converse is false but generically true.

Hidden intersections lemma:
$\delta_{\infty}(u, v)$ or $\delta_{\infty}(u)$ vanishes whenever all of the relevant asymptotic eigenvectors achieve their a priori bound (given by $\Omega^{\Phi}_{\pm}(\cdots)$).