

From pro- p Iwahori-Hecke modules to (φ, Γ) -modules I

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Abstract

Let \mathfrak{o} be the ring of integers in a finite extension K of \mathbb{Q}_p , let k be its residue field. Let G be a split reductive group over \mathbb{Q}_p , let T be a maximal split torus in G . Let $\mathcal{H}(G, I_0)$ be the pro- p -Iwahori Hecke \mathfrak{o} -algebra. Given a semiinfinite reduced chamber gallery (alcove walk) $C^{(\bullet)}$ in the T -stable apartment, a period $\phi \in N(T)$ of $C^{(\bullet)}$ of length r and a homomorphism $\tau : \mathbb{Z}_p^\times \rightarrow T$ compatible with ϕ , we construct a functor from the category $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ of finite length $\mathcal{H}(G, I_0)$ -modules to étale (φ^r, Γ) -modules over Fontaine's ring \mathcal{O}_E . If $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ there are essentially two choices of $(C^{(\bullet)}, \phi, \tau)$ with $r = 1$, both leading to a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to étale (φ, Γ) -modules and hence to $\text{Gal}_{\mathbb{Q}_p}$ -representations. Both induce a bijection between the set of absolutely simple supersingular $\mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$ -modules of dimension $d+1$ and the set of irreducible representations of $\text{Gal}_{\mathbb{Q}_p}$ over k of dimension $d+1$. We also compute these functors on modular reductions of tamely ramified locally unitary principal series representations of G over K . For $d = 1$ we recover Colmez' functor (when restricted to \mathfrak{o} -torsion $\text{GL}_2(\mathbb{Q}_p)$ -representations generated by their pro- p -Iwahori invariants).

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1 Introduction

In his remarkable opus [5] on the p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, Colmez established a bijection between certain representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and certain two-dimensional representations of the absolute Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$ of the field \mathbb{Q}_p of p -adic numbers. These representations have coefficients either in a finite extension of \mathbb{F}_p , or in a finite extension of \mathbb{Q}_p . In either case, the theory of (φ, Γ) -modules as developed by Fontaine [7] provides the required intermediate objects in order to pass from one side to the other. Prior to Colmez' work the characteristic p correspondence had been suggested by Breuil as an explicit "by hand" matching between the objects on either side; it was then astonishing to see this correspondence being realized even by a *functorial* relationship between $\mathrm{GL}_2(\mathbb{Q}_p)$ -representations and (φ, Γ) -modules. A certain functor \mathbf{D} from \mathfrak{o} -torsion representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to (φ, Γ) -modules over \mathfrak{o} constitutes one half of this relationship. Here \mathfrak{o} is the ring of integers in a finite extension K of \mathbb{Q}_p . Although Colmez does not phrase it in these terms, his functor \mathbf{D} may be viewed as factoring through a functor from certain coefficient systems on the Bruhat Tits tree \mathfrak{X} of $\mathrm{PGL}_2(\mathbb{Q}_p)$ to (φ, Γ) -modules. The purpose of the present paper is to suggest an extension of this latter functor to *certain* coefficient systems on the Bruhat Tits building X of a general split reductive group G over \mathbb{Q}_p . Such coefficient systems can in particular be attached to $(\mathfrak{o}$ -finite-length) modules over the pro- p -Iwahori Hecke \mathfrak{o} -algebra $\mathcal{H}(G, I_0)$, formed with respect to a pro- p -Iwahori subgroup I_0 in G . The entire construction depends on a certain choice, and for each such choice we end up (Theorem 7.5) with an exact functor from such $\mathcal{H}(G, I_0)$ -modules to (φ^r, Γ) -modules, with $r \in \mathbb{N}$ depending on that choice.

Let \mathfrak{v}_0 denote the vertex of \mathfrak{X} fixed by $\mathrm{GL}_2(\mathbb{Z}_p)$. In $\mathrm{GL}_2(\mathbb{Q}_p)$ consider the element $\varphi = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and the subgroups $\mathfrak{N}_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and $\Gamma = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$. The orbit of \mathfrak{v}_0 under the submonoid $[\mathfrak{N}_0, \varphi, \Gamma]$ of G generated by \mathfrak{N}_0 , φ and Γ defines a halftree $\overline{\mathfrak{X}}_+$ inside \mathfrak{X} : its edges are those whose both vertices belong to that orbit. Adding the unique edge with only one vertex (namely \mathfrak{v}_0) in that orbit we obtain the half tree \mathfrak{X}_+ . Let \mathcal{V} be a $[\mathfrak{N}_0, \varphi, \Gamma]$ -equivariant \mathfrak{o} -torsion coefficient system on \mathfrak{X}_+ . Let $D(\mathcal{V}) = H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^*$ and $D'(\mathcal{V}) = H_0(\mathfrak{X}_+, \mathcal{V})^*$ (Pontryagin duals). Under a suitable finiteness conditions these are compact $\mathcal{O}_{\mathcal{E}}^+ = \mathfrak{o}[[\mathfrak{N}_0]]$ -modules and the natural map $D'(\mathcal{V}) \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{O}_{\mathcal{E}} \rightarrow D(\mathcal{V}) \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{O}_{\mathcal{E}} =: \mathbf{D}(\mathcal{V})$ is bijective, where $\mathcal{O}_{\mathcal{E}}$ denotes the p -adic completion of $\mathcal{O}_{\mathcal{E}}^+$ with respect to the complement of $\pi_K \mathcal{O}_{\mathcal{E}}^+$, where $\pi_K \in \mathfrak{o}$ is a uniformizer. The actions of φ and Γ then provide $\mathbf{D}(\mathcal{V})$ with the structure of an étale (φ, Γ) -module. (The $\mathcal{O}_{\mathcal{E}}^+$ -lattice $D'(\mathcal{V})$ carries the φ -operator, the $\mathcal{O}_{\mathcal{E}}^+$ -lattice $D(\mathcal{V})$ carries the ψ -operator.)

Now suppose we are given a G -equivariant coefficient system \mathcal{V} on X . By what we said, in order to pass from \mathcal{V} to an étale (φ, Γ) -module we might try to assign to \mathcal{V} a $[\mathfrak{N}_0, \varphi, \Gamma]$ -equivariant coefficient system on \mathfrak{X}_+ . Thinking sheaf theoretically, a very naive pattern would simply be: choose an embedding (a notion to be clarified) $\iota : \mathfrak{X}_+ \rightarrow X$ and take the pull back $\iota^{-1}\mathcal{V}$ of \mathcal{V} to \mathfrak{X}_+ . However, in order that $\iota^{-1}\mathcal{V}$ obtains a $[\mathfrak{N}_0, \varphi, \Gamma]$ -action from the G -action on \mathcal{V} one must ask an equivariance property of ι . At first one would think of such an equivariance

property with respect to a chosen embedding of $[\mathfrak{N}_0, \varphi, \Gamma]$ into G , but if $G \neq \mathrm{GL}_2(\mathbb{Q}_p)$ such a construction is apparently not available. Now the main point of the present paper is that this embedding idea can nevertheless be implemented if the equivariance property is negotiated down to the bare minimum which just suffices for such an equivariant pull back of coefficient systems — but only for equivariant coefficient systems of a very specific type: coefficient systems ”of level 1” in our terminology.

Let $T \subset G$ be a split maximal torus, $A \subset X$ the corresponding apartment, $C \subset A$ a chamber, $I \subset G$ the corresponding Iwahori subgroup, $I_0 \subset I$ its pro- p -Iwahori subgroup, NT a Borel subgroup with unipotent radical N and $N_0 = N \cap I$. We fix an infinite reduced chamber gallery $C^{(\bullet)} = (C = C^{(0)}, C^{(1)}, C^{(2)}, \dots)$ in A with $|N_0 \cdot C^{(i)}| = p^i$ for $i \geq 0$. Given $C^{(\bullet)}$, we choose an auxiliary datum, which we call Θ : for all $i \geq 0$ an identification of the orbit $N_0 \cdot C^{(i)}$ with the set of edges of \mathfrak{X}_+ ”at distance i ”, and of the orbit $N_0 \cdot (C^{(i)} \cap C^{(i+1)})$ with the set of vertices of \mathfrak{X}_+ ”at distance i ”. This should respect face inclusions.* Carefully chosen, such a Θ identifies the action of N_0 on the neighbourhood of any $n \cdot (C^{(i)} \cap C^{(i+1)})$ (for $n \in N_0$) — i.e. on the chambers containing $n \cdot (C^{(i)} \cap C^{(i+1)})$ — with the action of \mathfrak{N}_0 on the neighbourhood of the corresponding vertex in \mathfrak{X}_+ . See Theorem 3.2. Such a Θ given, N_0 -equivariant coefficient systems \mathcal{V} on X of level 1 give rise to \mathfrak{N}_0 -equivariant coefficient systems $\Theta_* \mathcal{V}$ on \mathfrak{X}_+ : this is essentially built into the level 1 property which exactly says that the N_0 -action on $\mathcal{V}(n \cdot (C^{(i)} \cap C^{(i+1)}))$ is insensitive to the subgroup fixing the above neighbourhood of $n \cdot (C^{(i)} \cap C^{(i+1)})$ pointwise.† See Theorem 4.2. The ambiguity in the particular choice of Θ is immaterial for our further purposes, as long as $C^{(\bullet)}$ is fixed.

If $\phi \in N(T)$ is a period of length $r \in \mathbb{N}$ for $C^{(\bullet)}$, i.e. if $\phi(C^{(i)}) = C^{(i+r)}$ for all $i \geq 0$, then Θ can be chosen in such a way that $\Theta_* \mathcal{V}$ carries an action of $[\mathfrak{N}_0, \varphi^r]$. Similarly, if for an embedding $\tau : \mathbb{Z}_p^\times \rightarrow T$ the image $\tau(\mathbb{Z}_p^\times)$ commutes with ϕ and acts semilinearly on \mathcal{V} , then $\Theta_* \mathcal{V}$ carries an action of $[\mathfrak{N}_0, \varphi^r, \Gamma]$.

For equivariant coefficient systems \mathcal{V} of level 1 on X satisfying a certain finiteness condition we thus obtain an étale (φ^r, Γ) -module $\mathbf{D}(\Theta_* \mathcal{V})$ by the functor analogous to the one discussed at the beginning (now with a φ^r -action instead of a φ -action).

To any admissible \mathfrak{o} -torsion representation V of G one can associate a G -equivariant coefficient system \mathcal{V} on X which indeed satisfies the said finiteness condition and which to a chamber C' of X with corresponding pro- p -Iwahori subgroup I'_0 assigns $\mathcal{V}(C') = V^{I'_0}$. But there is even a functorial construction of such coefficient systems depending only on the $\mathcal{H}(G, I_0)$ -module V^{I_0} and not on the G -representation V . In other words, there is a functor $M \mapsto \mathcal{V}_M$ from the category $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ of $\mathcal{H}(G, I_0)$ -modules which (as \mathfrak{o} -modules) are of finite length, to coefficient systems on X which satisfy the said finiteness condition, cf. Proposition 5.1. One

*By convention, our chambers are the *closures* of those open subsets usually referred to as the chambers of X .

†Notice that we only need values of \mathcal{V} on all the $n \cdot C^{(i)}$ and $n \cdot (C^{(i)} \cap C^{(i+1)})$, not on smaller facets. Therefore, deviating from usual terminology, a ’coefficient system’ in this paper is only defined on facets of codimension 0 or 1.

gets a functor

$$M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$$

from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of étale (φ^r, Γ) -modules; it is *exact*. See Theorem 7.5. If the \mathfrak{o} -action on M factors through k then the action of $\mathcal{O}_{\mathcal{E}}$ on $\mathbf{D}(\Theta_* \mathcal{V}_M)$ factors through the residue field $k_{\mathcal{E}} (\cong k((t)))$ of $\mathcal{O}_{\mathcal{E}}$ and we have

$$\dim_{k_{\mathcal{E}}} \mathbf{D}(\Theta_* \mathcal{V}_M) \leq \dim_k M.$$

The functor depends on the choice of $C^{(\bullet)}$, ϕ and τ . It is of course of interest to find $C^{(\bullet)}$, ϕ , τ such that the length r of the period ϕ is small.

Let Γ_0 be the maximal pro- p -subgroup of Γ . A slight simplification of the above construction produces an étale (φ^r, Γ_0) -module $\mathbf{D}(\Theta_* \mathcal{V}_M)$ for *any* choice of $(C^{(\bullet)}, \phi)$, not requiring a cocharacter τ as above. (The existence of such τ apparently is a fairly restrictive assumption on $(C^{(\bullet)}, \phi)$.) Thus, any pair $(C^{(\bullet)}, \phi)$ gives rise to a functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ to étale (φ^r, Γ_0) -modules, cf. Proposition 4.4, Corollary 7.3 (a), Theorem 7.5 (a).

Let $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ for some $d \in \mathbb{N}$. Then, asking for $C^{(\bullet)}$, ϕ , τ with $r = 1$ one has essentially just two choices, namely where $C^{(\bullet)}$ is interpolated by the translates of C by one of the two (if $d > 1$) extreme (in the Dynkin diagram) simple coroots. As $r = 1$ we obtain usual étale (φ, Γ) -modules $\mathbf{D}(\Theta_* \mathcal{V}_M)$, hence we can pass to their corresponding $\text{Gal}_{\mathbb{Q}_p}$ -representations $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$. We compute the resulting functor

$$(1) \quad M \mapsto W(\mathbf{D}(\Theta_* \mathcal{V}_M))$$

from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to $\text{Gal}_{\mathbb{Q}_p}$ -representations over \mathfrak{o} in important cases. Let $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$.

Theorem 8.8: *If $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ the functor (1) induces a bijection between*

- (a) *the set of isomorphism classes of absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -modules of dimension $d + 1$ and*
- (b) *the set of isomorphism classes of smooth irreducible representations of $\text{Gal}_{\mathbb{Q}_p}$ over k of dimension $d + 1$.*

The required input about $\mathcal{H}(G, I_0)_k$ -modules is provided by work of Ollivier [11] and Vignéras [15] while the facts needed about $\text{Gal}_{\mathbb{Q}_p}$ -representations we found in work of Berger [2]. These ingredients allow us to make the correspondence in Theorem 8.8 completely explicit, similarly to Breuil's matching correspondence in the case of $\text{GL}_2(\mathbb{Q}_p)$.

Still concentrating on $G = \text{GL}_{d+1}(\mathbb{Q}_p)$, we next evaluate the functor on *reduced standard* $\mathcal{H}(G, I_0)_k$ -modules (or $\mathcal{H}(G, I_0)_k$ -modules of *W-type*). Such modules admit a k -basis indexed by the finite Weyl group W in terms of which the $\mathcal{H}(G, I_0)_k$ -action can be given very neatly. One may expect reduced standard modules to play a similar role for the representation theory of $\mathcal{H}(G, I_0)_k$ as standard modules do in similar and more classical contexts. If Y is a locally unitary

tamely ramified principal series representation of G over K then the $\mathcal{H}(G, I_0)_k$ -module arising by modular reduction from the $\mathcal{H}(G, I_0) \otimes_o K$ -module Y^{I_0} of I_0 -invariants is a reduced standard $\mathcal{H}(G, I_0)_k$ -module, see [9]. (Here, by local unitarity of Y we precisely mean that Y^{I_0} admits an $\mathcal{H}(G, I_0)$ -stable \mathfrak{o} -lattice.) For any reduced standard $\mathcal{H}(G, I_0)_k$ -module M we explain how the $\text{Gal}_{\mathbb{Q}_p}$ -representation $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ can be filtered in such a way that the subquotients take the form $\text{ind}(\omega_{m+1}^h) \otimes \omega^s \mu_\beta$ with varying $m \geq 0$, $h, s \in \mathbb{Z}$, $\beta \in k^{\text{alg}}$ (in usual notations, e.g. as in [2]). Such a filtration is induced from a suitable k -vector space filtration on M which itself is induced from a suitable (set theoretical) filtration on W associated with M . In the special case where $M = V^{I_0}$ for a tamely ramified principal series representation V of G over k one can be even more precise. In that case we describe a filtration of $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ with d -dimensional subquotients of the form $\text{ind}(\omega_d^h) \otimes \omega^s \mu_\beta$ with varying h, s, β , and we find $\dim_k W(\mathbf{D}(\Theta_* \mathcal{V}_M)) = d!d$ (whereas $\dim_k(M) = (d+1)!$, as for any reduced standard $\mathcal{H}(G, I_0)_k$ -module). See Theorem 8.14 for the precise statement. It generalizes to $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ Colmez' computation for $G = \text{GL}_2(\mathbb{Q}_p)$.

The key step in computing the $\text{Gal}_{\mathbb{Q}_p}$ -representation $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ is the passage from the module $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ over the non commutative polynomial ring $k_{\mathcal{E}}^+[\varphi, \Gamma]$ over $k_{\mathcal{E}}^+ = k[[\mathfrak{N}_0]] (\cong k[[t]])$ to the étale (φ, Γ) -module $\mathbf{D}(\Theta_* \mathcal{V}_M)$ over $k_{\mathcal{E}}$. To this end, inspired by section 5 of [6] we introduce a notion of 'standard cyclic modules' over $k_{\mathcal{E}}^+[\varphi, \Gamma]$; these give rise to (φ, Γ) -modules over $k_{\mathcal{E}}$ whose associated $\text{Gal}_{\mathbb{Q}_p}$ -representations are of the form $\text{ind}(\omega_{m+1}^h) \otimes \omega^s \mu_\beta$.

The structure of the paper is as follows. We start, section 2, by recalling preliminaries about mod p representations of $\text{SL}_2(\mathbb{F}_p)$ as needed in sections 4, 5, 8. In section 3 we present our main geometric construction: an isomorphism Θ between the N_0 -orbit of a semiinfinite chamber gallery $C^{(\bullet)}$ in X as above, and the half tree \mathfrak{X}_+ ; this Θ can be chosen to be 'equivariant' in a sense suitable for our purposes. In section 4 we introduce our notion of equivariant coefficient systems as needed in this paper, and their (strict) level 1-property. We explain the passage from equivariant coefficient systems of level 1 on X to such on \mathfrak{X}_+ by means of Θ , and we prove the important Theorem 4.3 which says that the \mathfrak{N}_0 -invariants in $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ for an \mathfrak{N}_0 -equivariant coefficient system \mathcal{V} on \mathfrak{X}_+ of strict level 1 are just the obvious ones. The functor $M \mapsto \mathcal{V}_M$ from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to coefficient systems on X is the subject of section 5. Section 6 recalls basic facts on (φ^r, Γ) -modules, followed by a discussion of standard cyclic modules over $k_{\mathcal{E}}^+[\varphi, \Gamma]$ and their corresponding $\text{Gal}_{\mathbb{Q}_p}$ -representations. Section 7 explains the functors from equivariant coefficient systems on \mathfrak{X}_+ to modules over $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$, to (ψ^r, Γ) -modules and to (φ^r, Γ) -modules. This is almost formal and only rephrases the main construction of Colmez' functor \mathbf{D} in terms of the halftree \mathfrak{X}_+ . We end with our computations of the functor $M \mapsto W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ in the case $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ and $r = 1$, section 8.

Remarks: (a) Throughout, we could just as well replace G by the group of F -rational points of an F -split connected reductive group over F , for an arbitrary local field F with residue field \mathbb{F}_p (but still using the tree \mathfrak{X} of $\text{GL}_2(\mathbb{Q}_p)$ and the associated usual (φ, Γ) -modules).

(b) In the sequel [10] to this paper we discuss in more detail the functor \mathbf{D} introduced here for split reductive groups over \mathbb{Q}_p other than $\text{GL}_{d+1}(\mathbb{Q}_p)$.

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Notations: We fix a finite extension K/\mathbb{Q}_p with ring of integers \mathfrak{o} , prime element π_K and residue field k . We let $\mathbb{Q}_p^{\text{alg}}$ resp. k^{alg} denote an algebraic closure of \mathbb{Q}_p resp. of k . For $m \geq 1$ we write $\mathfrak{o}_m = \mathfrak{o}/\pi_K^m$. Recall that Pontryagin duality $V \mapsto V^* = \text{Hom}_{\mathfrak{o}}^{\text{ct}}(V, K/\mathfrak{o})$ sets up an equivalence between the category of all torsion \mathfrak{o} -modules and the category of all compact linear-topological \mathfrak{o} -modules.

2 Mod p representations of $\text{SL}_2(\mathbb{F}_p)$

In this section we recall well known facts on the representation theory of $\text{SL}_2(\mathbb{F}_p)$ and its quotient $\text{PSL}_2(\mathbb{F}_p)$ on k -vector spaces. We use usual conventions concerning induced representations, Hecke algebras and Hecke operators, just as they are recalled explicitly in the later section 5.

Lemma 2.1. *Let \overline{U} be a cyclic group with p elements. Let W be a $k[\overline{U}]$ -module, generated by a finite dimensional sub k -vector space W' of W with $\dim_k(W') = \dim_k(W^{\overline{U}})$. Let*

$$\eta : k[\overline{U}] \otimes_k W' \longrightarrow W$$

denote the surjective morphism of $k[\overline{U}]$ -modules induced from the inclusion $W' \hookrightarrow W$. Then the map $H^1(\mathbb{Z}_p, \eta) : H^1(\mathbb{Z}_p, k[\overline{U}] \otimes_k W') \rightarrow H^1(\mathbb{Z}_p, W)$ (continuous cohomology) is bijective for every surjection $\mathbb{Z}_p \rightarrow \overline{U}$.

PROOF: The same as for [8] Lemma 2.1 (ii) and (iii). For the convenience of the reader, we reproduce the proof.

(i) Any (finitely generated) $k[\overline{U}]$ -module admits a direct sum decomposition with summands isomorphic to quotients of $k[\overline{U}]$: this can be seen e.g. by applying the structure theorem for modules over the polynomial ring in one variable over k , of which $k[\overline{U}]$ is a quotient. From this we see that the minimal number of elements needed to generate such a $k[\overline{U}]$ -module is the same as the dimension of its space of \overline{U} -invariants. As $\dim_k(W^{\overline{U}}) = \dim_k(W')$ we therefore see that, as a $k[\overline{U}]$ -module, W can not be generated by fewer than $\dim_k(W')$ many elements.

(ii) Let

$$\epsilon : k[\overline{U}] \otimes_k W' \longrightarrow W', \quad \bar{u} \otimes w \mapsto w \text{ for } \bar{u} \in \overline{U}$$

denote the augmentation map. We claim that $\ker(\eta) \subset \ker(\epsilon)$.

To see this let $x \in \ker(\eta)$. If $x \notin \ker(\epsilon)$ then the class of x in $(k[\overline{\mathcal{U}}] \otimes_k W') \otimes_{k[\overline{\mathcal{U}}]} k$ does not vanish (here $k[\overline{\mathcal{U}}] \rightarrow k$ is the augmentation, its kernel is the maximal ideal of the local ring $k[\overline{\mathcal{U}}]$). Therefore, by Nakayama's Lemma, the quotient $(k[\overline{\mathcal{U}}] \otimes_k W')/(k[\overline{\mathcal{U}}].x)$ can be generated by fewer than $\dim_k(W')$ many elements. As $x \in \ker(\eta)$ this is a contradiction to what we saw in (i).

(iii) Let $\gamma : \mathbb{Z}_p \rightarrow k[\overline{\mathcal{U}}] \otimes_k W'$ be a 1-cocycle such that $\eta \circ \gamma : \mathbb{Z}_p \rightarrow W$ is a coboundary. As η is surjective we may modify γ by a coboundary such that now $\eta \circ \gamma = 0$. Let $c \in \mathbb{Z}_p$ be a topological generator. Since $\eta(\gamma(c)) = 0$ we have $\gamma(c) \in \ker(\epsilon)$ by (ii). But $\ker(\epsilon) = (c - 1)k[\overline{\mathcal{U}}] \otimes_k W'$, so $\gamma(c) = cf - f$ for some $f \in k[\overline{\mathcal{U}}] \otimes_k W'$. Since c generates \mathbb{Z}_p the cocycle condition on γ shows $\gamma(c') = c'f - f$ for any $c' \in \mathbb{Z}_p$, so γ is a coboundary. We have shown injectivity of $H^1(\mathbb{Z}_p, \eta)$. The surjectivity of $H^1(\mathbb{Z}_p, \eta)$ follows from the surjectivity of η and from $H^2(\mathbb{Z}_p, ?) = 0$. \square

Let either $\overline{\mathcal{S}} = \mathrm{SL}_2(\mathbb{F}_p)$ or $\overline{\mathcal{S}} = \mathrm{PSL}_2(\mathbb{F}_p)$. Let $\overline{\mathcal{U}}$ be the (group of \mathbb{F}_p -valued point of the) unipotent radical of a Borel subgroup in $\overline{\mathcal{S}}$. For $m \geq 1$ we consider the Hecke algebra $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_{\mathfrak{o}_m} = \mathrm{End}_{\mathfrak{o}_m[\overline{\mathcal{S}}]}(\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m})^{\mathrm{op}}$.

Lemma 2.2. *The universal module $\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m}$ is flat over $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_{\mathfrak{o}_m}$.*

PROOF: For the course of this proof let us write $\mathcal{H}_{\mathfrak{o}_m} = \mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_{\mathfrak{o}_m}$ and more specifically $\mathcal{H}_k = \mathcal{H}_{\mathfrak{o}_1}$. The flatness assertion is equivalent with the claim that for any left ideal \mathcal{I} in $\mathcal{H}_{\mathfrak{o}_m}$ the natural map

$$(2) \quad \mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes_{\mathcal{H}_{\mathfrak{o}_m}} \mathcal{I} \longrightarrow \mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m}$$

is injective. We proceed by induction on m . For $m = 1$ this is a variant of the proof given for $\mathrm{GL}_2(\mathbb{F}_p)$ in Prop. 2.2 of [13]. For the facts on \mathcal{H}_k stated below see e.g. [3]. Let $\overline{\mathcal{T}}$ be a maximal split torus in $\overline{\mathcal{S}}$ such that $\overline{\mathcal{U}}$ is the unipotent radical of the Borel subgroup $\overline{\mathcal{B}} = \overline{\mathcal{U}}\overline{\mathcal{T}} = \overline{\mathcal{T}}\overline{\mathcal{U}}$. We have the $\overline{\mathcal{S}}$ -equivariant decomposition

$$\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k \cong \bigoplus_{\beta \in \overline{\mathcal{T}}^c} \mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta$$

where β runs through the set $\overline{\mathcal{T}}^c$ of k^\times -valued characters β of $\overline{\mathcal{T}}$, viewed as characters of $\overline{\mathcal{B}}$. For $\beta \in \overline{\mathcal{T}}^c$ let $\epsilon_\beta \in \mathcal{H}_k$ denote the natural projection $\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k \rightarrow \mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta$ composed with the inclusion $\mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \rightarrow \mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k$. The ϵ_β are pairwise orthogonal idempotents summing up to the unity element in \mathcal{H}_k . Let $T_{n_s} \in \mathcal{H}_k$ denote the Hecke operator corresponding to a generator $n_s \in N(\overline{\mathcal{T}})$ of the Weyl group $N(\overline{\mathcal{T}})/\overline{\mathcal{T}}$. Then \mathcal{H}_k is generated by T_{n_s} together with all the ϵ_β . For $\beta \in \overline{\mathcal{T}}^c$ define $\beta^s \in \overline{\mathcal{T}}^c$ by $\beta^s(t) = \beta(n_s t n_s^{-1})$ for $t \in \overline{\mathcal{T}}$. We have $T_{n_s} \epsilon_\beta = \epsilon_{\beta^s} T_{n_s}$.

In view of the above orthogonal decomposition, to prove injectivity of (2) for $m = 1$ it is enough to show the following:

(a) For any $\beta \in \overline{\mathcal{T}}^c$ with $\beta \neq \beta^s$ and any left ideal \mathcal{I} in $\mathcal{H}_k \epsilon_\beta \oplus \mathcal{H}_k \epsilon_{\beta^s}$ the map

$$(\mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \oplus \mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta^s) \otimes_{\mathcal{H}_k \epsilon_\beta \oplus \mathcal{H}_k \epsilon_{\beta^s}} \mathcal{I} \longrightarrow \mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \oplus \mathrm{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta^s$$

is injective.

(b) For (the unique) $\beta \in \overline{\mathcal{T}}^c$ with $\beta = \beta^s$ and any left ideal \mathcal{I} in $\mathcal{H}_k \epsilon_\beta$ the map

$$(\text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta) \otimes_{\mathcal{H}_k \epsilon_\beta} \mathcal{I} \longrightarrow \text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta$$

is injective.

In (a) the image of T_{n_s} in $\mathcal{H}_k \epsilon_\beta \oplus \mathcal{H}_k \epsilon_{\beta^s}$ (which we again denote by T_{n_s}) satisfies $T_{n_s}^2 = 0$, and besides the relations already stated, there are no further ones in $\mathcal{H}_k \epsilon_\beta \oplus \mathcal{H}_k \epsilon_{\beta^s}$. As ϵ_β and ϵ_{β^s} are orthogonal idempotents, we may further split up the situation, and the only critical case to be considered is where $\mathcal{I} = \mathcal{H}_k \epsilon_{\beta^s} T_{n_s}$ (or symmetrically $\mathcal{I} = \mathcal{H}_k \epsilon_\beta T_{n_s}$). We may rewrite the inclusion $\mathcal{I} = \mathcal{H}_k \epsilon_{\beta^s} T_{n_s} \rightarrow \mathcal{H}_k \epsilon_\beta$ as the exact sequence

$$\mathcal{H}_k \epsilon_\beta \xrightarrow{T_{n_s}} \mathcal{H}_k \epsilon_{\beta^s} \xrightarrow{T_{n_s}} \mathcal{H}_k \epsilon_\beta$$

and need to verify that the resulting sequence

$$\text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \xrightarrow{T_{n_s}} \text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta^s \xrightarrow{T_{n_s}} \text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta$$

is exact. But this is well known. (This exactness is specific to our working with the prime field \mathbb{F}_p ; for non-prime finite fields it fails.) In (b) the argument is similar (but easier, and valid for any finite field); namely, it boils down to the exactness of

$$\text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \xrightarrow{T_{n_s}} \text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \xrightarrow{T_{n_s}+1} \text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta \xrightarrow{T_{n_s}} \text{ind}_{\overline{\mathcal{B}}}^{\overline{\mathcal{S}}} \beta$$

(for the trivial $\beta \in \overline{\mathcal{T}}^c$).

Now let $m > 1$. We apply $\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes_{\mathcal{H}_{\mathfrak{o}_m}} (\cdot)$ to the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} \cap \pi_K^{m-1} \mathcal{H}_{\mathfrak{o}_m} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{I}/\mathcal{I} \cap \pi_K^{m-1} \mathcal{H}_{\mathfrak{o}_m} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_K^{m-1} \mathcal{H}_{\mathfrak{o}_m} & \longrightarrow & \mathcal{H}_{\mathfrak{o}_m} & \longrightarrow & \mathcal{H}_{\mathfrak{o}_{m-1}} \longrightarrow 0 \end{array}$$

Observe that $\mathcal{H}_{\mathfrak{o}_1} \cong \pi_K^{m-1} \mathcal{H}_{\mathfrak{o}_m}$. The bottom row then becomes the exact sequence $0 \rightarrow \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k \rightarrow \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \rightarrow \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_{m-1}} \rightarrow 0$. The top row remains exact in the middle. By induction hypothesis, the outer vertical arrows remain injective. Therefore the middle vertical arrow remains injective.

□

Let $\chi_{\overline{\mathcal{U}}}$ denote the unique element in $\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m}$ supported on $\overline{\mathcal{U}}$ and taking constant value 1 in \mathfrak{o}_m there. For a $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_{\mathfrak{o}_m}$ -(left)module M consider the natural map

$$(3) \quad M \longrightarrow \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_{\mathfrak{o}_m}} M, \quad m \mapsto \chi_{\overline{\mathcal{U}}} \otimes m.$$

Lemma 2.3. *The map (3) is an isomorphism from M onto $(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_{\mathfrak{o}_m}} M)^{\overline{\mathcal{U}}}$.*

PROOF: The same as in the proof of [8] Proposition 4.1 (where $\mathrm{GL}_2(\mathbb{F}_p)$ instead of $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$ was considered). Specifically, using Lemma 2.2 this is reduced to the case where M is an irreducible $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -module. \square

Lemma 2.4. *Let $\overline{\mathcal{U}} \neq \overline{\mathcal{U}}'$ be the unipotent radicals of two opposite Borel subgroups in $\overline{\mathcal{S}}$. Let W be a $k[\overline{\mathcal{S}}]$ -module which is generated by $W^{\overline{\mathcal{U}}'}$. Then W is generated by $W^{\overline{\mathcal{U}}'}$ even as a $k[\overline{\mathcal{U}}]$ -module.*

PROOF: The same as in [8] Lemma 2.1 (i). (We remark that the analogous statement is true more generally for $\mathrm{SL}_2(\mathbb{F}_q)$ or $\mathrm{PSL}_2(\mathbb{F}_q)$, for any finite field \mathbb{F}_q .) \square

For concreteness, let now more specifically $\overline{\mathcal{U}}$ denote the subgroup of unipotent upper triangular matrices in $\mathrm{SL}_2(\mathbb{F}_p)$. We define

$$\nu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad n_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h_s(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

for $x \in \mathbb{F}_p^\times$. In case $\overline{\mathcal{S}} = \mathrm{PSL}_2(\mathbb{F}_p)$ we use the same symbols $\overline{\mathcal{U}}$, ν , n_s , $h_s(x)$ for the respective images in $\overline{\mathcal{S}}$. Let $t = [\nu] - 1$ in $k[\overline{\mathcal{U}}]$; this is a generator of the maximal ideal in the local ring $k[\overline{\mathcal{U}}] \cong k[t]/(t^p)$. The k -algebra $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ is generated by the Hecke operators $T_{h_s(x)}$ for $x \in \mathbb{F}_p^\times$ together with T_{n_s} .

Let $0 \leq r \leq p-1$; in case $\overline{\mathcal{S}} = \mathrm{PSL}_2(\mathbb{F}_p)$ suppose in addition that r is even. We define a character $\chi_r : \mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k \rightarrow k$ by requiring

$$(4) \quad \chi_r(T_{h_s(x)}) = x^{-r} \quad \text{for all } x \in \mathbb{F}_p^\times.$$

as well as $\chi_r(T_{n_s}) = -1$ if $r = p-1$, and $\chi_r(T_{n_s}) = 0$ if $0 \leq r < p-1$.

We denote the one-dimensional k -vector space underlying χ_r again by χ_r , and we let e denote a basis element of it. By Lemma 2.3 we may regard e as an element of $(\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} \chi_r$.

Lemma 2.5. *The subspace $k \cdot e = \chi_r$ of $(\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} \chi_r$ is preserved by the element $t^r n_s^{-1}$ of $k[\overline{\mathcal{S}}]$. We have $t^r n_s^{-1} e = r! e$.*

PROOF: It is well known that $(\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} \chi_r$ is isomorphic with $\mathrm{Sym}^r(k^2)$ as a $k[\overline{\mathcal{S}}]$ -module. By Lemma 2.3 the space of $\overline{\mathcal{U}}$ -invariants in $(\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} \chi_r$ is χ_r . Therefore the claims follow by a straightforward computation in $\mathrm{Sym}^r(k^2)$. \square

Let $1 \leq r \leq p-1$; in case $\overline{\mathcal{S}} = \mathrm{PSL}_2(\mathbb{F}_p)$ suppose in addition that r is even. We define an $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -module M_r with k -basis e, f by requiring

$$T_{n_s}(e) = f \quad \text{and} \quad T_{n_s}(f) = \begin{cases} -f & : & r = p-1 \\ 0 & : & r < p-1 \end{cases},$$

$$T_{h_s(x)}(e) = x^r e \quad \text{and} \quad T_{h_s(x)}(f) = x^{-r} f$$

for all $x \in \mathbb{F}_p^\times$. By Lemma 2.3 we may regard M_r as a subspace of $(\mathrm{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M_r$.

Lemma 2.6. In $(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M_r$ we have

$$\begin{aligned}
(5) \quad & t^{p-1} n_s^{-1} e = f, \\
(6) \quad & t^r n_s^{-1} f = r! f, \\
(7) \quad & n_s f - e \in \sum_{i \geq 0} k \cdot t^i n_s e, \\
(8) \quad & t^{p-1-r} n_s^{-1} e - (p-1-r)! e \in \sum_{i \geq 0} k \cdot t^i n_s f.
\end{aligned}$$

If $r = p-1$ we have

$$\begin{aligned}
(9) \quad & n_s^{-1} (f + e) = f + e, \\
(10) \quad & t^{p-1} n_s^{-1} e + e \in \sum_{i \geq 0} k \cdot t^i n_s (f + e).
\end{aligned}$$

PROOF: In $\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k$ we compute

$$t^{p-1} n_s^{-1} \chi_{\overline{\mathcal{U}}} = ([\nu] - 1)^{p-1} n_s^{-1} \chi_{\overline{\mathcal{U}}} = \sum_{i=0}^{p-1} [\nu]^i n_s^{-1} \chi_{\overline{\mathcal{U}}} = T_{n_s} \chi_{\overline{\mathcal{U}}},$$

for the last equality observe $\overline{\mathcal{U}} = \{[\nu]^i; 0 \leq i \leq p-1\}$. This proves formula (5). Let $\overline{\mathcal{T}} = \{h_s(x) \mid x \in \mathbb{F}_p^\times\}$, the torus of diagonal matrices in $\overline{\mathcal{S}}$. Let $\theta : \overline{\mathcal{T}} \rightarrow k^\times$ be the character defined by $\theta(h_s(x)^{-1}) = x^r$. Then we have an isomorphism of $k[\overline{\mathcal{S}}]$ -modules

$$(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M_r \cong \text{ind}_{\overline{\mathcal{T}}\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \theta$$

sending e (resp. f) to the element \mathfrak{e} (resp. \mathfrak{f}) of $(\text{ind}_{\overline{\mathcal{T}}\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \theta)^{\overline{\mathcal{U}}}$ supported on $\overline{\mathcal{T}}\overline{\mathcal{U}}$ (resp. supported on $\overline{\mathcal{S}} - \overline{\mathcal{T}}\overline{\mathcal{U}} = \overline{\mathcal{T}}\overline{\mathcal{U}} n_s \overline{\mathcal{U}}$) and with $\mathfrak{e}(1) = 1$ (resp. with $\mathfrak{f}(n_s) = 1$). Indeed, in $(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M_r$ we find

$$\begin{aligned}
x^r e &= T_{h_s(x)}(e) = h_s(x)^{-1} e, \\
x^r f &= T_{h_s(x)^{-1}}(f) = h_s(x) f,
\end{aligned}$$

whereas in $\text{ind}_{\overline{\mathcal{T}}\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \theta$ we find

$$\begin{aligned}
\theta(h_s(x)^{-1}) \mathfrak{e}(\cdot) &= \mathfrak{e}(h_s(x)^{-1} \cdot) = \mathfrak{e}(\cdot h_s(x)^{-1}) = (h_s(x)^{-1} \mathfrak{e})(\cdot), \\
\theta(h_s(x)^{-1}) \mathfrak{f}(\cdot) &= \mathfrak{f}(h_s(x)^{-1} \cdot) = \mathfrak{f}(\cdot h_s(x)) = (h_s(x) \mathfrak{f})(\cdot),
\end{aligned}$$

and to see that the scaling factor is correct we compare formula (5) with $(t^{p-1} n_s^{-1} \mathfrak{e})(n_s) = \mathfrak{e}(1) = 1$. Now $\{\mathfrak{e}\} \cup \{t^i n_s \mathfrak{e} \mid 0 \leq i \leq p-1\}$ is a k -basis of $\text{ind}_{\overline{\mathcal{T}}\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \theta$. Since we have $(n_s \mathfrak{f})(1) = \mathfrak{f}(n_s) = 1 = \mathfrak{e}(1)$ but $(t^i n_s \mathfrak{e})(1) = 0$ for all $i \geq 0$, formula (7) follows. Next, f generates a $k[\overline{\mathcal{S}}]$ -submodule isomorphic with $\text{Sym}^r k^2$: we get formula (6). Finally, this $k[\overline{\mathcal{S}}]$ -submodule is in fact generated by $n_s f$ even as a $k[\overline{\mathcal{U}}]$ -module (Lemma 2.4), i.e. it coincides with $\sum_{i \geq 0} k \cdot t^i n_s f$, and the quotient of $(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M_r$ by this submodule is isomorphic with $\text{Sym}^{p-1-r} k^2$ as a $k[\overline{\mathcal{S}}]$ -module. This shows formula (8). For the stated formulae in case $r = p-1$ observe that $f + e$ generates the trivial one dimensional $k[\overline{\mathcal{S}}]$ -submodule, and dividing it out we are left with a copy of $\text{Sym}^{p-1} k^2$. \square

3 Half trees in Bruhat Tits buildings

Let G be the group of \mathbb{Q}_p -rational points of a \mathbb{Q}_p -split connected reductive group over \mathbb{Q}_p . Let Z denote the center of G . Fix a maximal split torus T in G , let $N(T)$ be its normalizer in G . Let Φ denote the set of roots of T . For $\alpha \in \Phi$ let N_α be the corresponding root subgroup in G . Choose a positive system Φ^+ in Φ . Let $N = \prod_{\alpha \in \Phi^+} N_\alpha$.

Let X denote the semi simple Bruhat-Tits building of G , let A denote its apartment corresponding to T . Our notational and terminological convention is that the facets of A or X are *closed* in X (i.e. *contain* all their faces (the lower dimensional facets at their boundary)). A chamber is a facet of codimension 0. For a chamber C in A let I_C be the Iwahori subgroup in G fixing C . Suppose we are given a semiinfinite chamber gallery

$$(11) \quad C^{(0)}, C^{(1)}, C^{(2)}, C^{(3)}, \dots$$

in A such that, setting

$$N_0^{(i)} = I_{C^{(i)}} \cap N = \prod_{\alpha \in \Phi^+} I_{C^{(i)}} \cap N_\alpha,$$

we have

$$(12) \quad N_0^{(0)} \supset N_0^{(1)} \supset N_0^{(2)} \supset N_0^{(3)} \supset \dots \quad \text{with } [N_0^{(i)} : N_0^{(i+1)}] = p \text{ for all } i \geq 0.$$

We write

$$N_0 = N_0^{(0)}, \quad C = C^{(0)}, \quad I = I_C = I_{C^{(0)}}.$$

There is a unique sequence $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \dots$ in Φ^+ such that, setting

$$e[i, \alpha] = |\{0 \leq j \leq i-1 \mid \alpha = \alpha^{(j)}\}|$$

for $i \geq 0$ and $\alpha \in \Phi^+$, we have

$$(13) \quad N_0^{(i)} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{e[i, \alpha]}}.$$

Geometrically, $C^{(i+1)}$ and $C^{(i)}$ share a common facet of codimension 1 contained in a wall which belongs to the translation class of walls corresponding to $\alpha^{(i)}$.

Lemma 3.1. $N_0^{(i+1)}$ is a normal subgroup in $N_0^{(i)}$, for any $i \geq 0$.

PROOF: We claim that, more generally, for any inclusion $U_1 \subset U_2$ of open subgroups of N with $[U_2 : U_1] = p$ and with $U_i = \prod_{\alpha \in \Phi^+} (U_i \cap N_\alpha)$ for $i = 1, 2$ we have: U_1 is normal in U_2 . Indeed, there is a unique $\alpha \in \Phi^+$ with $U_2 \cap N_\beta = U_1 \cap N_\beta$ for all $\beta \neq \alpha$. By standard facts on reductive groups, for $u \in N_\alpha$ and $v \in N_\gamma$ (any γ) the commutator $uvu^{-1}v^{-1}$ is a product of elements of the N_β with $\beta \neq \alpha$. The claim follows. \square

Definition: We define the half tree Y , endowed with an action by the group N_0 , as follows. Both its set Y^0 of vertices and its set Y^1 of edges are identified with the set of chambers of X of the form $n \cdot C^{(i)}$ for $n \in N_0$ and $i \geq 0$. The N_0 -action is the obvious one. Whenever we view the chamber $C^{(i)}$ of X as a vertex of Y we denote it by \mathbf{v}_i ; whenever we view $C^{(i)}$ as an edge of Y we denote it by \mathbf{e}_i . The simplicial structure on Y is given by declaring that for $n \in N_0$ and $i \geq 0$ the edge $n \cdot \mathbf{e}_{i+1}$ contains the two vertices $n \cdot \mathbf{v}_i$ and $n \cdot \mathbf{v}_{i+1}$. An element in $\bar{Y}^1 = Y^1 - \{\mathbf{e}_0\}$ we also write as the unordered pair of vertices it contains, e.g. $n \cdot \mathbf{e}_{i+1} = \{n \cdot \mathbf{v}_i, n \cdot \mathbf{v}_{i+1}\}$. In addition we have the edge \mathbf{e}_0 of Y : it contains only the vertex \mathbf{v}_0 (hence is a 'loose' end of Y) and is fixed by N_0 .

The half tree \bar{Y} is obtained from Y by removing the edge \mathbf{e}_0 . Thus, its set of vertices is $\bar{Y}^0 = Y^0$, its set of edges is $\bar{Y}^1 = Y^1 - \{\mathbf{e}_0\}$, and the simplicial structure is the one induced from Y .

Remark: By definition, we have a natural bijection $Y^1 \cong Y^0$: to \mathbf{e}_i it assigns \mathbf{v}_i , and more generally, to any edge its 'outward pointing' vertex. Thus, instead of identifying Y^0 with the set of chambers $n \cdot C^{(i)}$ (with $n \in N_0$ and $i \geq 0$), one might just as well (or perhaps more appropriately) identify Y^0 with the set of facets of codimension 1 in X shared by two chambers of the form $n \cdot C^{(i)}$.

For $\mathbf{v} \in Y^0$ let

$$(14) \quad]\mathbf{v}[= \{\mathbf{w} \in Y^0 \mid \{\mathbf{v}, \mathbf{w}\} \in Y^1 \text{ and } \mathbf{v} \in [\mathbf{w}, \mathbf{v}_0]\}$$

where $[\mathbf{w}, \mathbf{v}_0] \subset Y^0$ denotes the set of vertices the unique geodesic from \mathbf{w} to \mathbf{v}_0 in Y is passing through.

Let \mathfrak{X} denote the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Let \mathfrak{X}^0 resp. \mathfrak{X}^1 denote the set of vertices, resp. the set of edges of \mathfrak{X} . Let $\mathbf{v}_0 \in \mathfrak{X}^0$ denote the vertex fixed by $\mathrm{PGL}_2(\mathbb{Z}_p)$. In $\mathrm{GL}_2(\mathbb{Q}_p)$ we define the elements

$$\varphi = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the subgroups

$$\Gamma = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 1+p\mathbb{Z}_p & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{N}_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}.$$

For $a \in \mathbb{Z}_p^\times$ let us write $\gamma(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma$. Notice that γ_0 , resp. ν , is a topological generator of Γ_0 , resp. of \mathfrak{N}_0 . For $r \in \mathbb{N}$ let $[\mathfrak{N}_0, \varphi^r]$ (resp. $[\mathfrak{N}_0, \varphi^r, \Gamma_0]$, resp. $[\mathfrak{N}_0, \varphi^r, \Gamma]$) denote the submonoid (*not*: subgroup) of $\mathrm{GL}_2(\mathbb{Q}_p)$ generated by the subgroup \mathfrak{N}_0 and the element φ^r (resp. by the subgroups \mathfrak{N}_0 and Γ_0 and the element φ^r , resp. by the subgroups \mathfrak{N}_0 and Γ and the

element φ^r).

For $i \in \mathbb{Z}$ we put $\mathbf{v}_i = \varphi^i(\mathbf{v}_0) \in \mathfrak{X}^0$ and $\mathbf{e}_i = \{\mathbf{v}_i, \mathbf{v}_{i-1}\} \in \mathfrak{X}^1$. We denote by \mathfrak{X}_+ the half tree with sets of vertices, resp. edges,

$$\mathfrak{X}_+^0 = \coprod_{i \geq 0} \mathfrak{N}_0 \cdot \mathbf{v}_i, \quad \mathfrak{X}_+^1 = \coprod_{i \geq 0} \mathfrak{N}_0 \cdot \mathbf{e}_i.$$

Thus \mathfrak{X}_+ is one of the halves obtained by cutting \mathfrak{X} at the edge \mathbf{e}_0 into two pieces, where by definition \mathbf{e}_0 belongs to \mathfrak{X}_+ as its only loose end (i.e. an edge containing only one vertex).

The half tree $\overline{\mathfrak{X}}_+$ is obtained from \mathfrak{X}_+ by removing the edge \mathbf{e}_0 . Thus, its set of vertices is $\overline{\mathfrak{X}}_+^0 = \mathfrak{X}_+^0$, its set of edges is $\overline{\mathfrak{X}}_+^1 = \mathfrak{X}_+^1 - \{\mathbf{e}_0\}$, and the simplicial structure is the one induced from \mathfrak{X}_+ .

For $\mathbf{v} \in \mathfrak{X}_+^0$ we put, similarly to definition (14),

$$]\mathbf{v}[= \{\mathfrak{w} \in \mathfrak{X}_+^0 \mid \{\mathbf{v}, \mathfrak{w}\} \in \mathfrak{X}_+^1 \text{ and } \mathbf{v} \in [\mathfrak{w}, \mathbf{v}_0]\}.$$

Theorem 3.2. (a) *There exists an isomorphism of simplicial complexes*

$$\Theta : Y \xrightarrow{\cong} \mathfrak{X}_+$$

such that for all $\mathbf{v} \in Y^0$ and all $b \in \mathbb{Z}_{\geq 0}$ the composition of bijections

$$(15) \quad]\mathbf{v}[\xrightarrow{\Theta}]\Theta(\mathbf{v})[\xrightarrow{\nu^b}]\nu^b(\Theta(\mathbf{v}))[\xrightarrow{\Theta^{-1}}]\Theta^{-1}(\nu^b(\Theta(\mathbf{v})))[$$

is induced by the action of some element $g(\mathbf{v}, b)$ of N_0 .

(b) *Let $\phi \in N(T)$ and $r \in \mathbb{N}$ such that $\phi(C^{(i)}) = C^{(i+r)}$ for all $i \geq 0$. Then $\phi(Y^1) \subset Y^1$, and the isomorphism Θ and the $g(\mathbf{v}, b)$ in (a) can be chosen in a way such that*

$$(16) \quad \varphi^r \circ \Theta = \Theta \circ \phi$$

and such that in G we have

$$(17) \quad g(\phi(\mathbf{v}), p^r) \cdot \phi = \phi \cdot g(\mathbf{v}, 1).$$

(c) *Let $\tau : \mathbb{Z}_p^\times \rightarrow T$ be a homomorphism such that $\alpha^{(i)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}$ for all $i \geq 0$. Then there exists for each $\mathbf{v} \in Y^0$ and each $a \in \mathbb{Z}_p^\times$ some element $h(\mathbf{v}, a)$ in $N_0 \cdot \tau(a)$ which induces the composition of bijections*

$$(18) \quad]\mathbf{v}[\xrightarrow{\Theta}]\Theta(\mathbf{v})[\xrightarrow{\gamma(a)}]\gamma(a)(\Theta(\mathbf{v}))[\xrightarrow{\Theta^{-1}}]\Theta^{-1}(\gamma(a)(\Theta(\mathbf{v})))[.$$

For $a \in \mathbb{N} \cap \mathbb{Z}_p^\times$ these $h(\mathbf{v}, a)$ can be chosen in a way such that in G we have

$$(19) \quad h(g(\mathbf{v}, 1)\mathbf{v}, a) \cdot g(\mathbf{v}, 1) = g(h(\mathbf{v}, a)\mathbf{v}, a) \cdot h(\mathbf{v}, a).$$

(d) *Let $\phi \in N(T)$ be as in (b), let $\tau : \mathbb{Z}_p^\times \rightarrow T$ be as in (c) such that for all $a \in \mathbb{Z}_p^\times$ we have $\tau(a)\phi = \phi\tau(a)$ in G . The isomorphism Θ , the $g(\mathbf{v}, b)$ and the $h(\mathbf{v}, a)$ can be chosen in a way such that in G we have*

$$(20) \quad h(\phi(\mathbf{v}), a) \cdot \phi = \phi \cdot h(\mathbf{v}, a).$$

PROOF: (a) For any $j \geq 0$ we choose a topological generator ν_j of $N_{\alpha^{(j)}} \cap N_0 \cong \mathbb{Z}_p$. For $m \in \mathbb{Z}_{\geq 0}$ we define the elements m_0, m_1, m_2, \dots of $\{0, \dots, p-1\}$ to be the digits of m in its p -adic expansion, i.e. $m = \sum_{j \geq 0} m_j p^j$. We then define the element

$$\nabla(m) = \nu_0^{m_0} \cdots \nu_j^{m_j p^{e[j, \alpha^{(j)}]}} \cdot \nu_{j+1}^{m_{j+1} p^{e[j+1, \alpha^{(j+1)}]}} \cdots$$

of N_0 . We write $[0, p^i[= \{m \in \mathbb{Z}_{\geq 0} \mid 0 \leq m < p^i\}$ for $i \geq 0$. An easy induction on i shows that $\{\nabla(m) \mid m \in [0, p^i[\}$ is a set of representatives for the set of cosets $N_0/N_0^{(i)}$. Since $N_0^{(i)}$ is the stabilizer of \mathbf{v}_i in N_0 it follows that the map

$$\beta_{Y,i} : [0, p^i[\longrightarrow N_0 \cdot \mathbf{v}_i, \quad m \mapsto \nabla(m) \cdot \mathbf{v}_i$$

is bijective. On the other hand we have the bijection

$$\beta_{\mathfrak{X}_+,i} : [0, p^i[\longrightarrow \mathfrak{N}_0 \cdot \mathbf{v}_i, \quad m \mapsto \nu^m \cdot \mathbf{v}_i.$$

We put

$$\Theta_i = \beta_{\mathfrak{X}_+,i} \circ \beta_{Y,i}^{-1} : N_0 \cdot \mathbf{v}_i \xrightarrow{\cong} \mathfrak{N}_0 \cdot \mathbf{v}_i.$$

Taken for all $i \geq 0$ this is a bijection $\Theta : Y^0 \cong \mathfrak{X}_+^0$ and is easily seen to define an isomorphism of simplicial complexes $Y \cong \mathfrak{X}_+$. In the following, for $i \geq 0$ and $m \in \mathbb{Z}$ we write $\beta_{\mathfrak{X}_+,i}(m) = \beta_{\mathfrak{X}_+,i}(m')$ and $\beta_{Y,i}(m) = \beta_{Y,i}(m')$ where $m' \in [0, p^i[$ is such that $m - m' \in p^i \mathbb{Z}$. For $m \in [0, p^i[$ we find

$$(21) \quad]\beta_{Y,i}(m)[= \{\beta_{Y,i+1}(m + p^i t) \mid 0 \leq t \leq p-1\},$$

$$(22) \quad]\beta_{\mathfrak{X}_+,i}(m)[= \{\beta_{\mathfrak{X}_+,i+1}(m + p^i t) \mid 0 \leq t \leq p-1\}.$$

For all $b \in \mathbb{Z}_{\geq 0}$ we have $\nu^b(\beta_{\mathfrak{X}_+,i}(m)) = \beta_{\mathfrak{X}_+,i}(b+m)$ in $\mathfrak{N}_0 \cdot \mathbf{v}_i$ and

$$\nu^b(\beta_{\mathfrak{X}_+,i+1}(m + p^i t)) = \beta_{\mathfrak{X}_+,i+1}(b+m + p^i t) \quad \text{in } \mathfrak{N}_0 \cdot \mathbf{v}_{i+1}$$

for all $0 \leq t \leq p-1$. On the other hand, if for $\mathbf{v} = \beta_{Y,i}(m)$ with $m \in [0, p^i[$ we put

$$(23) \quad g(\mathbf{v}, b) = \nabla(b+m) \cdot \nabla(m)^{-1} \in N_0$$

then $g(\mathbf{v}, b) \cdot \beta_{Y,i}(m) = \beta_{Y,i}(b+m)$ in $N_0 \cdot \mathbf{v}_i$ and

$$g(\mathbf{v}, b) \cdot \beta_{Y,i+1}(m + p^i t) = \beta_{Y,i+1}(b+m + p^i t)$$

in $N_0 \cdot \mathbf{v}_{i+1}$. To see this last equation we use Lemma 3.1. Together we obtain

$$\nu^b \cdot \Theta_{i+1}(\mathbf{w}) = \Theta_{i+1}(g(\mathbf{v}, b) \cdot \mathbf{w})$$

for all $\mathbf{w} \in]\mathbf{v}[$, as desired.

(b) For a chamber D contained in A let $\pi_D : X \rightarrow A$ denote the retraction from X to A centered at D , i.e. the unique polysimplicial map restricting to the identity on A and with $\pi_D^{-1}(D) = D$. We have $Y^1 = N_0\{C^{(i)}\}_{i \geq 0} = I\{C^{(i)}\}_{i \geq 0} = \pi_C^{-1}(\{C^{(i)}\}_{i \geq 0})$ and hence

$$\phi(Y^1) = \phi(\pi_C^{-1}(\{C^{(i)}\}_{i \geq 0})) = \pi_{C(r)}^{-1}(\{C^{(i)}\}_{i \geq r}) \subset \pi_C^{-1}(\{C^{(i)}\}_{i \geq 0}) = Y^1.$$

We also see that $\pi_{C^{(j+r)}}^{-1}(\{C^{(i)}\}_{i \geq j+r}) = \phi(\pi_{C^{(j)}}^{-1}(\{C^{(i)}\}_{i \geq j}))$ is stable under $\phi(N_{\alpha^{(j)}} \cap N_0)\phi^{-1}$, for any $j \geq 0$. Now ϕ moves the wall containing $C^{(j)} \cap C^{(j+1)}$ to the one containing $C^{(j+r)} \cap C^{(j+r+1)}$, therefore $\phi(N_{\alpha^{(j)}} \cap N_0)\phi^{-1}$ must be the stabilizer of $C^{(j+r)} \cap C^{(j+r+1)}$ in either $N_{\alpha^{(j+r)}}$ or in $N_{-\alpha^{(j+r)}}$. But the stabilizer of $C^{(j+r)} \cap C^{(j+r+1)}$ in $N_{-\alpha^{(j+r)}}$ does not stabilize $\pi_{C^{(j+r)}}^{-1}(\{C^{(i)}\}_{i \geq j+r})$. Thus

$$(24) \quad \phi(N_{\alpha^{(j)}} \cap N_0)\phi^{-1} = (N_{\alpha^{(j+r)}} \cap N_0)^{p^{e[r, \alpha^{(j+r)]}}}.$$

Now we refine the construction in (a) by choosing the ν_j more specifically. First we claim

$$(25) \quad e[r, \alpha^{(j+r)}] + e[i, \alpha^{(j)}] = e[i+r, \alpha^{(j+r)}]$$

for all $i, j \geq 0$. Indeed, the above discussion shows in particular that for all $j, t \geq 0$ we have $\alpha^{(j)} = \alpha^{(t)}$ if and only if $\alpha^{(j+r)} = \alpha^{(t+r)}$, or in other words

$$\{0 \leq t \leq i-1 \mid \alpha^{(j+r)} = \alpha^{(t+r)}\} = \{0 \leq t \leq i-1 \mid \alpha^{(j)} = \alpha^{(t)}\}.$$

The set on the left hand side contains $e[i+r, \alpha^{(j+r)}] - e[r, \alpha^{(j+r)}]$ elements, as can be seen by performing the bijective shift $t \mapsto t+r$. The set on the right hand side contains $e[i, \alpha^{(j)}]$ elements. The claim follows. By formula (24) we may choose the topological generators ν_j of the $N_{\alpha^{(j)}} \cap N_0$ in such a way that

$$(26) \quad \phi\nu_j\phi^{-1} = \nu_{j+r}^{p^{e[r, \alpha^{(j+r)]}}}$$

for all $j \geq 0$. An induction using formulae (25) and (26) then shows

$$(27) \quad \phi\nabla(m)\phi^{-1} = \nabla(mp^r)$$

for all $m \in \mathbb{Z}_{\geq 0}$. We claim that for all $i \geq 0$ we have

$$(28) \quad \phi \circ \beta_{Y,i} = \beta_{Y,i+r} \cdot p^r,$$

$$(29) \quad \varphi^r \circ \beta_{\mathfrak{X}_+,i} = \beta_{\mathfrak{X}_+,i+r} \cdot p^r$$

(equality of maps $[0, p^i[\rightarrow N_0 \cdot \mathbf{v}_{i+r}$ resp. $[0, p^i[\rightarrow \mathfrak{N}_0 \cdot \mathbf{v}_{i+r}$). Indeed, formula (28) follows from formula (27) and $\phi(\mathbf{v}_i) = \mathbf{v}_{i+r}$, while formula (29) follows from $\varphi^r \nu \varphi^{-r} = \nu^{p^r}$ and $\varphi^r(\mathbf{v}_i) = \mathbf{v}_{i+r}$. Combining formulae (28) and (29) gives formula (16), as desired.

To prove formula (17) we write $\mathbf{v} = \beta_{Y,i}(m)$ with $m \in [0, p^i[$ as before. By formula (28) we have $\phi(\mathbf{v}) = \beta_{Y,i+r}(p^r m)$ and hence $g(\phi(\mathbf{v}), p^r) = \nabla(p^r(m+1)) \cdot \nabla(p^r m)^{-1}$, while on the other hand we have $g(\mathbf{v}, 1) = \nabla(m+1) \cdot \nabla(m)^{-1}$. Therefore it will be enough to prove

$$\phi\nabla(m+1) \cdot \nabla(m)^{-1}\phi^{-1} = \nabla(p^r(m+1)) \cdot \nabla(p^r m)^{-1}$$

in N_0 . But this follows from formula (27) applied to both m and $m+1$.

(c) Let us first remark that, since $\tau(a) \in T$ in fact belongs to the maximal compact subgroup of T fixing A pointwise, we have $\tau(a)C^{(i)} = C^{(i)}$, i.e. $\tau(a)\mathbf{v}_i = \mathbf{v}_i$ for all $i \geq 0$.

We may and do assume that $a \in \mathbb{Z} \cap \mathbb{Z}_p^\times$. We write $\mathbf{v} = \beta_{Y,i}(m)$ with $m \in [0, p^i[$ as before. We first claim that

$$(30) \quad \Theta^{-1} \circ \gamma(a) \circ \Theta = \tau(a) \quad \text{in } \text{Aut}(\mathbf{v}_i).$$

To see this observe (cf. formulae (21), (22)) that

$$\begin{aligned} \mathbf{v}_i[&= \{(\nu_i^{p^{e[i, \alpha^{(i)}}]})^b \mathbf{v}_{i+1} \mid 0 \leq b \leq p-1\}, \\ \mathbf{v}_i[&= \{(\nu^{p^i})^b \mathbf{v}_{i+1} \mid 0 \leq b \leq p-1\}, \end{aligned}$$

and that the bijection $\Theta : \mathbf{v}_i[\rightarrow \mathbf{v}_i[$ sends $(\nu_i^{p^{e[i, \alpha^{(i)}}]})^b \mathbf{v}_{i+1}$ to $(\nu^{p^i})^b \mathbf{v}_{i+1}$. The hypothesis $\alpha^{(i)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}$ implies $\tau(a)n = n^a \tau(a)$ in G for all $n \in N_{\alpha^{(i)}} \cap N_0$. Specifically,

$$\tau(a)(\nu_i^{p^{e[i, \alpha^{(i)}}]})^b = ((\nu_i^{p^{e[i, \alpha^{(i)}}]})^b)^a \tau(a).$$

We thus get

$$\tau(a)(\nu_i^{p^{e[i, \alpha^{(i)}}]})^b \mathbf{v}_{i+1} = ((\nu_i^{p^{e[i, \alpha^{(i)}}]})^b)^a \tau(a) \mathbf{v}_{i+1} = (\nu_i^{p^{e[i, \alpha^{(i)}}]})^{ab} \mathbf{v}_{i+1}.$$

On the other hand we have

$$\gamma(a)(\nu^{p^i})^b \mathbf{v}_{i+1} = ((\nu^{p^i})^b)^a \gamma(a) \mathbf{v}_{i+1} = (\nu^{p^i})^{ab} \mathbf{v}_{i+1}.$$

Comparing these formulae we get our claim (30). Now $g(\mathbf{v}_i, am)$ resp. $g(\mathbf{v}_i, m)$ induces

$$\begin{aligned} \Theta^{-1} \circ \nu^{am} \circ \Theta &: \mathbf{v}_i[\longrightarrow \mathbf{v}_i[\Theta^{-1}(\nu^{am}(\Theta(\mathbf{v}_i)))[\\ \text{resp.} \quad \Theta^{-1} \circ \nu^m \circ \Theta &: \mathbf{v}_i[\longrightarrow \mathbf{v}_i[\Theta^{-1}(\nu^m(\Theta(\mathbf{v}_i)))[= \mathbf{v}_i[. \end{aligned}$$

We may therefore rewrite the arrow (18) as

$$\begin{aligned} \Theta^{-1} \circ \gamma(a) \circ \Theta &= \Theta^{-1} \circ \nu^{am} \circ \gamma(a) \circ \nu^{-m} \circ \Theta \\ &= \Theta^{-1} \circ \nu^{am} \circ \gamma(a) \circ \Theta \circ g(\mathbf{v}_i, m)^{-1} \\ &\stackrel{(i)}{=} \Theta^{-1} \circ \nu^{am} \circ \Theta \circ \tau(a) \circ g(\mathbf{v}_i, m)^{-1} \\ &= g(\mathbf{v}_i, am) \circ \tau(a) \circ g(\mathbf{v}_i, m)^{-1} \end{aligned}$$

where (i) uses formula (30). Inserting formula (23) for $g(\mathbf{v}_i, am)$ and $g(\mathbf{v}_i, m)$ we find that the element

$$(31) \quad h(\mathbf{v}, a) = \nabla(am) \cdot \tau(a) \cdot \nabla(m)^{-1}$$

of G induces the arrow (18). It belongs to $N_0 \cdot \tau(a)$ because $\nabla(am)$ and $\nabla(m)^{-1}$ belong to N_0 , and $\tau(a)$ normalizes N_0 .

To show formula (19) we write $\mathbf{v} = \beta_{Y,i}(m) = \nabla(m)\mathbf{v}_i$ with $m \in [0, p^i[$ as before. From $g(\mathbf{v}, 1) = \nabla(m+1)\nabla(m)^{-1}$ we get $g(\mathbf{v}, 1)\mathbf{v} = \beta_{Y,i}(m+1)$, from formula (31) and $\tau(a)\mathbf{v}_i = \mathbf{v}_i$ we get $h(\mathbf{v}, a)\mathbf{v} = \beta_{Y,i}(am)$. Thus, inserting formula (31) again, formula (19) reads

$$\begin{aligned} & \nabla(a(m+1)) \cdot \tau(a) \cdot \nabla(m+1)^{-1} \cdot \nabla(m+1) \cdot \nabla(m)^{-1} \\ &= \nabla(am+a) \cdot \nabla(am)^{-1} \cdot \nabla(am) \cdot \tau(a) \cdot \nabla(m)^{-1} \end{aligned}$$

which is obviously correct.

(d) To show formula (20) for $\mathbf{v} = \beta_{Y,i}(m)$ we notice that $\phi(\mathbf{v}) = \beta_{Y,i}(p^r m)$ by formula (28). Inserting formula (31) for $h(\phi(\mathbf{v}), a)$ and $h(\mathbf{v}, a)$ we see that formula (20) becomes

$$\nabla(ap^r m) \cdot \tau(a) \cdot \nabla(p^r m)^{-1} \cdot \phi = \phi \cdot \nabla(am) \cdot \tau(a) \cdot \nabla(m)^{-1}.$$

That this is correct follows from formula (27), applied both on the left hand side and on the right hand side, and our assumption $\tau(a)\phi = \phi\tau(a)$. \square

Remarks: Let N'_0 denote the subgroup of G generated by all the $N_{\alpha(j)} \cap N_0$ for $j \geq 0$. The proof of Theorem 3.2 shows that the elements $g(\mathbf{v}, b)$ (resp. $h(\mathbf{v}, a)$) of G in fact belong to N'_0 (resp. to the subgroup of G generated by N'_0 and $\tau(a)$).

4 Coefficient systems on half trees

Let \mathfrak{T} be a half tree with set of vertices \mathfrak{T}^0 and set of edges \mathfrak{T}^1 , as considered in section 3.

Definition: (a) A (homological) coefficient system \mathcal{V} in \mathfrak{o} -modules on \mathfrak{T} is a collection of \mathfrak{o} -modules $\mathcal{V}(\tau)$ for each simplex τ of \mathfrak{T} , and a collection of \mathfrak{o} -linear transition maps $r_y^\tau : \mathcal{V}(\tau) \rightarrow \mathcal{V}(y)$ for each $y \in \mathfrak{T}^0$, $\tau \in \mathfrak{T}^1$ with $y \in \tau$. The \mathfrak{o} -modules $H_0(\mathfrak{T}, \mathcal{V})$ and $H_1(\mathfrak{T}, \mathcal{V})$ are defined by the exact sequence

$$0 \longrightarrow H_1(\mathfrak{T}, \mathcal{V}) \longrightarrow \bigoplus_{\tau \in \mathfrak{T}^1} \mathcal{V}(\tau) \longrightarrow \bigoplus_{y \in \mathfrak{T}^0} \mathcal{V}(y) \longrightarrow H_0(\mathfrak{T}, \mathcal{V}) \longrightarrow 0$$

where $v \in \mathcal{V}(\tau)$ is sent to $\sum_{\substack{y \in \mathfrak{T}^0 \\ y \in \tau}} r_y^\tau(v)$.

Morphisms of coefficient systems are defined in the obvious way. A sequence of coefficient systems $0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_2 \rightarrow \mathcal{V}_3 \rightarrow 0$ is called exact if for any simplex τ of \mathfrak{T} the sequence $0 \rightarrow \mathcal{V}_1(\tau) \rightarrow \mathcal{V}_2(\tau) \rightarrow \mathcal{V}_3(\tau) \rightarrow 0$ is exact.

(b) Let H be a monoid (with neutral element 1) acting on \mathfrak{T} . A coefficient system \mathcal{V} on \mathfrak{T} is called H -equivariant if in addition we are given an \mathfrak{o} -linear map $g_\tau : \mathcal{V}(\tau) \rightarrow \mathcal{V}(g\tau)$ for each simplex τ and each $g \in H$, subject to the following conditions:

- (a) $g_{h\tau} \circ h_\tau = (gh)_\tau$ for simplices τ and $g, h \in H$,
- (b) $1_\tau = \text{id}_{\mathcal{V}(\tau)}$ for simplices τ ,
- (c) $r_{gy}^{g\tau} \circ g_y = g_\tau \circ r_y^\tau$ for $y \in \mathfrak{T}^0$ and $\tau \in \mathfrak{T}^1$ with $y \in \tau$, and $g \in H$.

We then we have a natural action of H on $H_0(\mathfrak{T}, \mathcal{V})$ if (at least) one of the following conditions is satisfied:

- \mathfrak{T} has no loose ends (to each edge two vertices are assigned), or
- H acts by automorphisms of \mathfrak{T} .

Lemma 4.1. (a) *An exact sequence of coefficient systems in \mathfrak{o} -modules $0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V}_2 \rightarrow \mathcal{V}_3 \rightarrow 0$ induces an exact sequence of \mathfrak{o} -modules*

$$0 \longrightarrow H_1(\mathfrak{T}, \mathcal{V}_1) \longrightarrow H_1(\mathfrak{T}, \mathcal{V}_2) \longrightarrow H_1(\mathfrak{T}, \mathcal{V}_3) \longrightarrow H_0(\mathfrak{T}, \mathcal{V}_1) \longrightarrow H_0(\mathfrak{T}, \mathcal{V}_2) \longrightarrow H_0(\mathfrak{T}, \mathcal{V}_3) \longrightarrow 0.$$

(b) *If the coefficient system \mathcal{V} has injective transition maps r_y^τ , then $H_1(\mathfrak{T}, \mathcal{V}) = 0$.*

PROOF: Both statements are very easy to prove. □

Definition: We say that a N_0 -equivariant (resp. \mathfrak{N}_0 -equivariant) coefficient system \mathcal{V} on Y (resp. on \mathfrak{X}_+) is of level 1 if for any $i \geq 0$ the action of $N_0^{(i+1)}$ on $\mathcal{V}(\mathbf{v}_i)$ and the action of $N_0^{(i)}$ on $\mathcal{V}(\mathbf{e}_i)$ (resp. the action of $\mathfrak{N}_0^{p^{i+1}}$ on $\mathcal{V}(\mathbf{v}_i)$ and the action of \mathfrak{N}_0^i on $\mathcal{V}(\mathbf{e}_i)$) are trivial.

We fix a gallery (11) and choose an isomorphism $\Theta : Y \xrightarrow{\cong} \mathfrak{X}_+$ as in Theorem 3.2.

Theorem 4.2. *Let \mathcal{V} be a N_0 -equivariant coefficient system of level 1 on Y .*

(a) *The push forward $\Theta_*\mathcal{V}$ of \mathcal{V} to \mathfrak{X}_+ is in a natural way a \mathfrak{N}_0 -equivariant coefficient system of level 1 on \mathfrak{X}_+ .*

(b) *Let $\phi \in N(T)$ and $r \in \mathbb{N}$ be as in Theorem 3.2(b). If the action of N_0 on \mathcal{V} extends to an action of the submonoid of G generated by N_0 and the element ϕ , then $\Theta_*\mathcal{V}$ is in a natural way $[\mathfrak{N}_0, \varphi^r]$ -equivariant.*

(c) *Let $\phi \in N(T)$, $r \in \mathbb{N}$ and $\tau : \mathbb{Z}_p^\times \rightarrow T$ be as in Theorem 3.2(d). If the action of N_0 on \mathcal{V} extends to an action of the submonoid of G generated by N_0 , by the image of τ and by the element ϕ , then $\Theta_*\mathcal{V}$ is in a natural way $[\mathfrak{N}_0, \varphi^r, \Gamma]$ -equivariant.*

(d) *In (a), resp. (b), resp. (c), the isomorphism class of $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V})$, as an \mathfrak{o} -module acted on by the respective monoid, depends on the choice of (11) alone, resp. of (11) and ϕ alone, resp. of (11) and ϕ and τ alone, but not on the choice of Θ .*

PROOF: (a) Let $g \in \mathfrak{N}_0$ and $\mathbf{v} \in \mathfrak{X}_+^0$. As \mathfrak{N}_0 is topologically generated by ν we find, by Theorem 3.2(a), some $g' \in N_0$ which induces the bijection

$$(32) \quad]\Theta^{-1}(\mathbf{v})[\xrightarrow{\Theta}]\mathbf{v}[\xrightarrow{g}]g\mathbf{v}[\xrightarrow{\Theta^{-1}}]\Theta^{-1}(g\mathbf{v})[.$$

We define $g_{\mathbf{v}} : \Theta_*\mathcal{V}(\mathbf{v}) \rightarrow \Theta_*\mathcal{V}(g\mathbf{v})$ to be the map

$$g_{\mathbf{v}} = \Theta \circ g'_{\Theta^{-1}\mathbf{v}} \circ \Theta^{-1} \quad : \quad \Theta_*\mathcal{V}(\mathbf{v}) \longrightarrow \Theta_*\mathcal{V}(\Theta(g'\Theta^{-1}(\mathbf{v}))) = \Theta_*\mathcal{V}(g\mathbf{v}).$$

This definition is independent on the choice of g' . Indeed, let also $g'' \in N_0$ induce the bijection (32). Then $g'^{-1}g''$ belongs to the pointwise stabilizer H of $]\Theta^{-1}(\mathbf{v})[$ in N_0 . But the action of

H on $\mathcal{V}(\Theta^{-1}\mathfrak{v})$ is trivial. To see this we may assume, by N_0 -equivariance of \mathcal{V} , that $\Theta^{-1}\mathfrak{v} = \mathfrak{v}_i$ for some i . Then $H = N_0^{(i+1)}$ and this indeed acts trivially on $\mathcal{V}(\Theta^{-1}\mathfrak{v}) = \mathcal{V}(\mathfrak{v}_i)$ by the level 1 assumption. Therefore g' and $g'' = g'(g'^{-1}g'')$ give the same map $\mathcal{V}(\Theta^{-1}\mathfrak{v}) \rightarrow \mathcal{V}(g\Theta^{-1}\mathfrak{v})$. It follows that $g_{\mathfrak{v}}$ as defined above is well defined.

Now let $g \in \mathfrak{N}_0$ and $\mathfrak{e} \in \mathfrak{X}_+^1$. Choose some $g' \in N_0$ with $g'(\Theta^{-1}(\mathfrak{e})) = \Theta^{-1}(g(\mathfrak{e}))$ and define $g_{\mathfrak{e}} : \Theta_*\mathcal{V}(\mathfrak{e}) \rightarrow \Theta_*\mathcal{V}(g\mathfrak{e})$ to be the map

$$g_{\mathfrak{e}} = \Theta \circ g'_{\Theta^{-1}\mathfrak{e}} \circ \Theta^{-1} \quad : \quad \Theta_*\mathcal{V}(\mathfrak{e}) \longrightarrow \Theta_*\mathcal{V}(\Theta(g'\Theta^{-1}(\mathfrak{e}))) = \Theta_*\mathcal{V}(g\mathfrak{e}).$$

Again this definition is independent on the choice of g' . The definitions immediately show that we have defined an \mathfrak{N}_0 -action on $\Theta_*\mathcal{V}$ which is again of level 1.

(b) By formula (16), the ϕ -action on \mathcal{V} induces a φ^r -action on $\Theta_*\mathcal{V}$. It follows from formula (17) — which corresponds to the formula $\nu^{p^r} \cdot \varphi^r = \varphi^r \cdot \nu$ in $[\mathfrak{N}_0, \varphi^r]$ — that the actions of \mathfrak{N}_0 and φ^r merge as desired.

(c) Let $a \in \mathbb{Z}_p^\times$ and $\mathfrak{v} \in \mathfrak{X}_+^0$. We define $\gamma(a)_{\mathfrak{v}} : \Theta_*\mathcal{V}(\mathfrak{v}) \rightarrow \Theta_*\mathcal{V}(\gamma(a)\mathfrak{v})$ to be the map $\gamma(a)_{\mathfrak{v}} = \Theta \circ g'_{\Theta^{-1}\mathfrak{v}} \circ \Theta^{-1}$ where g' is some element of $N_0 \cdot \tau(a)$ which induces the arrow

$$(33) \quad]\Theta^{-1}(\mathfrak{v})[\xrightarrow{\Theta}]\mathfrak{v}[\xrightarrow{\gamma(a)}]\gamma(a)\mathfrak{v}[\xrightarrow{\Theta^{-1}}]\Theta^{-1}(\gamma(a)\mathfrak{v})[.$$

Such a g' does exist, by Theorem 3.2(c). As in the proof of (a) we see that $\gamma(a)_{\mathfrak{v}}$ does not depend on the choice of $g' \in N_0 \cdot \tau(a)$ (i.e. on its left hand factor $g' \cdot \tau(a)^{-1} \in N_0$, as long as the arrow (33) is induced as indicated).

Similarly, for $\mathfrak{e} \in \mathfrak{X}_+^1$ and $a \in \mathbb{Z}_p^\times$ choose some $g' \in N_0 \cdot \tau(a)$ with $g'(\Theta^{-1}(\mathfrak{e})) = \Theta^{-1}(\gamma(a)(\mathfrak{e}))$ and define $\gamma(a)_{\mathfrak{e}} : \Theta_*\mathcal{V}(\mathfrak{e}) \rightarrow \Theta_*\mathcal{V}(\gamma(a)\mathfrak{e})$ to be the map $\gamma(a)_{\mathfrak{e}} = \Theta \circ g'_{\Theta^{-1}\mathfrak{e}} \circ \Theta^{-1}$. Again this is independent on the choice of g' .

We have defined an action of Γ on $\Theta_*\mathcal{V}$ (notice that $N_0 \cdot \tau(a) \cdot N_0 \cdot \tau(a') = N_0 \cdot \tau(aa')$ for $a, a' \in \mathbb{Z}_p^\times$). It follows from formula (20) — which corresponds to the formula $\gamma(a) \cdot \varphi^r = \varphi^r \cdot \gamma(a)$ in $[\mathfrak{N}_0, \varphi^r, \Gamma]$ — and from formula (19) — which corresponds to the formula $\gamma(a) \cdot \nu = \nu^a \cdot \gamma(a)$ in $[\mathfrak{N}_0, \varphi^r, \Gamma]$ — that the actions of $[\mathfrak{N}_0, \varphi^r]$ and of Γ merge as desired.

(d) Let $\Xi : Y \xrightarrow{\cong} \mathfrak{X}_+$ be another choice. The automorphism $\Delta = \Theta \circ \Xi^{-1}$ of \mathfrak{X}_+ is covered by the isomorphism $\Xi_*\mathcal{V} \rightarrow \Theta_*\mathcal{V}$ which on facets σ of \mathfrak{X}_+ is given by the identity maps

$$(\Xi_*\mathcal{V})(\sigma) = \mathcal{V}(\Xi^{-1}\sigma) = \mathcal{V}(\Theta^{-1}\Delta\sigma) = (\Theta_*\mathcal{V})(\Delta\sigma).$$

It induces a canonical isomorphism $H_0(\overline{\mathfrak{X}}_+, \Xi_*\mathcal{V}) \rightarrow H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V})$. That it commutes with the actions of \mathfrak{N}_0 (resp. of φ^r , resp. of Γ) is a *tautological* consequence of the definition of these actions. \square

Remark: Theorem 3.2, and then Theorem 4.2 correspondingly, hold true if the topological generator ν of \mathfrak{N}_0 is replaced by any other topological generator of \mathfrak{N}_0 . In general, this choice does affect the isomorphism class of $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V})$ as a $[\mathfrak{N}_0, \varphi^r]$ - resp. $[\mathfrak{N}_0, \varphi^r, \Gamma]$ -representation (but not as a \mathfrak{N}_0 -representation).

Remark: Let N'_0 denote the subgroup of G generated by all the $N_{\alpha(j)} \cap N_0$ for $j \geq 0$. By the remark following Theorem 3.2 we see that Theorem 4.2 may be sharpened as follows: to obtain a \mathfrak{N}_0 -action on $\Theta_*\mathcal{V}$ we do not need a level 1-action of the full group N_0 on \mathcal{V} but only a level 1-action (same definition) of N'_0 ; similarly for $[\mathfrak{N}_0, \varphi^r]$ -, resp. $[\mathfrak{N}_0, \varphi^r, \Gamma]$ -actions.

Remark: It is straightforward to generalize Theorem 3.2 (c) and (d) as follows. In the setting of Theorem 3.2 (a) let $\tau : \mathbb{Z}_p^\times \rightarrow T$ be a homomorphism such that there is some $w \in \mathbb{N}$ with $\alpha^{(i)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}^w$ for all $i \geq 0$. Then there exists for each $\mathbf{v} \in Y^0$ and each $a \in \mathbb{Z}_p^\times$ some element $h(\mathbf{v}, a)$ in $N_0 \cdot \tau(a)$ which induces the composition of bijections

$$] \mathbf{v} [\xrightarrow{\Theta}] \Theta(\mathbf{v}) [\xrightarrow{\gamma(a)^w}] \gamma(a)^w(\Theta(\mathbf{v})) [\xrightarrow{\Theta^{-1}}] \Theta^{-1}(\gamma(a)^w(\Theta(\mathbf{v}))) [.$$

For $a \in \mathbb{N} \cap \mathbb{Z}_p^\times$ these $h(\mathbf{v}, a)$ can be chosen in a way such that in G we have

$$h(g(\mathbf{v}, 1)\mathbf{v}, a) \cdot g(\mathbf{v}, 1) = g(h(\mathbf{v}, a)\mathbf{v}, a^w) \cdot h(\mathbf{v}, a).$$

Moreover, if $\phi \in N(T)$ is as in Theorem 3.2 (b) such that for all $a \in \mathbb{Z}_p^\times$ we have $\tau(a)\phi = \phi\tau(a)$ in G , then, as before, we can achieve $h(\phi(\mathbf{v}), a) \cdot \phi = \phi \cdot h(\mathbf{v}, a)$. Given this setting, Theorem 4.2 (c) can be generalized as follows. If the action of N_0 on \mathcal{V} extends to an action of the submonoid of G generated by N_0 , by the image of τ and by ϕ , then $\Theta_*\mathcal{V}$ is in a natural way both $[\mathfrak{N}_0, \varphi^r]$ -equivariant and Γ -equivariant, such that

$$\gamma(a) \cdot \varphi^r = \varphi^r \cdot \gamma(a) \quad \text{and} \quad \gamma(a) \cdot \nu = \nu^{a^w} \cdot \gamma(a)$$

as endomorphisms of $\Theta_*\mathcal{V}$. Therefore, if in addition $\tau(a)$ acts trivially on \mathcal{V} for all $a \in \mathbb{Z}_p^\times$ with $a^w = 1$, then by extracting w -th roots of the above $\gamma(a)$ -operators we obtain an action of the submonoid of $[\mathfrak{N}_0, \varphi^r, \Gamma]$ generated by \mathfrak{N}_0 , by φ^r and by $\Gamma^w = \{\gamma^w \mid \gamma \in \Gamma\}$.

Definition: Let $m \geq 1$, let \mathcal{V} be a \mathfrak{N}_0 -equivariant coefficient system in \mathfrak{o}_m -modules on \mathfrak{X}_+ .

(1) We say that \mathcal{V} is strictly of level 1 over k if the \mathfrak{o}_m -action factors through k and if the following conditions (a), (b) and (c) are satisfied:

(a) All transition maps $r_y^\tau : \mathcal{V}(\tau) \rightarrow \mathcal{V}(y)$ for $y \in \mathfrak{X}_+^0$, $\tau \in \mathfrak{X}_+^1$ with $y \in \tau$ are injective; in the following we view them as inclusions.

(b) For all $i \geq 0$ we have

$$(34) \quad \mathcal{V}(\mathbf{v}_i) = \mathfrak{N}_0^{p^i} \cdot \mathcal{V}(\mathbf{e}_{i+1}),$$

i.e. as a $\mathfrak{N}_0^{p^i}$ -representation, $\mathcal{V}(\mathbf{v}_i)$ is generated by $\mathcal{V}(\mathbf{e}_{i+1})$, and

$$(35) \quad \mathcal{V}(\mathbf{e}_i) = \mathcal{V}(\mathbf{v}_i)^{\mathfrak{N}_0^{p^i}},$$

i.e. $\mathcal{V}(\mathbf{e}_i)$ is the submodule of $\mathfrak{N}_0^{p^i}$ -invariants in $\mathcal{V}(\mathbf{v}_i)$.

(c) The number $\dim_k(\mathcal{V}(\mathbf{e}_i))$ is finite and independent of $i \geq 0$.

(2) We say that \mathcal{V} is strictly of level 1 if it admits a finite filtration such that all the subquotients are strictly of level 1 over k .

Theorem 4.3. *Let \mathcal{V} be an \mathfrak{N}_0 -equivariant coefficient system in \mathfrak{o}_m -modules on \mathfrak{X}_+ which is strictly of level 1. We have a natural isomorphism*

$$\mathcal{V}(\mathfrak{e}_0) \xrightarrow{\cong} H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^{\mathfrak{N}_0}.$$

PROOF: (i) Suppose first that \mathcal{V} is strictly of level 1 over k . We have canonical isomorphisms of \mathfrak{N}_0 -representations

$$(36) \quad \bigoplus_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^i}} \mathcal{V}(n \cdot \mathfrak{v}_i) \cong \text{ind}_{\mathfrak{N}_0^{p^i}}^{\mathfrak{N}_0} \mathcal{V}(\mathfrak{v}_i),$$

$$(37) \quad \bigoplus_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^i}} \mathcal{V}(n \cdot \mathfrak{e}_i) \cong \text{ind}_{\mathfrak{N}_0^{p^i}}^{\mathfrak{N}_0} \mathcal{V}(\mathfrak{e}_i).$$

By formula (35) the transition map $\mathcal{V}(\mathfrak{e}_0) \rightarrow \mathcal{V}(\mathfrak{v}_0)$ is an isomorphism between $\mathcal{V}(\mathfrak{e}_0)$ and $\mathcal{V}(\mathfrak{v}_0)^{\mathfrak{N}_0}$. Hence we need to show that the natural map $\mathcal{V}(\mathfrak{v}_0)^{\mathfrak{N}_0} \rightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^{\mathfrak{N}_0}$ is an isomorphism. The injectivity even of $\mathcal{V}(\mathfrak{v}_0) \rightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ follows from the injectivity of all the transition maps of \mathcal{V} . Next, Lemma 4.1 (which of course relies on the same argument) shows the exactness of the sequence

$$0 \longrightarrow \bigoplus_{\tau \in \overline{\mathfrak{X}}_+^1} \mathcal{V}(\tau) \longrightarrow \bigoplus_{y \in \overline{\mathfrak{X}}_+^0} \mathcal{V}(y) \longrightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V}) \longrightarrow 0.$$

Looking at the long exact cohomology sequence obtained by applying the group cohomology functor $H^*(\mathfrak{N}_0, \cdot)$ we see that it is now enough to prove that the natural map

$$(38) \quad \left(\bigoplus_{\tau \in \overline{\mathfrak{X}}_+^1} \mathcal{V}(\tau) \right)^{\mathfrak{N}_0} \bigoplus \mathcal{V}(\mathfrak{v}_0)^{\mathfrak{N}_0} \longrightarrow \left(\bigoplus_{y \in \overline{\mathfrak{X}}_+^0} \mathcal{V}(y) \right)^{\mathfrak{N}_0}$$

is bijective and that the natural map

$$(39) \quad H^1(\mathfrak{N}_0, \bigoplus_{\tau \in \overline{\mathfrak{X}}_+^1} \mathcal{V}(\tau)) \longrightarrow H^1(\mathfrak{N}_0, \bigoplus_{y \in \overline{\mathfrak{X}}_+^0} \mathcal{V}(y))$$

is injective. We recognize the map (38) as the natural map

$$(40) \quad \left(\bigoplus_{\tau \in \overline{\mathfrak{X}}_+^1} \mathcal{V}(\tau) \right)^{\mathfrak{N}_0} \longrightarrow \left(\bigoplus_{y \in \overline{\mathfrak{X}}_+^0} \mathcal{V}(y) \right)^{\mathfrak{N}_0}.$$

In view of the isomorphisms (36), (37) and Shapiro's Lemma, we may rewrite it as

$$\bigoplus_{i \geq 0} \mathcal{V}(\mathfrak{e}_i)^{\mathfrak{N}_0^{p^i}} \longrightarrow \bigoplus_{i \geq 0} \mathcal{V}(\mathfrak{v}_i)^{\mathfrak{N}_0^{p^i}}$$

(with $\mathcal{V}(\mathfrak{e}_0)^{\mathfrak{N}_0^{p^0}}$ mapping to $\mathcal{V}(\mathfrak{v}_0)^{\mathfrak{N}_0^{p^0}}$ and with $\mathcal{V}(\mathfrak{e}_i)^{\mathfrak{N}_0^{p^i}}$ for $i \geq 1$ mapping to both $\mathcal{V}(\mathfrak{v}_i)^{\mathfrak{N}_0^{p^i}}$ and $\mathcal{V}(\mathfrak{v}_{i-1})^{\mathfrak{N}_0^{p^{i-1}}}$). Since by hypothesis all the maps

$$\mathcal{V}(\mathfrak{e}_i)^{\mathfrak{N}_0^{p^i}} = \mathcal{V}(\mathfrak{e}_i) \longrightarrow \mathcal{V}(\mathfrak{v}_i)^{\mathfrak{N}_0^{p^i}}$$

are bijective we have proven the bijectivity of the map (38). Similarly as with the map (40) we proceed with the map (39): we use Shapiro's Lemma to rewrite it as

$$\bigoplus_{i \geq 0} H^1(\mathfrak{N}_0^{p^i}, k[\frac{\mathfrak{N}_0^{p^i}}{\mathfrak{N}_0^{p^{i+1}}}] \otimes_k \mathcal{V}(\mathfrak{e}_{i+1})) \longrightarrow \bigoplus_{i \geq 0} H^1(\mathfrak{N}_0^{p^i}, \mathcal{V}(\mathfrak{v}_i)).$$

To see its injectivity it is enough to show that the maps

$$H^1(\mathfrak{N}_0^{p^i}, k[\frac{\mathfrak{N}_0^{p^i}}{\mathfrak{N}_0^{p^{i+1}}}] \otimes_k \mathcal{V}(\mathfrak{e}_{i+1})) \longrightarrow H^1(\mathfrak{N}_0^{p^i}, \mathcal{V}(\mathfrak{v}_i))$$

are injective for all $i \geq 0$. But in view of our hypotheses on \mathcal{V} this follows from Lemma 2.1, applied to $\mathfrak{N}_0^{p^i} \cong \mathbb{Z}_p$ acting through its quotient $\mathfrak{N}_0^{p^i}/\mathfrak{N}_0^{p^{i+1}} \cong \mathbb{F}_p$ on $\mathcal{V}(\mathfrak{v}_i)$.

(ii) For general \mathcal{V} strictly of level 1 we argue by induction on the minimal length of a filtration with subquotients strictly of level 1 over k . Consider an exact sequence $0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{V} \rightarrow \mathcal{V}_2 \rightarrow 0$ where \mathcal{V}_1 and \mathcal{V}_2 are strictly of level 1 and different from \mathcal{V} . The strict level 1 property of \mathcal{V}_2 implies that \mathcal{V}_2 has injective transition maps. By Lemma 4.1 we obtain the exactness of

$$(41) \quad 0 \longrightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V}_1) \longrightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V}) \longrightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V}_2) \longrightarrow 0.$$

It follows that in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V}_1(\mathfrak{e}_0) & \longrightarrow & \mathcal{V}(\mathfrak{e}_0) & \longrightarrow & \mathcal{V}_2(\mathfrak{e}_0) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_0(\overline{\mathfrak{X}}_+, \mathcal{V}_1)^{\mathfrak{N}_0} & \longrightarrow & H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^{\mathfrak{N}_0} & \longrightarrow & H_0(\overline{\mathfrak{X}}_+, \mathcal{V}_2)^{\mathfrak{N}_0} \end{array}$$

the bottom sequence is exact. For the top sequence this is obvious. The outer vertical arrows are bijective by induction hypothesis. Hence the middle vertical arrow is bijective. \square

Proposition 4.4. *Let $r \in \mathbb{N}$, let \mathcal{V} be a $[\mathfrak{N}_0, \varphi^r]$ -equivariant coefficient system on \mathfrak{X}_+ whose \mathfrak{N}_0 -action is of level 1. The $[\mathfrak{N}_0, \varphi^r]$ -action on \mathcal{V} naturally extends to a $[\mathfrak{N}_0, \varphi^r, \Gamma_0]$ -action such that Γ_0 acts trivially on $\mathcal{V}(\mathfrak{e}_0)$.*

PROOF: The action of Γ_0 on \mathfrak{X}_+^0 respects the orbits $\mathfrak{N}_0 \cdot \mathfrak{v}_i$ for all $i \geq 0$. On such an orbit $\mathfrak{N}_0 \cdot \mathfrak{v}_i$ it is described by the formula

$$(42) \quad \gamma_0(\nu^n(\mathfrak{v}_i)) = \nu^{(p+1)n}(\mathfrak{v}_i)$$

for all $n \geq 0$. Applied to the elements of

$$[\nu^n(\mathfrak{v}_i)] = \{\nu^{n+tp^i}(\mathfrak{v}_{i+1}) \mid 0 \leq t < p\},$$

formula (42) (with $i+1$ instead of i) shows that the restrictions of γ_0 and ν^{np} to the subset $[\nu^n(\mathfrak{v}_i)]$ of $\mathfrak{N}_0 \cdot \mathfrak{v}_{i+1}$ coincide, because $(p+1)(n+tp^i) \equiv pn + n + tp^i$ modulo (p^{i+1}) . Thus we

obtain that for all $j \in \mathbb{Z}$ and all $\mathfrak{v} \in \mathfrak{X}_+^0$ there is some $\beta \in \mathfrak{N}_0$ such that the restrictions of γ_0^j and β to the subset $]\mathfrak{v}[$ of \mathfrak{X}_+^0 coincide. Therefore, if we define the map

$$\gamma_{0,\mathfrak{v}}^j : \mathcal{V}(\mathfrak{v}) \longrightarrow \mathcal{V}(\gamma_0^j(\mathfrak{v}))$$

as the map $\beta_{\mathfrak{v}} : \mathcal{V}(\mathfrak{v}) \rightarrow \mathcal{V}(\beta(\mathfrak{v})) = \mathcal{V}(\gamma_0^j(\mathfrak{v}))$ then, arguing as in the proof of Theorem 4.2, we see that by the level 1 property of \mathcal{V} this definition is independent on the choice of β . In the same way we define maps $\gamma_{0,\mathfrak{e}}^j : \mathcal{V}(\mathfrak{e}) \longrightarrow \mathcal{V}(\gamma_0^j(\mathfrak{e}))$ for $\mathfrak{e} \in \mathfrak{X}_+^1$. We have defined an action of Γ_0 on \mathcal{V} .

Now let again $\nu^n(\mathfrak{v}_i)$ (some $n \geq 0$, some $i \geq 0$) be an arbitrary element of \mathfrak{X}_+^0 . Let $x \in \mathcal{V}(\nu^n(\mathfrak{v}_i))$. We compute (dropping the names of vertices in subscripts)

$$\gamma_0(\nu(x)) \stackrel{(i)}{=} \nu^{p(n+1)}(\nu(x)) = \nu^{p+1}(\nu^{pn}(x)) \stackrel{(ii)}{=} \nu^{p+1}(\gamma_0(x))$$

where in (i) we used that γ_0 and $\nu^{p(n+1)}$ coincide on $]\nu^{n+1}(\mathfrak{v}_i)[$ whereas in (ii) we used that γ_0 and ν^{pn} coincide on $]\nu^n(\mathfrak{v}_i)[$. Similarly we compute

$$\gamma_0(\varphi^r(x)) \stackrel{(i)}{=} \nu^{p^{r+1}n}(\varphi^r(x)) = \varphi^r(\nu^{pn}(x)) \stackrel{(ii)}{=} \varphi^r(\gamma_0(x))$$

where in (i) we used that γ_0 and $\nu^{p^{r+1}n}$ coincide on $]\nu^{p^r n}(\mathfrak{v}_{i+r})[$ whereas in (ii) we used that γ_0 and ν^{pn} coincide on $]\nu^n(\mathfrak{v}_i)[$. A similar computation can be done on the values of \mathcal{V} at all $\mathfrak{e} \in \mathfrak{X}_+^1$. We have shown

$$(43) \quad \gamma_0 \circ \varphi^r = \varphi^r \circ \gamma_0 \quad \text{and} \quad \gamma_0 \circ \nu = \nu^{p+1} \circ \gamma_0$$

as endomorphisms of \mathcal{V} . This means that the actions of $[\mathfrak{N}_0, \varphi^r]$ and of Γ_0 merge to an action of $[\mathfrak{N}_0, \varphi^r, \Gamma_0]$ on \mathcal{V} . \square

5 Pro- p Iwahori-Hecke modules and coefficient systems

Let I_0 denote the maximal pro- p subgroup in I . Let $\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ denote the \mathfrak{o} -module of \mathfrak{o} -valued compactly supported functions f on G such that $f(ig) = f(g)$ for all $g \in G$, all $i \in I_0$. It is a G -representation by means of $(g'f)(g) = f(gg')$ for $g, g' \in G$. Let

$$\mathcal{H}(G, I_0) = \text{End}_{\mathfrak{o}[G]}(\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{\text{op}}$$

denote the corresponding pro- p -Iwahori Hecke algebra with coefficients in \mathfrak{o} . Then $\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ is naturally a right $\mathcal{H}(G, I_0)$ -module. For a subset H of G let χ_H denote the characteristic function of H . For $g \in G$ let $T_g \in \mathcal{H}(G, I_0)$ denote the Hecke operator corresponding to the double coset $I_0 g I_0$. It sends $f : G \rightarrow \mathfrak{o}$ to

$$T_g(f) : G \longrightarrow \mathfrak{o}, \quad h \mapsto \sum_{x \in I_0 \backslash G} \chi_{I_0 g I_0}(hx^{-1})f(x).$$

In particular we have

$$(44) \quad T_g(\chi_{I_0}) = \chi_{I_0 g} = g^{-1} \chi_{I_0} \quad \text{if } gI_0 = I_0 g.$$

For any facet F of X let I_0^F denote the 'pro-unipotent radical' of the stabilizer of F . More precisely, following [12] section 3 (where instead the notation I_F is used), I_0^F consists of all $g \in \mathbf{G}_F^0(\mathbb{Z}_p)$ mapping to the unipotent radical in $\overline{\mathbf{G}}_F^0$. Here \mathbf{G}_F is the smooth affine \mathbb{Z}_p -group scheme whose general fibre is (the reductive \mathbb{Q}_p -group scheme underlying) G and such that $\mathbf{G}_F(\mathbb{Z}_p)$ is the pointwise stabilizer of the preimage of F in the enlarged building of G . For example, $I_0 = I_0^C$. Since $F_1 \subset F_2$ implies $I_0^{F_1} \subset I_0^{F_2}$ the assignment $F \mapsto (\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{I_0^F}$ is a G -equivariant coefficient system on X . Since the right action of $\mathcal{H}(G, I_0)$ on $\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ commutes with the left G -action this is, in fact, also a coefficient system of right $\mathcal{H}(G, I_0)$ -modules. Given a left $\mathcal{H}(G, I_0)$ -module M we therefore obtain a new G -equivariant coefficient system \mathcal{V}_M^X on X by putting

$$\mathcal{V}_M^X(F) = (\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{I_0^F} \otimes_{\mathcal{H}(G, I_0)} M.$$

In fact we will only be interested in the restriction of \mathcal{V}_M^X to facets of codimension 0 and 1. We may regard the restriction of \mathcal{V}_M^X to Y as a coefficient system \mathcal{V}_M on Y . To be explicit, for $n \in N_0$, $i \geq 0$, to the vertex $n \cdot \mathbf{v}_i$, resp. the edge $n \cdot \mathbf{e}_i$, of Y , we assign the codimension-1-facet, resp. codimension-0-facet,

$$\eta^0(n \cdot \mathbf{v}_i) = nC^{(i)} \cap nC^{(i+1)}, \quad \text{resp.} \quad \eta^1(n \cdot \mathbf{e}_i) = nC^{(i)},$$

of X . Both η^0 and η^1 are injective mappings, equivariant under all sub monoids of G which respect Y , and (η^0, η^1) respects facet inclusions. We now put

$$\mathcal{V}_M(\mathbf{v}) = \mathcal{V}_M^X(\eta^0(\mathbf{v})), \quad \text{resp.} \quad \mathcal{V}_M(\mathbf{e}) = \mathcal{V}_M^X(\eta^1(\mathbf{e})),$$

for $\mathbf{v} \in Y^0$, resp. $\mathbf{e} \in Y^1$. This defines a coefficient system \mathcal{V}_M on Y , equivariant under all sub monoids of G which respect Y .

It is clear that all these constructions are covariantly functorial in M .

For $m \geq 1$ let us write $\mathcal{H}(G, I_0)_{\mathfrak{o}_m} = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} \mathfrak{o}_m$. We denote by $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_{\mathfrak{o}_m})$ the category of $\mathcal{H}(G, I_0)_{\mathfrak{o}_m}$ -modules which are finitely generated as \mathfrak{o}_m -modules.

We fix a gallery (11) and choose an isomorphism $\Theta : Y \xrightarrow{\cong} \mathfrak{X}_+$ as in Theorem 3.2.

Let F be a codimension-1-face of C . There is a unique quotient $\overline{\mathcal{S}}$ of I_0^F which is isomorphic with either $\text{SL}_2(\mathbb{F}_p)$ or $\text{PSL}_2(\mathbb{F}_p)$. The image of I_0 in $\overline{\mathcal{S}}$ is the unipotent radical $\overline{\mathcal{U}}$ of a Borel subgroup in $\overline{\mathcal{S}}$.

Proposition 5.1. (a) $\Theta_* \mathcal{V}_M$ is in a natural way a \mathfrak{N}_0 -equivariant coefficient system of level 1 on \mathfrak{X}_+ .

(b) For any left- $\mathcal{H}(G, I_0)_{\mathfrak{o}_m}$ -module M the Hecke algebra $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}}) = \text{End}_{\mathfrak{o}[\overline{\mathcal{S}}]}(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}})^{\text{op}}$ naturally acts on M , and we have natural isomorphisms $M \cong \mathcal{V}_M^X(C)$ and

$$(45) \quad \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}} \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})} M \cong \mathcal{V}_M^X(F)$$

such that the transition map $\mathcal{V}_M^X(C) \rightarrow \mathcal{V}_M^X(F)$ gets identified with the natural map

$$(46) \quad M \longrightarrow \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}} \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})} M.$$

(c) If $M \in \text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_{\mathfrak{o}_m})$ then $\Theta_* \mathcal{V}_M$ is strictly of level 1.

(d) If $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is an exact sequence of $\mathcal{H}(G, I_0)_{\mathfrak{o}_m}$ -modules for some $m \in \mathbb{N}$ then the induced sequence $0 \rightarrow \Theta_* \mathcal{V}_{M_1} \rightarrow \Theta_* \mathcal{V}_M \rightarrow \Theta_* \mathcal{V}_{M_2} \rightarrow 0$ is exact.

PROOF: (a) This follows from Theorem 4.2.

(b) For a facet D of X let J_D denote the stabilizer of D in G . Put

$$\mathcal{H}_F = \text{End}_{\mathfrak{o}[J_F]}(\text{ind}_{I_0}^{J_F} \mathbf{1}_{\mathfrak{o}})^{\text{op}}.$$

As $I_0^F \subset J_F$ we have natural embeddings

$$\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}} \cong \text{ind}_{I_0}^{I_0^F} \mathbf{1}_{\mathfrak{o}} \hookrightarrow \text{ind}_{I_0}^{J_F} \mathbf{1}_{\mathfrak{o}} \hookrightarrow \text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}.$$

Thus Frobenius reciprocity provides inclusions of Hecke algebras

$$\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}}) \subset \mathcal{H}_F \subset \mathcal{H}(G, I_0).$$

We obtain a natural map

$$(47) \quad \text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}} \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})} M \longrightarrow \text{ind}_{I_0}^{J_F} \mathbf{1}_{\mathfrak{o}} \otimes_{\mathcal{H}_F} M.$$

The elements of J_C normalize I_0 , hence we have the group homomorphism $J_C \rightarrow \mathcal{H}(G, I_0)^{\times}$, $g \mapsto T_{g^{-1}}$, into the group $\mathcal{H}(G, I_0)^{\times}$ of invertible elements of $\mathcal{H}(G, I_0)$. [Indeed, for $g_1, g_2 \in J_C$ we form the composition $T_{g_1} \circ T_{g_2}$ in $\text{End}_{\mathfrak{o}[G]}(\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})$ and compute $(T_{g_1} \circ T_{g_2})(\chi_{I_0}) = T_{g_1}(\chi_{I_0 g_2}) = T_{g_1}(g_2^{-1} \chi_{I_0}) = g_2^{-1} T_{g_1}(\chi_{I_0}) = g_2^{-1} \chi_{I_0 g_1} = g_2^{-1} g_1^{-1} \chi_{I_0} = \chi_{I_0 g_1 g_2}$.] Similarly we have a homomorphism $J_C \cap J_F \rightarrow \mathcal{H}_F^{\times}$. As $J_C \cap J_F$ and I_0^F together generate J_F and as their intersection is I_0 we deduce that the map (47) is an isomorphism. But also the map

$$(48) \quad (\text{ind}_{I_0}^{J_F} \mathbf{1}_{\mathfrak{o}}) \otimes_{\mathcal{H}_F} \mathcal{H}(G, I_0)_{\mathfrak{o}_m} \longrightarrow (\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}_m})^{I_0^F}, \quad f \otimes h \mapsto h(f)$$

is an isomorphism. This is proven in Lemma 3.11.i and section 4.9 of [12] in the setting where the coefficient ring is a field (instead of our \mathfrak{o}_m). However, the proof given in loc. cit. for coefficient fields of characteristic p also applies to the coefficient ring \mathfrak{o}_m . Composing the isomorphism (47) with the isomorphism obtained from applying $(\cdot) \otimes_{\mathcal{H}(G, I_0)_{\mathfrak{o}}} M$ to (48) we obtain the desired isomorphism (45). On the other hand, as $I_0 = I_0^C$ we have $\mathcal{H}(G, I_0) \cong (\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{I_0^C}$ and hence $M \cong \mathcal{V}_M^X(C)$ naturally. By construction, the transition map $\mathcal{V}_M^X(C) \rightarrow \mathcal{V}_M^X(F)$ gets identified with the map (46).

(c) By Lemma 2.3 the map (46) is an isomorphism between M and $(\text{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}} \otimes_{\mathcal{H}(\bar{\mathcal{S}}, \bar{\mathcal{U}})} M)^{\bar{\mathcal{U}}}$. In other words, M is the submodule of I_0 -invariants in the I_0^F -representation $\mathcal{V}_M^X(F)$. By G -equivariance of \mathcal{V}_M^X we deduce property (35) for $\Theta_* \mathcal{V}_M$. Similarly, Lemma 2.4 ensures property (34) for $\Theta_* \mathcal{V}_M$. All this applies of course similarly to the subquotients of M with respect to the filtration $\{p^i M\}_{i \geq 0}$. The other properties required for being strictly of level 1 are clear.

(d) The exactness at any edge of \mathfrak{X}_+ (which corresponds to a chamber of Y) follows immediately from the definitions. To prove exactness at any vertex of \mathfrak{X}_+ it is enough, by G -equivariance, to prove exactness of $0 \rightarrow \mathcal{V}_{M_1}^X(F) \rightarrow \mathcal{V}_M^X(F) \rightarrow \mathcal{V}_{M_2}^X(F) \rightarrow 0$ for all codimension-1-faces F of C . From what we learned in the proof of (b) and (c) we see that here we need to prove exactness of

$$0 \longrightarrow \text{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes M_1 \longrightarrow \text{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes M \longrightarrow \text{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m} \otimes M_2 \longrightarrow 0$$

where all tensor products are taken over $\mathcal{H}(\bar{\mathcal{S}}, \bar{\mathcal{U}})_{\mathfrak{o}_m} = \text{End}_{\mathfrak{o}_m[\bar{\mathcal{S}}]}(\text{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m})^{\text{op}}$. This exactness follows from the flatness of $\text{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_{\mathfrak{o}_m}$ over $\mathcal{H}(\bar{\mathcal{S}}, \bar{\mathcal{U}})_{\mathfrak{o}_m}$, Lemma 2.2. \square

Remark: An alternative description of \mathcal{V}_M^X (which we do not need), at least when restricted to facets of codimension 0 and 1, and hence of \mathcal{V}_M can be given if M is realized inside a smooth G -representation V . Recall that for such V the submodule V^{I_0} of I_0 -invariants is in a natural way a (left) module over $\mathcal{H}(G, I_0)$. Let M be an $\mathcal{H}(G, I_0)$ -sub module of V^{I_0} . For any facet D of X we let

$$\mathcal{V}_{(V, M)}^X(D) = \sum_{\substack{g \in G \\ gC \supset D}} gM$$

(sum inside V). This defines a G -equivariant coefficient system $\mathcal{V}_{(V, M)}^X$ on X . If $M \in \text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_{\mathfrak{o}_m})$ for some $m \geq 1$ then there is a natural G -equivariant morphism $\mathcal{V}_M^X \rightarrow \mathcal{V}_{(V, M)}^X$ which, at least when restricted to facets of codimension 0 and 1, is an isomorphism. To see this one can use arguments from the proof of Proposition 5.1 (the starting point is to see, using Lemmata 2.2 and 2.3, that for any codimension-1-face F of C with corresponding subgroup I_0^F of G we have $(I_0^F M)^{I_0} = M$).

6 (φ^r, Γ) -modules

6.1 (ψ^r, Γ) -modules and (φ^r, Γ) -modules

Let $\mathcal{O}_{\mathcal{E}}^+ = \mathfrak{o}[[\mathfrak{N}_0]]$ denote the completed group ring of \mathfrak{N}_0 over \mathfrak{o} . Let $\mathcal{O}_{\mathcal{E}}$ denote the p -adic completion of the localization of $\mathcal{O}_{\mathcal{E}}^+$ with respect to the complement of $\pi_K \mathcal{O}_{\mathcal{E}}^+$.

Let $\varphi_{\mathcal{O}_{\mathcal{E}}^+}$ denote the endomorphism of $\mathcal{O}_{\mathcal{E}}^+$ induced by the endomorphism $n \mapsto \varphi n \varphi^{-1}$ of \mathfrak{N}_0 . Let $\varphi_{\mathcal{O}_{\mathcal{E}}}$ denote the endomorphism of $\mathcal{O}_{\mathcal{E}}$ induced from $\varphi_{\mathcal{O}_{\mathcal{E}}^+}$ by functoriality. As an \mathfrak{o} -module, $\mathcal{O}_{\mathcal{E}}$ decomposes as $\mathcal{O}_{\mathcal{E}} = \text{im}(\varphi_{\mathcal{O}_{\mathcal{E}}}) \oplus (\mathfrak{N}_0 - \mathfrak{N}_0^p) \text{im}(\varphi_{\mathcal{O}_{\mathcal{E}}})$ (notice that $\varphi \mathfrak{N}_0 \varphi^{-1} = \mathfrak{N}_0^p$). We denote by $\psi_{\mathcal{O}_{\mathcal{E}}}$ the \mathfrak{o} -linear endomorphism of $\mathcal{O}_{\mathcal{E}}$ with $\psi_{\mathcal{O}_{\mathcal{E}}} \circ \varphi_{\mathcal{O}_{\mathcal{E}}} = \text{id}$ and with kernel $\ker(\psi_{\mathcal{O}_{\mathcal{E}}}) = (\mathfrak{N}_0 - \mathfrak{N}_0^p) \text{im}(\varphi_{\mathcal{O}_{\mathcal{E}}})$. Similarly (or simply by restriction) we define the \mathfrak{o} -linear

endomorphism $\psi_{\mathcal{O}_{\mathcal{E}}^+}$ of $\mathcal{O}_{\mathcal{E}}^+$. The conjugation action $(\gamma, n) \mapsto \gamma n \gamma^{-1}$ of Γ on \mathfrak{N}_0 induces an action $(\gamma, a) \mapsto \gamma \cdot a$ of Γ on $\mathcal{O}_{\mathcal{E}}^+$ and on $\mathcal{O}_{\mathcal{E}}$.

On $k_{\mathcal{E}}^+ = k[[\mathfrak{N}_0]] = \mathcal{O}_{\mathcal{E}}^+ \otimes_{\mathfrak{o}} k$ we have the φ -operator $\varphi_{k_{\mathcal{E}}^+} = \varphi_{\mathcal{O}_{\mathcal{E}}^+} \otimes_{\mathfrak{o}} k$, the ψ -operator $\psi_{k_{\mathcal{E}}^+} = \psi_{\mathcal{O}_{\mathcal{E}}^+} \otimes_{\mathfrak{o}} k$ and the induced action of Γ , and similarly on $k_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{o}} k$.

Let $r \in \mathbb{N}$. We need the non-commutative polynomial ring $\mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r]$ over $\mathcal{O}_{\mathcal{E}}^+$ in which relations are as imposed by multiplication in $\mathrm{GL}_2(\mathbb{Q}_p)$ (i.e. $\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r \cdot [n] = [n]^{p^r} \cdot \varphi_{\mathcal{O}_{\mathcal{E}}^+}^r$ for $n \in \mathfrak{N}_0$) and the twisted group ring $\mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r, \Gamma]$ over $\mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r]$, again with relations as imposed by multiplication in $\mathrm{GL}_2(\mathbb{Q}_p)$. (I.e. for $\gamma \in \Gamma$ and $n \in \mathfrak{N}_0$ we have $\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r \cdot \gamma = \gamma \cdot \varphi_{\mathcal{O}_{\mathcal{E}}^+}^r$ and $\gamma \cdot [n] = [\gamma n \gamma^{-1}] \cdot \gamma$. Specifically, for $a \in \mathbb{Z}_p^\times$, setting $t = [\nu] - 1$ we have $\gamma(a) \cdot t = ((t+1)^a - 1) \cdot \gamma(a)$.) Similarly we define $\mathcal{O}_{\mathcal{E}}[\varphi_{\mathcal{O}_{\mathcal{E}}}^r]$ and $\mathcal{O}_{\mathcal{E}}[\varphi_{\mathcal{O}_{\mathcal{E}}}^r, \Gamma]$, as well as $k_{\mathcal{E}}^+[\varphi_{k_{\mathcal{E}}^+}^r]$ and $k_{\mathcal{E}}^+[\varphi_{k_{\mathcal{E}}^+}^r, \Gamma]$ (equivalently, $\mathcal{O}_{\mathcal{E}}[\varphi_{\mathcal{O}_{\mathcal{E}}}^r] = \mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r] \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{O}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}[\varphi_{\mathcal{O}_{\mathcal{E}}}^r, \Gamma] = \mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r, \Gamma] \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{O}_{\mathcal{E}}$ as well as $k_{\mathcal{E}}^+[\varphi_{k_{\mathcal{E}}^+}^r] = \mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r] \otimes_{\mathfrak{o}} k$ and $k_{\mathcal{E}}^+[\varphi_{k_{\mathcal{E}}^+}^r, \Gamma] = \mathcal{O}_{\mathcal{E}}^+[\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r, \Gamma] \otimes_{\mathfrak{o}} k$). We will often just write $k_{\mathcal{E}}^+[\varphi^r]$ and $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$, and also drop the exponent r in case $r = 1$.

Definition: (a) An étale (φ^r, Γ) -module over $\mathcal{O}_{\mathcal{E}}$ is a module \mathbf{D} over $\mathcal{O}_{\mathcal{E}}[\varphi_{\mathcal{O}_{\mathcal{E}}}^r, \Gamma]$, finitely generated as a module over $\mathcal{O}_{\mathcal{E}}$ and such that the structure map $\varphi_{\mathbf{D}}^r$ is étale, i.e. $\varphi_{\mathbf{D}}^r$ is injective and satisfies $\mathbf{D} = \bigoplus_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \varphi_{\mathbf{D}}^r(\mathbf{D})$. In this situation there is a unique \mathfrak{o} -linear endomorphism $\psi_{\mathbf{D}}^r$ of \mathbf{D} with $\psi_{\mathbf{D}}^r \circ \varphi_{\mathbf{D}}^r = \mathrm{id}_{\mathbf{D}}$ and with kernel $\bigoplus_{n \in (\mathfrak{N}_0 / \mathfrak{N}_0^{p^r}) - \{\mathfrak{N}_0^{p^r}\}} n \varphi_{\mathbf{D}}^r(\mathbf{D})$. A (φ^r, Γ) -module over $k_{\mathcal{E}}$ is an étale (φ^r, Γ) -module over $\mathcal{O}_{\mathcal{E}}$ whose $\mathcal{O}_{\mathcal{E}}$ -action factors through the quotient $k_{\mathcal{E}}$ of $\mathcal{O}_{\mathcal{E}}$.

(b) Similarly we define étale (φ^r, Γ_0) -modules over $\mathcal{O}_{\mathcal{E}}$ and over $k_{\mathcal{E}}$.

Definition: A (ψ^r, Γ) -module over $k_{\mathcal{E}}^+$ is a finitely generated free $k_{\mathcal{E}}^+$ -module D^{\sharp} , together with a k -linear endomorphism $\psi_{D^{\sharp}}^r$ satisfying $\psi_{D^{\sharp}}^r(\varphi_{k_{\mathcal{E}}^+}^r(\alpha)x) = \alpha \psi_{D^{\sharp}}^r(x)$ and a continuous semi-linear action of Γ that commutes with $\psi_{D^{\sharp}}^r$.

It is called non degenerate if $\ker(\psi_{D^{\sharp}}^r)$ contains no non zero $k_{\mathcal{E}}^+$ -sub module.

- Let D^{\sharp} be a (ψ^r, Γ) -module over $k_{\mathcal{E}}^+$. Viewing D^{\sharp} as a linearly compact k -vector space we endow the topological dual $(D^{\sharp})^* = \mathrm{Hom}_k^{\mathrm{ct}}(D^{\sharp}, k)$ with the structure of a $k_{\mathcal{E}}^+$ -module by setting $(\alpha \cdot \ell)(x) = \ell(\alpha \cdot x)$ for $\ell \in (D^{\sharp})^*$, $x \in D^{\sharp}$ and $\alpha \in k_{\mathcal{E}}^+$.[‡] Notice that $(D^{\sharp})^*$ is a torsion $k_{\mathcal{E}}^+$ -module. We define a k -linear endomorphism $\varphi_{(D^{\sharp})^*}^r$ on $(D^{\sharp})^*$ through $(\varphi_{(D^{\sharp})^*}^r(\ell))(x) = \ell(\psi_{D^{\sharp}}^r(x))$ and an action by Γ through $(\gamma(\ell))(x) = \ell(\gamma^{-1}(x))$. One checks that this defines on $(D^{\sharp})^*$ the structure of a module over $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$.

- Let D^{\sharp} be a (ψ^r, Γ) -module over $k_{\mathcal{E}}^+$ which is non degenerate and such that $\psi_{D^{\sharp}}^r$ is surjective. Then $\mathbf{D} = D^{\sharp} \otimes_{k_{\mathcal{E}}^+} k_{\mathcal{E}}$ carries a unique structure of a (φ^r, Γ) -module over $k_{\mathcal{E}}$ compatible with the (ψ^r, Γ) -structure on D^{\sharp} (i.e. the Γ -actions coincide, and $\psi_{\mathbf{D}}^r$ extends $\psi_{D^{\sharp}}^r$). For $r = 1$ this is

[‡]We choose not to use the involution on $k_{\mathcal{E}}^+$ induced by inversion in \mathfrak{N}_0 when defining the $k_{\mathcal{E}}^+$ -action on the dual, taking advantage of the commutativity of \mathfrak{N}_0 .

well known [for the uniqueness one may use Proposition II.3.4(ii) of [4]; see also [14] Proposition 3.3.24.]. For general $r \geq 1$ the proofs are the same. (For the (ψ^r, Γ) -modules met later on in this paper, the corresponding (φ^r, Γ) -modules over $k_{\mathcal{E}}$ come along simultaneously and explicitly, see section 7.)

Let $\text{Gal}_{\mathbb{Q}_p} = \text{Gal}(\mathbb{Q}_p^{\text{alg}}/\mathbb{Q}_p)$ denote the absolute Galois group of \mathbb{Q}_p .

Theorem 6.1. (*Fontaine*) *There is an equivalence functor $\mathbf{D} \mapsto W(\mathbf{D})$ from the category of (φ, Γ) -modules over $k_{\mathcal{E}}$ to the category of finite dimensional (smooth) k -representations of $\text{Gal}_{\mathbb{Q}_p}$.*

PROOF: This is shown in [7], for a brief account see [2] Theorem 2.1.2. \square

Let $\mathcal{I}_{\mathbb{Q}_p}$ denote the inertia subgroup of $\text{Gal}_{\mathbb{Q}_p}$. For $m \geq 0$ let $\omega_{m+1} : \mathcal{I}_{\mathbb{Q}_p} \rightarrow \overline{\mathbb{F}}_p^{\times}$ denote the fundamental character of level $m+1$. It is given by the formula $\omega_{m+1}(g) = \overline{g(\pi_{m+1})/\pi_{m+1}}$ where $\pi_{m+1} \in \mathbb{Q}_p^{\text{alg}}$ is chosen such that $\pi_{m+1}^{p^{m+1}-1} = -p$. We denote by ω the cyclotomic character modulo p of $\text{Gal}_{\mathbb{Q}_p}$.

For $0 \leq h \leq p^{m+1} - 1$ let $\text{ind}(\omega_{m+1}^h)$ be the $(m+1)$ -dimensional $\text{Gal}_{\mathbb{Q}_p}$ -representation over k with $\det(\text{ind}(\omega_{m+1}^h)) = \omega^h$ and $\text{ind}(\omega_{m+1}^h)|_{\mathcal{I}_{\mathbb{Q}_p}} = \bigoplus_{j=0}^m \omega_{m+1}^{p^j h}$ described in [2]. For $\beta \in (k^{\text{alg}})^{\times}$ let μ_{β} denote the unique unramified (i.e. trivial on $\mathcal{I}_{\mathbb{Q}_p}$) character of $\text{Gal}_{\mathbb{Q}_p}$ sending the geometric Frobenius (i.e. lifting $x \mapsto x^{-p}$) to β . If $\beta^{m+1} \in k^{\times}$ then $\text{ind}(\omega_{m+1}^h) \otimes \mu_{\beta}$ is defined over k .

One says that an integer $1 \leq h \leq p^{m+1} - 2$ is primitive (with respect to $m+1$) if there is no $n < m+1$ dividing $m+1$ such that h is a multiple of $(p^{m+1} - 1)/(p^n - 1)$. If h is primitive then $\text{ind}(\omega_{m+1}^h)$ is absolutely irreducible. For all this see [2] section 2.1.

6.2 Standard cyclic $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$ -modules

We put $t = [\nu] - 1 \in k_{\mathcal{E}}^+ = k[[\mathfrak{N}_0]]$. This is a uniformizer in the complete discrete valuation ring $k_{\mathcal{E}}^+$, thus $k_{\mathcal{E}}^+ = k[[t]]$ and $k_{\mathcal{E}} = k((t))$.

We often identify elements of \mathbb{F}_p^{\times} with their Teichmüller lifting in \mathbb{Z}_p^{\times} . In particular, for $x \in \mathbb{F}_p^{\times}$ we may consider the element $\gamma(x) \in \Gamma$.

Definition: (a) We say that a $k_{\mathcal{E}}^+[\varphi^r]$ -module H is standard cyclic of perimeter $m+1 \in \mathbb{N}$ if it is generated by $\ker(t|_H) = H^{\mathfrak{N}_0}$, if it is a torsion $k_{\mathcal{E}}^+$ -module and if there are a k -basis e_0, \dots, e_m of $\ker(t|_H)$, integers $0 \leq k_0, \dots, k_m \leq p^r - 1$ and units $\varrho_0, \dots, \varrho_m \in k^{\times}$ with

$$t^{k_i} \varphi^r e_{i-1} = \varrho_i e_i$$

for all $0 \leq i \leq m$. Here and below we extend the indexing of the k_i , e_i , ϱ_i by $i \in \{0, \dots, m\}$ to an indexing by $i \in \mathbb{Z}$ such that $k_i = k_{i+m+1}$, $e_i = e_{i+m+1}$, $\varrho_i = \varrho_{i+m+1}$.

(b) We say that a $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$ -module H is standard cyclic of perimeter $m+1 \in \mathbb{N}$ if it is standard cyclic of perimeter $m+1$ as a $k_{\mathcal{E}}^+[\varphi^r]$ -module, in such a way that all the above e_i can be chosen as eigenvectors for the action of Γ , with eigenvalues in k^{\times} .

(c) Giving H the discrete topology, we endow the topological dual $H^* = \text{Hom}_k^{\text{ct}}(H, k)$ with the structure of a $k_{\mathcal{E}}^+$ -module by setting $(\alpha \cdot \ell)(x) = \ell(\alpha \cdot x)$ for $\ell \in H^*$, $x \in H$ and $\alpha \in k_{\mathcal{E}}^+$. We define a k -linear endomorphism $\psi_{H^*}^r$ on H^* through $(\psi_{H^*}^r(\ell))(x) = \ell(\varphi^r(x))$ and an action by Γ through $(\gamma(\ell))(x) = \ell(\gamma^{-1}(x))$.

Proposition 6.2. (a) If $k_i > 0$ for at least one $0 \leq i \leq m$ then H^* is a non degenerate (ψ^r, Γ) -module over $k_{\mathcal{E}}^+$ with surjective operator $\psi_{H^*}^r$, free of rank $m+1$ as a $k_{\mathcal{E}}^+$ -module.

(b) Suppose that for any $1 \leq j \leq m$ there is some $0 \leq i \leq m$ with $k_i \neq k_{i+j}$. Then H is irreducible as a $k_{\mathcal{E}}^+[\varphi^r]$ -module.

(c) For $0 \leq i \leq m$ let $\eta_i : \Gamma \rightarrow k^\times$ be the character with $\gamma e_i = \eta_i(\gamma) e_i$ for all $\gamma \in \Gamma$. Suppose that for any $1 \leq j \leq m$ which satisfies $k_i = k_{i+j}$ for all $0 \leq i \leq m$ there is some $0 \leq i \leq m$ with $\eta_i \neq \eta_{i+j}$. Then H is irreducible as a $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$ -module.

(d) If $k_i = 0$ for all $0 \leq i \leq m$ then $t = 0$ on H , and e_0, \dots, e_m is a k -basis of H .

PROOF: Statement (d) is clear. We prove statement (a).

Claim: The action of t on H is surjective.

For any $x \in H$ we need to find some $y \in H$ with $ty = x$. As $\ker(t|_H)$ generates H we may assume, by additivity, that $x = t^{n_1} \varphi^{r n'_1} \dots t^{n_l} \varphi^{r n'_l} e_i$ for some $0 \leq i \leq d$, some $l \geq 0$, some $n_j, n'_j \geq 0$. Using $\varphi^r t = t^{p^r} \varphi^r$ we rewrite this as $x = t^n \varphi^{r n'} e_i$ (some $n, n' \geq 0$). If $n \geq 1$ we are done. Otherwise we put $w_i = k_i + p^r k_{i-1} + \dots + p^{ri} k_0 + p^{r(i+1)} k_m + \dots + p^{rm} k_{i+1}$ and substitute $t^{w_i} \varphi^{r(m+1)} e_i$ for e_i — up to a scalar in k^\times these are the same, as follows from $\varphi^r t = t^{p^r} \varphi^r$. We use $\varphi^r t = t^{p^r} \varphi^r$ again to rewrite the result as $x = t^n \varphi^{r n'} e_i$ for some new $n, n' \geq 0$. Now we have $n \geq 1$ because $w_i \geq 1$ for any i , as follows from the hypothesis that $k_i > 0$ for at least one i . The claim is proven.

Let $\ell \in H^*$, $\ell \neq 0$. Choose $x \in H$ with $\ell(x) \neq 0$. By the above claim (iterated n times) we find for any $n \geq 0$ some $y \in H$ with $t^n y = x$, hence $(t^n \ell)(y) = \ell(t^n y) = \ell(x) \neq 0$, hence $t^n \ell \neq 0$. It follows that H^* is a torsion free $k_{\mathcal{E}}^+$ -module. On the other hand, we know $H^* \otimes_{k_{\mathcal{E}}^+} k$ (with $t \mapsto 0$) because this is dual to $\ker(t|_H)$. In view of Nakayama's Lemma we obtain that H^* is $k_{\mathcal{E}}^+$ -free of rank $m+1$.

Now φ^r is injective on H [as $\varphi^r t = t^{p^r} \varphi^r$ we have $t \cdot \ker(\varphi^r) \subset \ker(\varphi^r)$, thus if we had $\ker(\varphi^r) \neq 0$ then also $\ker(\varphi^r) \cap \ker(t) \neq 0$, but this is false], hence $\psi_{H^*}^r$ is surjective; similarly, $\ker(\psi_{H^*}^r)$ contains no non zero $k_{\mathcal{E}}^+$ -sub module.

(b) Let $0 \neq Z \subset H$ be a non zero $k_{\mathcal{E}}^+[\varphi^r]$ -sub module. As H is a torsion $k_{\mathcal{E}}^+$ -module so is Z , hence $\ker(t|_Z)$ is non zero. Let us put

$$\eta(z) = \eta\left(\sum_{0 \leq i \leq m} x_i e_i\right) = \max\{k_i \mid 0 \leq i \leq m, x_i \neq 0\}$$

for non zero elements $z = \sum_{0 \leq i \leq m} x_i e_i \in \ker(t|_Z)$ (with $x_i \in k$). For such $0 \neq z \in \ker(t|_Z)$ let us put $\Lambda(z) = t^{\eta(z)} \varphi^r z$. This is again an element in $\ker(t|_Z)$ (as Z is stable under t and φ^r); moreover $\Lambda(z) \neq 0$. We apply Λ repeatedly: The hypothesis shows that for sufficiently

large $n \geq 0$ we have $\Lambda^n(z) \in k^\times \cdot e_i$ for some $0 \leq i \leq m$. But then we further see that $\Lambda^{n+j}(z) \in k^\times \cdot e_{i+j}$ for all $j \geq 0$. It follows that Z contains all the e_i , hence $Z = H$ as the e_i generate the $k_{\mathcal{E}}^+[\varphi^r]$ -module H .

(c) Let $0 \neq Z \subset H$ be a non zero $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$ -sub module. If H is not already irreducible as a $k_{\mathcal{E}}^+[\varphi^r]$ -module then, by the proof of (b), there is some $0 \neq z \in \ker(t|_Z)$ such that for all $n \geq 0$, if we write $\Lambda^n(z) = \sum_{0 \leq i \leq m} x_{i,n} e_i$, then the number $|\{i \mid x_{i,n} \neq 0\}|$ is larger than 1 and independent on n . Thus $x_{i,0} \neq 0$ and $x_{i+j,0} \neq 0$ for some i, j , and this j violates the hypothesis in (b). By the hypothesis in (c), replacing z by $\Lambda^n(z)$ for a suitable n (and replacing i by $i+n$) we may assume that $\eta_i \neq \eta_{i+j}$. This allows us to produce a non-zero element $\sum_{0 \leq i \leq m} y_i e_i$ in $\ker(t|_Z)$ such that $|\{i \mid y_i \neq 0\}| < |\{i \mid x_{i,0} \neq 0\}|$. Proceeding by induction we obtain that $e_i \in \ker(t|_Z)$ for some $0 \leq i \leq m$ and hence, as in (b), that $Z = H$. \square

For $0 \leq j \leq m$ let $i_j = p^r - 1 - k_{m+1-j}$ and

$$w_j = k_i + p^r k_{j-1} + \dots + p^{rj} k_0 + p^{r(j+1)} k_m + \dots + p^{rm} k_{j+1}.$$

Let $h_0 = 0$ and for $1 \leq j \leq m+1$ let $h_j = \sum_{i=0}^{j-1} i_{m+i+1-j} p^{ri}$. Let $\varrho = \prod_{i=0}^m \varrho_i$; then $t^{w_i} \varphi^{r(m+1)} e_i = \varrho e_i$ for all $0 \leq i \leq m$ (use $\varphi^r t = t^{p^r} \varphi^r$).

Lemma 6.3. (a) *There is some $0 \leq s \leq p-2$ such that $\gamma(x)e_i = x^{-h_i-s} e_i$ for all $x \in \mathbb{F}_p^\times$, all $0 \leq i \leq m$.*

(b) *$h = h_{m+1}/(p-1)$ is an integer.*

PROOF: (a) For $a \in \mathbb{N} \cap \mathbb{Z}_p^\times$ we formally compute $[\nu]^a - 1 = \sum_{j=1}^a \binom{a}{j} ([\nu] - 1)^j = \sum_{j=1}^a \binom{a}{j} t^j$, thus $\gamma(a)t\gamma(a^{-1}) - at = ([\nu]^a - 1) - at \in t^2 k_{\mathcal{E}}^+$ and therefore also

$$(49) \quad \gamma(a)t^k \gamma(a^{-1}) - a^k t^k \in t^{k+1} k_{\mathcal{E}}^+$$

for all $k \in \mathbb{Z}_{\geq 0}$. For $0 \leq i \leq m$ let $n_i \in \mathbb{Z}$ such that $\gamma(a)e_i = a^{n_i} e_i$ for all $a \in \mathbb{N} \cap \mathbb{Z}_p^\times$. Now let $1 \leq i \leq m$. Then

$$\varrho_i a^{n_i} e_i = \varrho_i \gamma(a) e_i = \gamma(a) \varrho_i e_i = \gamma(a) t^{k_i} \varphi^r e_{i-1} \stackrel{(i)}{=} t^{k_i} \varphi^r a^{k_i} \gamma(a) e_{i-1} = t^{k_i} \varphi^r a^{k_i+n_{i-1}} e_{i-1}$$

where in (i) we used formula (49) and the fact that $t^{k_i} \varphi^r e_{i-1}$ belongs to $\ker(t|_H)$ (as it equals $\varrho_i e_i$). Comparing this with $\varrho_i e_i = t^{k_i} \varphi^r e_{i-1}$ we obtain $k_i \equiv n_i - n_{i-1}$ modulo $(p-1)\mathbb{Z}$. On the other hand, the definition of k_i shows $k_i \equiv -i_{m+1-i} \equiv h_{i-1} - h_i$ modulo $(p-1)\mathbb{Z}$. Together this means $n_i + h_i \equiv n_{i-1} + h_{i-1}$ modulo $(p-1)\mathbb{Z}$ and statement (a) follows.

(b) The fraction h is an integer if and only if $p-1$ divides h_{m+1} , if and only if $p-1$ divides $\sum_{j=0}^m i_j$, if and only if $p-1$ divides $\sum_{j=0}^m k_j$, if and only if $p-1$ divides w_i for any $0 \leq i \leq m$. For any such i and any $\gamma \in \Gamma$ we have

$$\gamma t^{w_i} \gamma^{-1} \varphi^{r(m+1)} \gamma(e_i) = \gamma(t^{w_i} \varphi^{r(m+1)} e_i) = \varrho \gamma(e_i) = t^{w_i} \varphi^{r(m+1)} \gamma(e_i)$$

and this is a non zero element in H . Specifically, taking $\gamma = \gamma(a)$ and using formula (49) we obtain $a^{w_i} = 1$ in \mathbb{F}_p^\times for all $a \in \mathbb{N} \cap \mathbb{Z}_p^\times$. This implies that $p-1$ divides w_i . \square

Lemma 6.4. *Suppose that $k_i > 0$ for at least one i . The (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ associated with the (ψ^r, Γ) -module H^* admits a $k_{\mathcal{E}}$ -basis g_0, \dots, g_m such that*

$$(50) \quad \varphi_{\mathbf{D}}^r(g_j) = g_{j+1} \quad \text{for } 0 \leq j \leq m-1,$$

$$(51) \quad \varphi_{\mathbf{D}}^r(g_m) = \varrho^{-1} t^{-h_{m+1}} g_0,$$

$$(52) \quad \gamma(x)(g_j) - x^s g_j \in t \cdot k_{\mathcal{E}}^+ \cdot g_j \quad \text{for } 0 \leq j \leq m \text{ and } x \in \mathbb{F}_p^\times.$$

These formulae completely characterize the actions of $\varphi_{\mathbf{D}}^r$ and Γ .

PROOF: Let \mathbf{D} denote the φ^r -module over $k_{\mathcal{E}}$ with $k_{\mathcal{E}}$ -basis g_0, \dots, g_m and with φ^r -operator $\varphi_{\mathbf{D}}^r$ given by formulae (50) and (51). It is étale and hence admits the usual canonical left inverse $\psi_{\mathbf{D}}^r$. For $0 \leq j \leq m$ let $f_j = t^{h_j} g_j$. For $1 \leq j \leq m$ and $\alpha \in k_{\mathcal{E}}^+$ we compute

$$(53) \quad \psi_{\mathbf{D}}^r(\alpha f_j) = t^{h_{j-1}} \psi_{\mathbf{D}}^r(\alpha t^{i_{m+1-j}} g_j) = t^{h_{j-1}} \psi_{k_{\mathcal{E}}^+}^r(\alpha t^{i_{m+1-j}}) g_{j-1} = \psi_{k_{\mathcal{E}}^+}^r(\alpha t^{i_{m+1-j}}) f_{j-1}$$

where in the first equation we used $h_j = p^r h_{j-1} + i_{m+1-j}$. We also have $h_{m+1} = p^r h_m + i_0$ and therefore we similarly compute

$$(54) \quad \begin{aligned} \psi_{\mathbf{D}}^r(\alpha f_0) &= \psi_{\mathbf{D}}^r(\alpha g_0) = \varrho \psi_{\mathbf{D}}^r(\alpha t^{h_{m+1}} \varphi_{\mathbf{D}}^r(g_m)) = \varrho t^{h_m} \psi_{k_{\mathcal{E}}^+}^r(\alpha t^{i_0}) g_m \\ &= \varrho \psi_{k_{\mathcal{E}}^+}^r(\alpha t^{i_0}) f_m. \end{aligned}$$

Let D^\sharp be the free $k_{\mathcal{E}}^+$ -submodule of \mathbf{D} with basis f_0, \dots, f_m . Formulae (53) and (54) show that D^\sharp is stable under $\psi_{\mathbf{D}}^r$, hence that D^\sharp is a ψ^r -module over $k_{\mathcal{E}}$. Its ψ^r -operator $\psi_{D^\sharp}^r = \psi_{\mathbf{D}}^r|_{D^\sharp}$ is surjective, as follows from formulae (53) and (54) in view of

$$(55) \quad \psi_{k_{\mathcal{E}}^+}^r(t^{\sum_{i=0}^r n_i p^i}) = (-1)^{\sum_{i=0}^{r-1} n_i} t^{n_r} \quad \text{for } 0 \leq n_0, \dots, n_{r-1} \leq p-1 \text{ and } 0 \leq n_r.$$

For $0 \leq i \leq m$ we define $e'_i \in (D^\sharp)^*$ by $e'_i(f_j) = \delta_{ij}$ and by $e'_i|_{t \cdot D^\sharp} = 0$. The set $\{e'_0, \dots, e'_m\}$ is a k -basis of $\ker(t|_{(D^\sharp)^*})$, and it generates $(D^\sharp)^*$ as a $k_{\mathcal{E}}^+[\varphi^r]$ -module as will follow from formulae (56) and (57) below. We claim

$$(56) \quad t^{k_i} \varphi_{(D^\sharp)^*}^r e'_{i-1} = e'_i \quad \text{for } 1 \leq i \leq m,$$

$$(57) \quad t^{k_0} \varphi_{(D^\sharp)^*}^r e'_m = \varrho e'_0,$$

$$(58) \quad t^{w_i} \varphi_{(D^\sharp)^*}^{r(m+1)} e'_i = \varrho e'_i \quad \text{for } 0 \leq i \leq m.$$

For $n \geq 0$ we compute

$$\begin{aligned} (t^{k_i} \varphi_{(D^\sharp)^*}^r e'_{i-1})(t^n f_j) &= (\varphi_{(D^\sharp)^*}^r e'_{i-1})(t^{k_i+n} f_j) \\ &= e'_{i-1}(\psi_{D^\sharp}^r(t^{k_i+n} f_j)). \end{aligned}$$

Inserting formula (53) we thus get

$$(59) \quad (t^{k_i} \varphi_{(D^\sharp)^*}^r e'_{i-1})(t^n f_j) = e'_{i-1}(\psi_{k_{\mathcal{E}}^+}^r(t^{k_i+n+p^r-1-k_j}) f_{j-1})$$

for $1 \leq j \leq m$ while for $j = 0 \equiv m + 1$, inserting formula (54) we get

$$(60) \quad (t^{k_i} \varphi^r_{(D^\sharp)^*} e'_{i-1})(t^n f_0) = e'_{i-1}(\varrho \psi^r_{k_\mathcal{E}^+}(t^{k_i+n+p^r-1-k_0}) f_m).$$

The right hand side in formula (59) (resp. in formula (60)) vanishes if $i \neq j$ (resp. if $i \neq m+1 \equiv 0$). To evaluate the right hand side if $1 \leq i = j \leq m$ (resp. if $i = m+1 \equiv 0$) we again use formula (55). It shows that for $n > 0$ the right hand side in formula (59) (resp. in formula (60)) vanishes and that for $n = 0$ and $1 \leq i = j \leq m$ its value is 1 (resp. for $n = 0$ and $i = m+1 \equiv 0$ its value is ϱ). We have proven formulae (56) and (57). Formula (58) follows by iteration (as $\varphi^r t = t^{p^r} \varphi^r$).

As both H^* and D^\sharp are $k_\mathcal{E}^+$ -free of the same rank, a comparison of formulae (56), (57) with those describing the action of φ^r on $H \cong (H^*)^*$ shows that there is an isomorphism of $k_\mathcal{E}^+[\varphi^r]$ -modules $H \cong (D^\sharp)^*$ sending the e_i to suitable k^\times -rescalings of the e'_i .

We use this isomorphism to transport the Γ -action on H to a Γ -action on $(D^\sharp)^*$. For $0 \leq i, j \leq m$ and $x \in \mathbb{F}_p$ it satisfies

$$e'_i(\gamma(x)(f_j) - x^{h_j+s} f_j) = (\gamma(x^{-1})e'_i)(f_j) - x^{h_j+s} e'_i(f_j).$$

We claim that this vanishes. Indeed, if $i = j$ then this follows from $\gamma(x^{-1})e'_i = x^{h_i+s} e'_i$ (the definition of s), whereas if $i \neq j$ even both summands vanish individually. The claim proven we infer

$$\gamma(x)(f_j) - x^{h_j+s} f_j \in \bigcap_i \ker(e'_i) = t \cdot D^\sharp.$$

On the other hand, formula (49) says $\gamma(x)(f_j) - x^{h_j} t^{h_j} \gamma(x)(g_j) \in t^{h_j+1} k_\mathcal{E}^+ \cdot g_j \subset t \cdot D^\sharp$. Together we obtain

$$\gamma(x)(g_j) - x^s g_j \in t^{1-h_j} \cdot D^\sharp.$$

As both $\gamma(x)(g_j)$ and $x^s g_j$ belong to $k_\mathcal{E}^+ \cdot g_j$ and as $k_\mathcal{E}^+ \cdot g_j \cap t^{1-h_j} \cdot D^\sharp = t \cdot k_\mathcal{E}^+ \cdot g_j$ we have proven formula (52).

For the final statement it is enough to show that formulae (50), (51) and (52) characterize the $k_\mathcal{E}^+[\varphi^r, \Gamma]$ -module $(D^\sharp)^*$. We have already seen that they imply formulae (56), (57), (58), i.e. they characterize the action of $\varphi^r_{(D^\sharp)^*}$. It therefore remains to see that they characterize the action of Γ on the $k_\mathcal{E}^+[\varphi^r]$ -generators e'_i of $(D^\sharp)^*$. We claim

$$\gamma(x)e'_i = x^{-h_i-s} e'_i.$$

Indeed, both sides vanish on $t \cdot D^\sharp$. To compare their values on an argument in D^\sharp we may ignore summands belonging to $t \cdot D^\sharp$. Thus formula (52) gives us

$$\begin{aligned} (\gamma(x)e'_i)(f_j) &= e'_i(\gamma(x)^{-1} f_j) \\ &= e'_i(x^{-h_j-s} f_j) \end{aligned}$$

for any $0 \leq i, j \leq m$. We are done. □

Let $\beta \in (k^{\text{alg}})^\times$ such that $(-1)^m \beta^{-m-1} = \varrho = \prod_{i=0}^m \varrho_i$.

Proposition 6.5. (Berger) Suppose that $r = 1$ and that $k_i > 0$ for at least one i . Let \mathbf{D} be the (φ, Γ) -module over $k_{\mathcal{E}}$ associated with the (ψ, Γ) -module H^* . We have an isomorphism of $\text{Gal}_{\mathbb{Q}_p}$ -representations

$$W(\mathbf{D}) \cong \text{ind}(\omega_{m+1}^h) \otimes \omega^s \mu_{\beta}.$$

PROOF: In [2] section 2.2 it is shown that the (φ, Γ) -module \mathbf{D}' over $k_{\mathcal{E}}$ with $W(\mathbf{D}') \cong \text{ind}(\omega_{m+1}^h) \otimes \omega^s \mu_{\beta}$ admits a basis in which the actions of φ and Γ satisfy the formulae (50), (51), (52), hence we conclude with Lemma 6.4. \square

Remarks: (a) In [2] section 2.2 even the precise formula for $\gamma(g_j)$, for $\gamma \in \Gamma$, is worked out, sharpening formula (52).

(b) The results in [2] are in fact stated there only under the hypothesis that h be primitive. However, it is easily checked that those statements of [2] which we are using in Proposition 6.5 hold true without that primitivity assumption.

6.3 (φ^r, Γ) -modules and (φ, Γ) -modules

Here we briefly explain the interest in étale (φ^r, Γ) -modules for *any* $r \in \mathbb{N}$ (we will not need this later on in the present paper): There is an exact functor from the category of étale (φ^r, Γ) -modules to the category of étale (φ, Γ) -modules (the rank gets multiplied by the factor r). To the latter, of course, e.g. Theorem 6.1 applies.

Let $\mathbf{D} = (\mathbf{D}, \varphi_{\mathbf{D}}^r)$ be an étale φ^r -module over $\mathcal{O}_{\mathcal{E}}$. For $0 \leq i \leq r-1$ let $\mathbf{D}^{(i)} = \mathbf{D}$ be a copy of \mathbf{D} . For $1 \leq i \leq r-1$ define $\varphi_{\tilde{\mathbf{D}}}^{(i)} : \mathbf{D}^{(i)} \rightarrow \mathbf{D}^{(i-1)}$ to be the identity map on \mathbf{D} , and define $\varphi_{\tilde{\mathbf{D}}} : \mathbf{D}^{(0)} \rightarrow \mathbf{D}^{(r-1)}$ to be the structure map $\varphi_{\mathbf{D}}^r$ on \mathbf{D} . Together we obtain a \mathbb{Z}_p -linear endomorphism $\varphi_{\tilde{\mathbf{D}}}$ on

$$\tilde{\mathbf{D}} = \bigoplus_{i=0}^{r-1} \mathbf{D}^{(i)}.$$

Define an $\mathcal{O}_{\mathcal{E}}$ -action on $\tilde{\mathbf{D}}$ by the formula

$$(61) \quad x \cdot ((d_i)_{0 \leq i \leq r-1}) = (\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x) d_i)_{0 \leq i \leq r-1}.$$

Lemma 6.6. The endomorphism $\varphi_{\tilde{\mathbf{D}}}$ of $\tilde{\mathbf{D}}$ is semilinear with respect to the $\mathcal{O}_{\mathcal{E}}$ -action (61), hence it defines on $\tilde{\mathbf{D}}$ the structure of an étale φ -module over $\mathcal{O}_{\mathcal{E}}$.

PROOF:

$$\begin{aligned} \varphi_{\tilde{\mathbf{D}}}(x \cdot ((d_i)_i)) &= \varphi_{\tilde{\mathbf{D}}}((\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x) d_i)_i) \\ &= ((\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x) d_{i+1})_{0 \leq i \leq r-2}, (\varphi_{\mathbf{D}}^r(x \cdot d_0))_{r-1}) \\ &= ((\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x) d_{i+1})_{0 \leq i \leq r-2}, (\varphi_{\mathcal{O}_{\mathcal{E}}}^r(x) \varphi_{\mathbf{D}}^r(d_0))_{r-1}) \\ &= \varphi_{\mathcal{O}_{\mathcal{E}}}(x)((d_{i+1})_{0 \leq i \leq r-2}, (\varphi_{\mathbf{D}}^r(d_0))_{r-1}) \\ &= \varphi_{\mathcal{O}_{\mathcal{E}}}(x) \varphi_{\tilde{\mathbf{D}}}((d_i)_i). \end{aligned}$$

□

Let Γ' be an open subgroup of Γ , let \mathbf{D} be an étale (φ^r, Γ') -module over $\mathcal{O}_{\mathcal{E}}$. Define an action of Γ' on $\tilde{\mathbf{D}}$ by

$$\gamma \cdot ((d_i)_{0 \leq i \leq r-1}) = (\gamma \cdot d_i)_{0 \leq i \leq r-1}.$$

Lemma 6.7. *The Γ' -action on $\tilde{\mathbf{D}}$ commutes with $\varphi_{\tilde{\mathbf{D}}}$ and is semilinear with respect to the $\mathcal{O}_{\mathcal{E}}$ -action (61), hence we obtain on $\tilde{\mathbf{D}}$ the structure of an étale (φ, Γ') -module over $\mathcal{O}_{\mathcal{E}}$. We thus obtain an exact functor from the category of étale (φ^r, Γ') -modules to the category of étale (φ, Γ') -modules over $\mathcal{O}_{\mathcal{E}}$.*

PROOF: This is immediate from the respective properties of the Γ' -action on \mathbf{D} . □

7 The functor \mathbf{D}

The topological dual V^* of a smooth \mathfrak{N}_0 -representation on a torsion \mathfrak{o} -module V (endowed with the discrete topology) is a compact left $\mathcal{O}_{\mathcal{E}}^+$ -module, with $\mathcal{O}_{\mathcal{E}}^+$ acting through $(a \cdot f)(v) = f(a \cdot v)$ for $a \in \mathcal{O}_{\mathcal{E}}^+$ for $f \in V^*$ and $v \in V$.[§]

Let \mathcal{V} be a \mathfrak{N}_0 -equivariant coefficient system on \mathfrak{X}_+ of level 1. Clearly the \mathfrak{N}_0 -action on both $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ and $H_0(\mathfrak{X}_+, \mathcal{V})$ is smooth, thus

$$D(\mathcal{V}) = H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^* \quad \text{and} \quad D'(\mathcal{V}) = H_0(\mathfrak{X}_+, \mathcal{V})^*$$

are compact $\mathcal{O}_{\mathcal{E}}^+$ -modules. We put

$$\mathbf{D}(\mathcal{V}) = \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} D(\mathcal{V}).$$

Proposition 7.1. *Suppose that \mathcal{V} is strictly of level 1.*

(a) *$D(\mathcal{V})$ can be generated as an $\mathcal{O}_{\mathcal{E}}^+$ -module by $\dim_k(\mathcal{V}(\mathfrak{e}_0) \otimes_{\mathfrak{o}} k)$ many elements. In particular, $\mathbf{D}(\mathcal{V})$ can be generated as an $\mathcal{O}_{\mathcal{E}}$ -module by $\dim_k(\mathcal{V}(\mathfrak{e}_0) \otimes_{\mathfrak{o}} k)$ many elements.*

(b) *The natural map $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} D'(\mathcal{V}) \rightarrow \mathbf{D}(\mathcal{V})$ is bijective.*

PROOF: By Theorem 4.3 we know $\mathcal{V}(\mathfrak{e}_0) \cong H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^{\mathfrak{N}_0}$. It follows that the k -vector space $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^{\mathfrak{N}_0, \pi_K=0}$ can be generated by $\dim_k(\mathcal{V}(\mathfrak{e}_0) \otimes_{\mathfrak{o}} k)$ many elements. By duality this means that the $k = \mathcal{O}_{\mathcal{E}}^+/\mathfrak{m}$ -vector space $D(\mathcal{V})/\mathfrak{m}D(\mathcal{V})$ can be generated by $\dim_k(\mathcal{V}(\mathfrak{e}_0) \otimes_{\mathfrak{o}} k)$ many elements; here \mathfrak{m} denotes the maximal ideal in the local ring $\mathcal{O}_{\mathcal{E}}^+$. Now we conclude with the topological Nakayama Lemma (see [1]).

[§]As \mathfrak{N}_0 is commutative this formula indeed defines an $\mathcal{O}_{\mathcal{E}}^+$ -action. One might argue that it would be more natural to endow the dual with the $\mathcal{O}_{\mathcal{E}}^+$ -action $(a \cdot f)(v) = f(\tilde{a} \cdot v)$ where $\tilde{(\cdot)} : \mathcal{O}_{\mathcal{E}}^+ \rightarrow \mathcal{O}_{\mathcal{E}}^+$ denotes the involution induced by the inversion map on \mathfrak{N}_0 . Of course, everything we are going to develop holds true with this alternative $\mathcal{O}_{\mathcal{E}}^+$ -action as well.

(b) We have an exact sequence $0 \rightarrow \mathcal{V}(\mathfrak{e}_0) \rightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V}) \rightarrow H_0(\mathfrak{X}_+, \mathcal{V}) \rightarrow 0$ giving rise to an exact sequence of $\mathcal{O}_{\mathcal{E}}^+$ -modules (taking the Pontryagin dual is exact)

$$0 \longrightarrow D'(\mathcal{V}) \longrightarrow D(\mathcal{V}) \longrightarrow \mathcal{V}(\mathfrak{e}_0)^* \longrightarrow 0.$$

Tensoring its first non trivial arrow with $\mathcal{O}_{\mathcal{E}}$ over $\mathcal{O}_{\mathcal{E}}^+$ gives the map in question. Its surjectivity follows from $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{V}(\mathfrak{e}_0)^* = 0$ which holds true because $\mathcal{V}(\mathfrak{e}_0)$ and hence $\mathcal{V}(\mathfrak{e}_0)^*$ is finitely generated over \mathfrak{o} . \square

Now suppose that \mathcal{V} is $[\mathfrak{N}_0, \varphi^r]$ -equivariant for some $r \in \mathbb{N}$ such that the structure maps $\varphi_x^r : \mathcal{V}(x) \rightarrow \mathcal{V}(\varphi^r x)$ and $\varphi_\tau^r : \mathcal{V}(\tau) \rightarrow \mathcal{V}(\varphi^r \tau)$ are bijective for all $x \in \mathfrak{X}_+^0$ and $\tau \in \mathfrak{X}_+^1$. The φ^r -action provides an endomorphism $\varphi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r$ of $C_0(\mathfrak{X}_+, \mathcal{V}) = \bigoplus_{x \in \mathfrak{X}_+^0} \mathcal{V}(x)$ and hence an endomorphism $\varphi_{H_0(\overline{\mathfrak{X}}_+, \mathcal{V})}^r$ of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$. We define

$$\psi_{D(\mathcal{V})}^r : D(\mathcal{V}) \longrightarrow D(\mathcal{V}), \quad d \mapsto d \circ \varphi_{H_0(\overline{\mathfrak{X}}_+, \mathcal{V})}^r.$$

It is straightforward to check

$$(62) \quad \psi_{D(\mathcal{V})}^r(\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r(a) \cdot d) = a \cdot (\psi_{D(\mathcal{V})}^r(d)) \quad \text{for } a \in \mathcal{O}_{\mathcal{E}}^+, d \in D(\mathcal{V}).$$

As $\mathcal{O}_{\mathcal{E}} = \bigoplus_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \varphi_{\mathcal{O}_{\mathcal{E}}}^r(\mathcal{O}_{\mathcal{E}})$, any element in $\mathcal{O}_{\mathcal{E}}$ can be written as a sum of products $\varphi_{\mathcal{O}_{\mathcal{E}}}^r(b) \cdot c$ with $b \in \mathcal{O}_{\mathcal{E}}$ and $c \in \mathcal{O}_{\mathcal{E}}^+$. Thus any element in $\mathbf{D}(\mathcal{V})$ is a sum of elements of the form $\varphi_{\mathcal{O}_{\mathcal{E}}}^r(b) \otimes d$ with $b \in \mathcal{O}_{\mathcal{E}}$ and $d \in D(\mathcal{V})$. It therefore follows from formula (62) that there is a well defined \mathfrak{o} -linear map

$$\psi_{\mathbf{D}(\mathcal{V})}^r : \mathbf{D}(\mathcal{V}) \longrightarrow \mathbf{D}(\mathcal{V}), \quad \varphi_{\mathcal{O}_{\mathcal{E}}}^r(b) \otimes d \mapsto b \otimes \psi_{D(\mathcal{V})}^r(d).$$

The map $\varphi^r : \mathfrak{X}_+^0 \rightarrow \mathfrak{X}_+^0$ is injective. Thus the inverse of φ^r induces an isomorphism

$$\varphi^{-r} : \bigoplus_{x \in \varphi^r \mathfrak{X}_+^0} \mathcal{V}(x) \xrightarrow{\cong} \bigoplus_{x \in \mathfrak{X}_+^0} \mathcal{V}(x).$$

We extend it to a map

$$\psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r : C_0(\mathfrak{X}_+, \mathcal{V}) = \bigoplus_{x \in \mathfrak{X}_+^0} \mathcal{V}(x) \longrightarrow C_0(\mathfrak{X}_+, \mathcal{V}) = \bigoplus_{x \in \mathfrak{X}_+^0} \mathcal{V}(x)$$

by requiring that its restriction to $\bigoplus_{x \in \mathfrak{X}_+^0 - \varphi^r \mathfrak{X}_+^0} \mathcal{V}(x)$ vanishes. The definition implies

$$(63) \quad \psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r(\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r(a) \cdot h) = a \cdot \psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r(h) \quad \text{for } a \in \mathcal{O}_{\mathcal{E}}^+, h \in \bigoplus_{x \in \mathfrak{X}_+^0} \mathcal{V}(x).$$

The map $\psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r$ induces an endomorphism

$$\psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r : H_0(\mathfrak{X}_+, \mathcal{V}) \longrightarrow H_0(\mathfrak{X}_+, \mathcal{V})$$

and formula (63) becomes

$$(64) \quad \psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(\varphi_{\mathcal{O}_{\mathcal{E}}^+}^r(a) \cdot h) = a \cdot \psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(h)$$

for $a \in \mathcal{O}_\varepsilon^+$ and $h \in H_0(\mathfrak{X}_+, \mathcal{V})$. We define the endomorphism

$$\varphi_{D'(\mathcal{V})}^r : D'(\mathcal{V}) \longrightarrow D'(\mathcal{V}), \quad d \mapsto d \circ \psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r.$$

We claim

$$(65) \quad \varphi_{D'(\mathcal{V})}^r(a \cdot d) = \varphi_{\mathcal{O}_\varepsilon^+}^r(a) \cdot \varphi_{D'(\mathcal{V})}^r(d) \quad \text{for } a \in \mathcal{O}_\varepsilon^+, d \in D'(\mathcal{V}).$$

Indeed, for $h \in H_0(\mathfrak{X}_+, \mathcal{V})$ we compute

$$\begin{aligned} (\varphi_{D'(\mathcal{V})}^r(a \cdot d))(h) &= (a \cdot d)(\psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(h)) \\ &= d(a \cdot \psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(h)) \\ &\stackrel{(i)}{=} d(\psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(\varphi_{\mathcal{O}_\varepsilon^+}^r(a) \cdot h)) \\ &= (\varphi_{D'(\mathcal{V})}^r(d))(\varphi_{\mathcal{O}_\varepsilon^+}^r(a) \cdot h) \\ &= (\varphi_{\mathcal{O}_\varepsilon^+}^r(a) \cdot \varphi_{D'(\mathcal{V})}^r(d))(h) \end{aligned}$$

where in (i) we used formula (64). Because of formula (65) we may proceed to define

$$\varphi_{\mathbf{D}(\mathcal{V})}^r = \varphi_{\mathcal{O}_\varepsilon}^r \otimes \varphi_{D'(\mathcal{V})}^r : \mathbf{D}(\mathcal{V}) \longrightarrow \mathbf{D}(\mathcal{V})$$

where we use the isomorphism $\mathcal{O}_\varepsilon \otimes_{\mathcal{O}_\varepsilon^+} D'(\mathcal{V}) \cong \mathbf{D}(\mathcal{V})$ of Proposition 7.1 as an identification.

Proposition 7.2. *We have the formulae*

$$(66) \quad \psi_{\mathbf{D}(\mathcal{V})}^r \circ (b \cdot \varphi_{\mathbf{D}(\mathcal{V})}^r) = \psi_{\mathcal{O}_\varepsilon}^r(b) \cdot \text{id}_{\mathbf{D}(\mathcal{V})} \quad \text{for } b \in \mathcal{O}_\varepsilon,$$

$$(67) \quad \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{pr}} n \circ \varphi_{\mathbf{D}(\mathcal{V})}^r \circ \psi_{\mathbf{D}(\mathcal{V})}^r \circ n^{-1} = \text{id}_{\mathbf{D}(\mathcal{V})}.$$

In particular, we have $\psi_{\mathbf{D}(\mathcal{V})}^r \circ \varphi_{\mathbf{D}(\mathcal{V})}^r = \text{id}_{\mathbf{D}(\mathcal{V})}$ and $\varphi_{\mathbf{D}(\mathcal{V})}^r$ is an étale map.

PROOF: To prove formula (66) we first remark that for $b \in \mathcal{O}_\varepsilon^+$ we have

$$\psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r \circ (b \cdot \varphi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r) = \psi_{\mathcal{O}_\varepsilon^+}^r(b) \cdot \text{id}.$$

Indeed, this is true already on 0-chains. To check this the 0-chain may be assumed to be supported on a single vertex, and b may be assumed to belong either to $\text{im}(\varphi_{\mathcal{O}_\varepsilon^+}^r)$ or to $\text{ker}(\psi_{\mathcal{O}_\varepsilon^+}^r)$; in either case the claim follows easily from the definitions. We use this to see

$$\begin{aligned} \psi_{\mathbf{D}(\mathcal{V})}^r(b \cdot \varphi_{\mathbf{D}(\mathcal{V})}^r(d))(h) &= (b \cdot \varphi_{\mathbf{D}(\mathcal{V})}^r(d))(\varphi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(h)) \\ &= d(\psi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(b \cdot \varphi_{H_0(\mathfrak{X}_+, \mathcal{V})}^r(h))) \\ &= d(\psi_{\mathcal{O}_\varepsilon^+}^r(b) \cdot h) \\ &= \psi_{\mathcal{O}_\varepsilon^+}^r(b) \cdot d(h). \end{aligned}$$

Any element of \mathcal{O}_ε can be written as a sum of products $b = b_1 \cdot b_2$ with $b_1 \in \mathcal{O}_\varepsilon$ and $b_2 \in \mathcal{O}_\varepsilon^+$ and $b_1 = \varphi_{\mathcal{O}_\varepsilon}^r(\psi_{\mathcal{O}_\varepsilon}^r(b_1))$. Inserting what we just saw we compute

$$\begin{aligned}
\psi_{\mathbf{D}(\mathcal{V})}^r(b \cdot \varphi_{\mathbf{D}(\mathcal{V})}^r(a \otimes d)) &= \psi_{\mathbf{D}(\mathcal{V})}^r(b \cdot \varphi_{\mathcal{O}_\varepsilon}^r(a) \otimes \varphi_{D'(\mathcal{V})}^r(d)) \\
&= \psi_{\mathbf{D}(\mathcal{V})}^r(\varphi_{\mathcal{O}_\varepsilon}^r(\psi_{\mathcal{O}_\varepsilon}^r(b_1)) \cdot \varphi_{\mathcal{O}_\varepsilon}^r(a) \otimes b_2 \cdot \varphi_{D'(\mathcal{V})}^r(d)) \\
&= \psi_{\mathbf{D}(\mathcal{V})}^r(\varphi_{\mathcal{O}_\varepsilon}^r(\psi_{\mathcal{O}_\varepsilon}^r(b_1) \cdot a) \otimes b_2 \cdot \varphi_{D'(\mathcal{V})}^r(d)) \\
&= \psi_{\mathcal{O}_\varepsilon}^r(b_1) \cdot a \otimes \psi_{D(\mathcal{V})}^r(b_2 \cdot \varphi_{D'(\mathcal{V})}^r(d)) \\
&= \psi_{\mathcal{O}_\varepsilon}^r(b_1) \cdot a \otimes \psi_{\mathcal{O}_\varepsilon^+}^r(b_2) \cdot d \\
&= \psi_{\mathcal{O}_\varepsilon}^r(b_1) \cdot \psi_{\mathcal{O}_\varepsilon}^r(b_2) \cdot a \otimes d \\
&= \psi_{\mathcal{O}_\varepsilon}^r(b) \cdot a \otimes d.
\end{aligned}$$

We have proven formula (66). To prove formula (67) we view the injective map $D'(\mathcal{V}) \rightarrow D(\mathcal{V})$ as an inclusion. We find some $N > 0$ such that $\psi_{D(\mathcal{V})}^r(t^N D(\mathcal{V})) \subset D'(\mathcal{V})$. The map $t^N D(\mathcal{V}) \rightarrow D(\mathcal{V})$ induces an isomorphism $\mathcal{O}_\varepsilon \otimes_{\mathcal{O}_\varepsilon^+} t^N D(\mathcal{V}) \cong \mathcal{O}_\varepsilon \otimes_{\mathcal{O}_\varepsilon^+} D(\mathcal{V}) = \mathbf{D}(\mathcal{V})$, therefore we may write an element in $\mathbf{D}(\mathcal{V})$ as a sum of elements $\varphi_{\mathcal{O}_\varepsilon}^r(b) \otimes d$ with $d \in D(\mathcal{V})$ such that $\psi_{D(\mathcal{V})}^r(d) \in D'(\mathcal{V})$, and then more generally $\psi_{D(\mathcal{V})}^r(n^{-1} \cdot d) \in D'(\mathcal{V})$ for $n \in \mathfrak{N}_0$. We compute

$$\begin{aligned}
\sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \circ \varphi_{\mathbf{D}(\mathcal{V})}^r \circ \psi_{\mathbf{D}(\mathcal{V})}^r \circ n^{-1}(\varphi_{\mathcal{O}_\varepsilon}^r(b) \otimes d) &= \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \circ \varphi_{\mathbf{D}(\mathcal{V})}^r \circ \psi_{\mathbf{D}(\mathcal{V})}^r(\varphi_{\mathcal{O}_\varepsilon}^r(b) \otimes n^{-1} \cdot d) \\
&= \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \circ \varphi_{\mathbf{D}(\mathcal{V})}^r(b \otimes \psi_{D(\mathcal{V})}^r(n^{-1} \cdot d)) \\
&= \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} \varphi_{\mathcal{O}_\varepsilon}^r(b) \otimes n \cdot (\varphi_{D'(\mathcal{V})}^r(\psi_{D(\mathcal{V})}^r(n^{-1} \cdot d))).
\end{aligned}$$

Therefore we need to show the equality

$$(68) \quad \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \cdot (\varphi_{D'(\mathcal{V})}^r(\psi_{D(\mathcal{V})}^r(n^{-1} \cdot d))) = d$$

of linear forms on $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$. An inductive argument using formula (34) shows that any element of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ can be represented by a 0-chain supported on

$$\mathfrak{N}_0 \varphi^r \mathfrak{X}_+^0 = \coprod_{n_0 \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n_0 \varphi^r \mathfrak{X}_+^0.$$

Thus, to prove (68) it is enough, by the definitions of $\varphi_{D'(\mathcal{V})}^r$ and $\psi_{D(\mathcal{V})}^r$, to prove

$$(69) \quad \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} (n^{-1} \circ \varphi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r \circ \psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r \circ n)(c) = c$$

for all $c \in C_0(\mathfrak{X}_+, \mathcal{V})$ supported on $n_0 \varphi^r \mathfrak{X}_+^0$ for some $n_0 \in \mathfrak{N}_0$. For $n \in \mathfrak{N}_0$ with $nn_0 \notin \mathfrak{N}_0^{p^r}$ we have $nn_0 \varphi^r \mathfrak{X}_+^0 \cap \varphi^r \mathfrak{X}_+^0 = \emptyset$ and then c is killed by $\psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r \circ n$. On the other hand, if $nn_0 \in \mathfrak{N}_0^{p^r}$ then c is fixed by $n^{-1} \circ \varphi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r \circ \psi_{C_0(\mathfrak{X}_+, \mathcal{V})}^r \circ n$. Formula (69) is proven. \square

Corollary 7.3. (a) The $\mathcal{O}_{\mathcal{E}}$ -module $\mathbf{D}(\mathcal{V})$ can be generated by $\dim_k(\mathcal{V}(\mathfrak{e}_0) \otimes_{\mathfrak{o}} k)$ many elements and carries a natural structure of an étale (φ^r, Γ_0) -module.

(b) If the $[\mathfrak{N}_0, \varphi^r]$ -action on \mathcal{V} extends to a $[\mathfrak{N}_0, \varphi^r, \Gamma]$ -action on \mathcal{V} , then $\mathbf{D}(\mathcal{V})$ is an étale (φ^r, Γ) -module.

PROOF: (a) Except for the Γ_0 -action everything else has already been established in Propositions 7.1 and 7.2. By proposition 4.4 the $[\mathfrak{N}_0, \varphi^r]$ -action on \mathcal{V} naturally extends to a $[\mathfrak{N}_0, \varphi^r, \Gamma_0]$ -action on \mathcal{V} . In particular it induces a Γ_0 -action on $H_0(\mathfrak{X}_+, \mathcal{V})$. We define a Γ_0 -action on $D'(\mathcal{V})$ by setting $(\gamma \cdot d)(h) = d(\gamma^{-1} \cdot h)$ for $\gamma \in \Gamma_0$, $d \in D'(\mathcal{V})$, $h \in H_0(\mathfrak{X}_+, \mathcal{V})$. For $a \in \mathcal{O}_{\mathcal{E}}^+$ we compute

$$\begin{aligned} (\gamma \cdot (a \cdot d))(h) &= (a \cdot d)(\gamma^{-1} \cdot h) = d(a \cdot \gamma^{-1} \cdot h) = (\gamma \cdot d)(\gamma a \gamma^{-1} \cdot h) \\ &= ((\gamma a \gamma^{-1}) \cdot (\gamma \cdot d))(h) = ((\gamma \cdot a) \cdot (\gamma \cdot d))(h) \end{aligned}$$

i.e. the action is semilinear. Moreover, it follows immediately from the definitions that it commutes with $\varphi_{D'(\mathcal{V})}^r$. Therefore we obtain a semilinear action of Γ_0 on $\mathbf{D}(\mathcal{V})$, commuting with $\varphi_{\mathbf{D}(\mathcal{V})}^r$, by putting $\gamma(a \otimes d) = \gamma \cdot a \otimes \gamma \cdot d$ for $a \in \mathcal{O}_{\mathcal{E}}$ and $d \in D'(\mathcal{V})$.

Of course, the same construction also endows $D(\mathcal{V})$ with a semilinear Γ_0 -action, commuting with $\psi_{D(\mathcal{V})}^r$, and extending to the same Γ_0 -action on $\mathbf{D}(\mathcal{V})$.

(b) This is the same argument (without invoking proposition 4.4). \square

Corollary 7.4. (a) The restriction of $\psi_{\mathbf{D}(\mathcal{V})}^r$ to $\cap_{n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}} \ker(\psi_{\mathbf{D}(\mathcal{V})}^r(n \cdot \cdot))$ is a bijection onto $\mathbf{D}(\mathcal{V})$. Its inverse, composed with the inclusion of $\cap_{n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}} \ker(\psi_{\mathbf{D}(\mathcal{V})}^r(n \cdot \cdot))$ into $\mathbf{D}(\mathcal{V})$, is the map $\varphi_{\mathbf{D}(\mathcal{V})}^r$; in particular, the latter can be reconstructed from $\psi_{\mathbf{D}(\mathcal{V})}^r$.

(b) Conversely, $\psi_{\mathbf{D}(\mathcal{V})}^r$ can be reconstructed from $\varphi_{\mathbf{D}(\mathcal{V})}^r$.

PROOF: This is a formal consequence of Proposition 7.2.

(a) (See [14] Proposition 3.3.24 for the abstract argument.) All we need to show is

$$\text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r) = \bigcap_{n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}} \ker(\psi_{\mathbf{D}(\mathcal{V})}^r(n \cdot \cdot)).$$

Let $d \in \text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r)$, say $d = \varphi_{\mathbf{D}(\mathcal{V})}^r(c)$. Let $n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}$. Then $\psi_{\mathbf{D}(\mathcal{V})}^r(n \cdot d) = \psi_{\mathbf{D}(\mathcal{V})}^r(n \cdot \varphi_{\mathbf{D}(\mathcal{V})}^r(c)) = \psi_{\mathcal{O}_{\mathcal{E}}}^r(n) \cdot c = 0$ where we used formula (66) and then $\psi_{\mathcal{O}_{\mathcal{E}}}^r(n) = 0$. Conversely, let $d \in \cap_{n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}} \ker(\psi_{\mathbf{D}(\mathcal{V})}^r(n \cdot \cdot))$. Formula (67) shows

$$d = \sum_{n \in \mathfrak{N}_0 / \mathfrak{N}_0^{p^r}} n \cdot \varphi_{\mathbf{D}(\mathcal{V})}^r(\psi_{\mathbf{D}(\mathcal{V})}^r(n^{-1} \cdot d)).$$

By hypothesis, only the summand for the coset $\mathfrak{N}_0^{p^r}$ survives, showing $d = \varphi_{\mathbf{D}(\mathcal{V})}^r(\psi_{\mathbf{D}(\mathcal{V})}^r(d))$, hence $d \in \text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r)$.

(b) Proposition 7.2 implies $\mathbf{D}(\mathcal{V}) = \text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r) \oplus \sum_{n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}} n \cdot \text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r)$ and that $\varphi_{\mathbf{D}(\mathcal{V})}^r$ is injective. Thus $\psi_{\mathbf{D}(\mathcal{V})}^r$ is the projection onto $\text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r)$ with kernel $\sum_{n \in \mathfrak{N}_0 - \mathfrak{N}_0^{p^r}} n \cdot \text{im}(\varphi_{\mathbf{D}(\mathcal{V})}^r)$,

composed with the inverse of $\varphi_{\mathbf{D}(\mathcal{V})}^r$. □

We fix a gallery (11) and choose an isomorphism $\Theta : Y \xrightarrow{\cong} \mathfrak{X}_+$ as in Theorem 3.2.

Theorem 7.5. (a) *Given ϕ as in Theorem 3.2(b), the assignment $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ is an exact contravariant functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_{\mathfrak{o}_m})$ to the category of étale (φ^r, Γ_0) -modules over $\mathcal{O}_{\mathcal{E}}$.*

(b) *Given ϕ and τ as in Theorem 3.2(d), the assignment $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ is an exact contravariant functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_{\mathfrak{o}_m})$ to the category of étale (φ^r, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$.*

(c) *The functors in (a) and (b) depend canonically on the choice of (11) and ϕ alone, resp. of (11) and ϕ and τ alone, not on the choice of Θ .*

(d) *For $M \in \text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_{\mathfrak{o}_1}) = \text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_k)$ we have*

$$\dim_{k_{\mathcal{E}}} \mathbf{D}(\Theta_* \mathcal{V}_M) \leq \dim_k M.$$

PROOF: The assignment $M \mapsto \mathcal{V}_M$ (and hence $M \mapsto \Theta_* \mathcal{V}_M$) is exact by Proposition 5.1. Therefore the assignment $M \mapsto H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ is exact, cf. the exact sequence (41). By exactness of taking Pontryagin duals it follows that $M \mapsto D(\Theta_* \mathcal{V}_M)$ is exact. Finally, by the flatness of $\mathcal{O}_{\mathcal{E}}^+ \rightarrow \mathcal{O}_{\mathcal{E}}$ we obtain that $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ is exact.

The independence on the choice of Θ follows from the corresponding independence statement in Theorem 4.2. □

8 The case $\text{GL}_{d+1}(\mathbb{Q}_p)$

We consider the case $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ for some $d \geq 1$ and keep all the previous notations $T, N(T), X, A$ etc.. We fix a chamber C in A , and as before we denote by I resp. I_0 the corresponding Iwahori subgroup, resp. pro- p -Iwahori subgroup of G . The (affine) reflections in the codimension-1-faces of C form a set S of Coxeter generators for the affine Weyl group which we view as a subgroup of the extended affine Weyl group $N(T)/Z(T \cap I)$. Put $W = N(T)/T$, the finite Weyl group.

We find elements $u, s_d \in N(T)$ such that $uC = C$ (equivalently, $uI = Iu$, or also $uI_0 = I_0u$), such that $u^{d+1} \in \{p \cdot \text{id}, p^{-1} \cdot \text{id}\}$ and such that, setting

$$s_i = u^{d-i} s_d u^{i-d} \quad \text{for } 0 \leq i \leq d$$

the set $\{s_0, s_1, \dots, s_d\}$ maps bijectively to S ; we henceforth regard this bijection as an identifications. Let $\iota : \text{SL}_2(\mathbb{Q}_p) \rightarrow G$ denote the embedding corresponding to s_d . For $0 \leq i \leq d$ we put

$$n_{s_i} = u^{d-i} \cdot \iota \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \cdot u^{i-d} \quad \text{and} \quad h_{s_i}(x) = u^{d-i} \cdot \iota \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \right) \cdot u^{i-d}$$

for $x \in \mathbb{F}_p^\times$ where we use the Teichmüller character to regard $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ as an element of $\mathrm{SL}_2(\mathbb{Q}_p)$. Similarly, by means of the Teichmüller character we regard the group $\bar{T} = I/I_0 = (T \cap I)/(T \cap I_0)$ as a subgroup of T .

We write $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$. Let $\mathcal{H}(G, I_0)_{\mathrm{aff}, k}$ denote the k -subalgebra of $\mathcal{H}(G, I_0)_k$ generated by the T_{n_s} for $s \in S$ and the T_t for $t \in \bar{T}$. Let $\mathcal{H}(G, I_0)'_{\mathrm{aff}, k}$ denote the k -subalgebra of $\mathcal{H}(G, I_0)_k$ generated by $\mathcal{H}(G, I_0)_{\mathrm{aff}, k}$ together with the elements $T_{p \cdot \mathrm{id}}$ and $T_{p^{-1} \cdot \mathrm{id}} = T_{p \cdot \mathrm{id}}^{-1}$. We put

$$\phi = s_d u \in N(T) \quad \text{and} \quad C^{(i)} = \phi^i C \quad \text{for } i \geq 0.$$

We have $\phi^d = \xi(p) \in T$ for some $\xi \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{G}_m, T)$. Let

$$\tau = (\cdot)^m \cdot \xi|_{\mathbb{Z}_p^\times} : \mathbb{Z}_p^\times \longrightarrow T, \quad a \mapsto a^m \cdot \xi(a)$$

for some $m \in \mathbb{Z}$. For $x \in \mathbb{F}_p^\times$ use the Teichmüller lifting to define $\tau(x) \in T$.

Lemma 8.1. *(For a suitable choice of N_0 we have:) $\{C^{(i)}\}_{i \geq 0}$, ϕ and τ satisfy the assumptions of Theorem 3.2 (with $r = 1$ there).*

PROOF: We choose a system Δ of simple roots in such a way that the image of $S - \{s_0\}$ in W is the set of simple reflections corresponding to Δ . (The above embedding $\iota : \mathrm{SL}_2(\mathbb{Q}_p) \rightarrow G$ is then taken, more precisely, to be the one corresponding to the simple root associated with s_d .) We take N_0 to be the group of \mathbb{Z}_p -valued points of the unipotent radical of the Borel subgroup containing T corresponding to Δ . Then ϕ^d belongs to T and we have $\phi^d N_0 \phi^{-d} \subset N_0$ with $[N_0 : \phi^d N_0 \phi^{-d}] = p^d$. From this the claims easily follow. (All this can also be checked by means of the concrete realizations of T , ϕ , u etc. given below.) \square

We have the isomorphism $\Theta : Y \cong \mathfrak{X}_+$ as constructed in Theorem 3.2, by choosing the element ν_0 in the proof of Theorem 3.2 to be $\nu_0 = \iota(\nu)$. Recall that the chamber C of X corresponds to the edge \mathbf{e}_0 of Y . The codimension-1-face of C corresponding to the vertex \mathbf{v}_0 of Y is

$$F = C \cap C^{(1)} = C \cap \phi C = C \cap s_d u C = C \cap s_d C.$$

Therefore s_d is the simple reflection corresponding to the simple root $\alpha^{(0)}$, and in the present setting we have

$$\varphi = \phi = s_d u.$$

Placing ourselves into the setting of sections 2 and 5 we observe:

Lemma 8.2. *(a) The image of $\mathfrak{N}_0 \subset \mathrm{SL}_2(\mathbb{Z}_p)$ in $\bar{\mathcal{S}} = \mathrm{SL}_2(\mathbb{F}_p)$ is $\mathfrak{N}_0/\mathfrak{N}_0^p$, and this is the unipotent radical $\bar{\mathcal{U}}$ of a Borel subgroup in $\bar{\mathcal{S}}$.*

(b) We have an isomorphism between $\bar{\mathcal{S}}$ and the maximal reductive (over \mathbb{F}_p) quotient of I_0^F , inducing an isomorphism between $\bar{\mathcal{U}}$ and the image of I_0 , hence an embedding of k -algebras

$$(70) \quad \begin{aligned} \mathcal{H}(\bar{\mathcal{S}}, \bar{\mathcal{U}})_k &= \mathrm{End}_{k[\bar{\mathcal{S}}]}(\mathrm{ind}_{\bar{\mathcal{U}}}^{\bar{\mathcal{S}}} \mathbf{1}_k)^{\mathrm{op}} \cong \mathrm{End}_{k[I_0^F]}(\mathrm{ind}_{I_0}^{I_0^F} \mathbf{1}_k)^{\mathrm{op}} \\ &\hookrightarrow \mathcal{H}(G, I_0)_{\mathrm{aff}, k} \subset \mathcal{H}(G, I_0)_k. \end{aligned}$$

It sends $T_{h_s(x)}$ to $T_{h_{s_d}(x)}$ (for $x \in \mathbb{F}_p^\times$) and T_{n_s} to $T_{n_{s_d}}$.

PROOF: (a) is clear. (b) The subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ generated by the stabilizers of the edges in $\overline{\mathfrak{X}}_+$ emanating from \mathfrak{v}_0 is $\mathrm{SL}_2(\mathbb{Z}_p)$, its maximal reductive (over \mathbb{F}_p) quotient is $\overline{\mathcal{S}} = \mathrm{SL}_2(\mathbb{F}_p)$. From this everything follows. \square

For a $\mathcal{H}(G, I_0)_k$ -module M , Theorem 4.2 and Proposition 5.1 tell us that $\Theta_* \mathcal{V}_M$ is in a natural way a $[\mathfrak{N}_0, \varphi, \Gamma]$ -equivariant coefficient system on \mathfrak{X}_+ , with \mathfrak{N}_0 -action strictly of level 1. For later use we remark that $\Theta_* \mathcal{V}_M(\mathfrak{v}_0) = \mathcal{V}_M(\mathfrak{v}_0) = \mathcal{V}_M^X(F)$ is stable under s_d (which acts on the G -equivariant coefficient system \mathcal{V}_M^X).

From now on, for concreteness, we specialize our discussion as follows. T is the subgroup of diagonal matrices in G and I_0 is the pro- p -Iwahori subgroup of G consisting of all matrices in $\mathrm{GL}_{d+1}(\mathbb{Z}_p)$ which (minus the identity) are strictly upper triangular modulo p . We further assume that we are in one of the following two 'opposite' cases (which we treat simultaneously).[¶] The first one is where

$$(71) \quad u = \begin{pmatrix} 0 & E_d \\ p & 0 \end{pmatrix}, \quad s_d = \begin{pmatrix} E_{d-1} & 0 \\ 0 & s \end{pmatrix}, \quad \tau(a) = \begin{pmatrix} E_d & 0 \\ 0 & a^{-1} \end{pmatrix}$$

for $a \in \mathbb{Z}_p^\times$. Here E_l denotes the identity $l \times l$ -matrix, and $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2$. In this case we have $N'_0 = \begin{pmatrix} E_d & * \\ 0 & 1 \end{pmatrix}$ (with column $*$ having entries in \mathbb{Z}_p), where N'_0 denotes the subgroup of G generated by all the $N_{\alpha(j)} \cap N_0$ for $j \geq 0$ (cf. the remark following Theorem 3.2).

The 'opposite' case (corresponding to the other end of the Dynkin diagram) is where

$$(72) \quad u = \begin{pmatrix} 0 & p^{-1} \\ E_d & 0 \end{pmatrix}, \quad s_d = \begin{pmatrix} -s & 0 \\ 0 & E_{d-1} \end{pmatrix}, \quad \tau(a) = \begin{pmatrix} a & 0 \\ 0 & E_d \end{pmatrix}.$$

In this case we have $N'_0 = \begin{pmatrix} 1 & * \\ 0 & E_d \end{pmatrix}$.

8.1 Supersingular $\mathcal{H}(G, I_0)_k$ -modules

For a character $\lambda : \overline{T} \rightarrow k^\times$ we denote by S_λ the set of all $s \in S$ with $\lambda(h_s(x)) = 1$ for all $x \in \mathbb{F}_p^\times$. Suppose we are given a character $\lambda : \overline{T} \rightarrow k^\times$ and a subset \mathcal{J} of S_λ . There is a uniquely determined character

$$\chi_{\lambda, \mathcal{J}} : \mathcal{H}(G, I_0)_{\mathrm{aff}, k} \longrightarrow k$$

[¶]Notice that our analysis requires explicit matrix computations only when we justify certain statements involving τ . For some of these statements our specific choice of τ is important, i.e. some of them may fail if τ is replaced by $a \mapsto a^m \cdot \tau(a)$ for some arbitrary $m \in \mathbb{Z}$. In general, the condition to be imposed on τ should be that the image of τ together with $\mathrm{SL}_{d+1}(\mathbb{F}_p)$ generates $\mathrm{GL}_{d+1}(\mathbb{F}_p)$.

which sends T_t to $\lambda(t^{-1})$ for $t \in \overline{T}$, which sends T_{n_s} to 0 for $s \in S - \mathcal{J}$ and which sends T_{n_s} to -1 for $s \in \mathcal{J}$ (see [15] Proposition 2). Let $b \in k^\times$. The character $\chi_{\lambda, \mathcal{J}}$ extends uniquely to a character

$$\chi_{\lambda, \mathcal{J}, b} : \mathcal{H}(G, I_0)'_{\text{aff}, k} \longrightarrow k$$

which sends $T_{u^{d+1}}$ to b (see the proof of [15] Proposition 3). We define the $\mathcal{H}(G, I_0)_k$ -module

$$M[\lambda, \mathcal{J}, b] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)'_{\text{aff}, k}} k.e$$

where $k.e$ denotes the one dimensional k -vector space on the basis element e , endowed with the action of $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ by the character $\chi_{\lambda, \mathcal{J}, b}$.

For $0 \leq i \leq d$ we let $e_i = T_{u^{-i}} \otimes e \in M[\lambda, \mathcal{J}, b]$. The pair $(\lambda^{[i]}, \mathcal{J}^{[i]})$ defined by $\lambda^{[i]}(t) = \lambda(u^{-i}tu^i)$ and $\mathcal{J}^{[i]} = u^i \mathcal{J} u^{-i}$ satisfies the same assumptions as (λ, \mathcal{J}) , hence gives rise to a corresponding character $\chi_{\lambda^{[i]}, \mathcal{J}^{[i]}, b}$ of $\mathcal{H}(G, I_0)'_{\text{aff}, k}$.

For $m \in \mathbb{Z}$ and $0 \leq i \leq d$ with $m - i \in (d+1)\mathbb{Z}$ we set $e_m = e_i$, $\mathcal{J}^{[m]} = \mathcal{J}^{[i]}$ and $\lambda^{[m]} = \lambda^{[i]}$, and similarly for all other objects defined below which are indexed by $0 \leq i \leq d$. Consider the condition

$$(73) \quad \text{The pairs } (\lambda^{[0]}, \mathcal{J}^{[0]}), \dots, (\lambda^{[d]}, \mathcal{J}^{[d]}) \text{ are pairwise distinct.}$$

Proposition 8.3. (a) If (λ, \mathcal{J}) satisfies (73) then $M[\lambda, \mathcal{J}, b]$ is an absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -module of k -dimension $d+1$. For another pair (λ', \mathcal{J}') satisfying (73) the $\mathcal{H}(G, I_0)_k$ -modules $M[\lambda, \mathcal{J}, b]$ and $M[\lambda', \mathcal{J}', b]$ are isomorphic if and only if (λ', \mathcal{J}') and (λ, \mathcal{J}) are conjugate by some power of u .

(b) Any absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -module of k -dimension $d+1$ is isomorphic with $M[\lambda, \mathcal{J}, b]$ for suitable λ, \mathcal{J}, b satisfying (73).

(c) As a $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ -module, $M[\lambda, \mathcal{J}, b]$ decomposes as

$$(74) \quad M[\lambda, \mathcal{J}, b] \cong \bigoplus_{0 \leq i \leq d} k.e_i$$

with $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ acting on $k.e_i$ by the character $\chi_{\lambda^{[i]}, \mathcal{J}^{[i]}, b}$.

PROOF: For (a) see [15] Proposition 3 and Theorem 5. Notice that conjugating the pair (λ, \mathcal{J}) by powers of u is equivalent with cyclically permuting the set of pairs $\{(\lambda^{[i]}, \mathcal{J}^{[i]})\}_i$. For (b) see [15] Theorem 5 together with [11] Theorem 7.3. For (c) see the proof of [15] Proposition 3. To see e.g. that T_t for $t \in \overline{T}$ acts by $\lambda^{[i]}(t^{-1})$ on $k.e_i$ we compute

$$\begin{aligned} T_t e_i &= T_t T_{u^{-i}} \otimes e = T_{u^{-i}t} \otimes e = T_{u^{-i}} T_{u^{-i}tu^i} \otimes e \\ &= T_{u^{-i}} \otimes T_{u^{-i}tu^i} e = T_{u^{-i}} \otimes \lambda(u^{-i}t^{-1}u^i) e = T_{u^{-i}} \otimes \lambda^{[i]}(t^{-1}) e = \lambda^{[i]}(t^{-1}) e_i. \end{aligned}$$

□

For $0 \leq i \leq d$ we define a number $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ such that

$$(75) \quad \lambda^{[i]}(h_{s_d}(x)) = \lambda(h_{s_{i-1}}(x)) = x^{k_i} \quad \text{for all } x \in \mathbb{F}_p^\times,$$

as follows. If $\lambda^{[i]} \circ h_{s_d}$ is not the constant character $\mathbf{1}$ then k_i is already uniquely determined by formula (75). Next notice that $\lambda^{[i]} \circ h_{s_d} = \mathbf{1}$ is equivalent with $s_{i-1} \in S_\lambda$. If $\lambda^{[i]} \circ h_{s_d} = \mathbf{1}$ and $s_{i-1} \in \mathcal{J}$ we put $k_i = p-1$, if $\lambda^{[i]} \circ h_{s_d} = \mathbf{1}$ and $s_{i-1} \notin \mathcal{J}$ we put $k_i = 0$. We put

$$(76) \quad w_i = w_i(\lambda, \mathcal{J}) = k_i + pk_{i-1} + p^2k_{i-2} + \dots + p^ik_0 + p^{i+1}k_d + \dots + p^dk_{i+1},$$

$$\delta(\lambda, \mathcal{J}) = (-1)^d \lambda(-\text{id}) \prod_{i=0}^d k_i!.$$

Put $\mathcal{V} = \Theta_* \mathcal{V}_{M[\lambda, \mathcal{J}, b]}$. The image of $e_i \in M[\lambda, \mathcal{J}, b]$ in $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ we denote again by e_i .

Proposition 8.4. (a) $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ is a standard cyclic $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module of perimeter $d+1$. The set $\{e_0, \dots, e_d\}$ is a k -basis of the kernel $\ker(t)$ of t , and we have the following formulae:

$$(77) \quad t^{k_i} \varphi e_{i-1} = k_i! \lambda^{[i]}(\tau(-1)) e_i \quad \text{for } 1 \leq i \leq d,$$

$$(78) \quad t^{k_0} \varphi e_d = k_0! \lambda^{[0]}(\tau(-1)) b^{-1} e_0,$$

$$(79) \quad t^{w_i} \varphi^{d+1} e_i = (-1)^d \delta(\lambda, \mathcal{J}) b^{-1} e_i \quad \text{for } 0 \leq i \leq d,$$

$$(80) \quad \gamma(x) e_i = \lambda^{[i]}(\tau(x)) e_i \quad \text{for } 0 \leq i \leq d \text{ and } x \in \mathbb{F}_p^\times.$$

PROOF: As $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ is an inductive limit of $k_{\mathcal{E}}^+$ -modules which are finite dimensional over k , it is a torsion $k_{\mathcal{E}}^+$ -module. From Theorem 4.3 we obtain $M[\lambda, \mathcal{J}, b] = \mathcal{V}(\mathfrak{e}_0) = H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^{\mathfrak{N}_0} = \ker(t)$. From this and the strict level 1 property (namely formula (34)) together with the fact that for any $\mathfrak{e} \in Y^1$ we find some $n \in N_0$ and $m \geq 0$ with $n\phi^m \mathfrak{e}_0 = \mathfrak{e}$ it follows that $\ker(t)$ generates $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$.

For $x \in \mathbb{F}_p^\times$ we compute $T_{h_{s_d}(x)} e_i = \lambda^{[i]}(h_{s_d}(x^{-1})) e_i = x^{-k_i} e_i$. On the other hand, in Lemma 2.5 (with $k_i = r$ there) we have $T_{h_s(x)} e = x^{-k_i} e$. Therefore, if we apply Lemma 2.5 to the character of $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ obtained by pulling back the character $\chi_{\lambda^{[i]}, \mathcal{J}^{[i]}, b}$ of $\mathcal{H}(G, I_0)_{\text{aff}, k}$ along the embedding (70), then we obtain $t^{k_i} n_{s_d}^{-1} e_i = k_i! e_i$ inside $M[\lambda, \mathcal{J}, b] = \mathcal{V}(\mathfrak{e}_0)$, the latter viewed (cf. Proposition 5.1) as a subspace of

$$\mathcal{V}(\mathfrak{v}_0) \cong (\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M.$$

A matrix computation shows $n_{s_d} s_d = \tau(-1) = \tau(-1)^{-1} \in \overline{T}$. We therefore see that $s_d e_i = \lambda^{[i]}(\tau(-1)) n_{s_d}^{-1} e_i$ since $\tau(-1)$ acts by $T_{\tau(-1)^{-1}}$, i.e. by $\lambda^{[i]}(\tau(-1)^{-1})$ on $k.e_i$. We get

$$(81) \quad \begin{aligned} t^{k_i} s_d e_i &= \lambda^{[i]}(\tau(-1)) t^{k_i} n_{s_d}^{-1} e_i \\ &= k_i! \lambda^{[i]}(\tau(-1)) e_i. \end{aligned}$$

Formula (44) gives $u e_{i-1} = T_{u^{-1}} e_{i-1}$. For $1 \leq i \leq d$ we therefore obtain

$$(82) \quad t^{k_i} \varphi e_{i-1} = t^{k_i} s_d u e_{i-1} = t^{k_i} s_d T_{u^{-1}} e_{i-1} = t^{k_i} s_d e_i$$

while for $i = 0$ we obtain

$$(83) \quad t^{k_0} \varphi e_d = t^{k_0} s_d u e_d = t^{k_0} s_d T_{u^{-1}} e_d = t^{k_0} s_d T_{u^{-1}-d} e_0 = t^{k_0} s_d b^{-1} e_0.$$

Formula (81) together with formula (82), resp. with formula (83), gives formula (77), resp. formula (78). Combining formulae (77) and (78) gives formula (79); for this observe $\varphi t = t^p \varphi$ and $\lambda(-\text{id}) = \prod_{i=0}^d \lambda^{[i]}(\tau(-1))$. To see formula (80) notice that the action of $\gamma(x) \in \Gamma$ is given by the action of $\tau(x) \in T$, and that $\tau(x)$ acts by $T_{\tau(x)^{-1}}$, i.e. by $\lambda^{[i]}(\tau(x))$ on $k.e_i$. \square

For $0 \leq j \leq d+1$ let $i_j = p-1-k_{d+1-j}$ and $h_j = \sum_{i=0}^{j-1} i_{d+i+1-j} p^i$. (Attention: In general, $0 = h_0$ need not be equal to h_{d+1} . The h_j must not be confused with the h_{s_j} .) Define $0 \leq s \leq p-2$ by the condition $x^{-s} = \lambda(\tau(x))$ for all $x \in \mathbb{F}_p^\times$. Let $\beta \in (k^{\text{alg}})^\times$ be such that $\beta^{d+1} = \delta(\lambda, \mathcal{J})^{-1} b$.

Theorem 8.5. *$h = h_{d+1}/(p-1)$ is an integer, and if $k_i > 0$ for some i then we have an isomorphism of k -linear $\text{Gal}_{\mathbb{Q}_p}$ -representations*

$$W(\mathbf{D}(\mathcal{V})) \cong \text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta.$$

PROOF: That h is an integer follows from Lemma 6.3. Alternatively, it follows from the divisibility of $\sum_{0 \leq i \leq d} k_i$ by $p-1$, which is a consequence of

$$x^{\sum_i k_i} = \prod_i x^{k_i} = \prod_i \lambda^{[i]}(h_{s_d}(x)) = \lambda\left(\prod_i h_{s_i}(x)\right) = \lambda(\text{id}) = 1$$

for all $x \in \mathbb{F}_p^\times$. We now conclude with Proposition 6.5, using the formulae of Proposition 8.4. (By Lemma 6.3, the formula $x^{-s-h_j} = \lambda^{[j]}(\tau(x))$ for $x \in \mathbb{F}_p^\times$ and all $0 \leq j \leq d$ follows from the case $j = 0$; alternatively it can be verified by a straightforward calculation showing $x^{h_j} = (\lambda/\lambda^{[j]})(\tau(x))$.) \square

Theorem 8.6. (a) *If for any $1 \leq j \leq d$ there is some $0 \leq i \leq d$ such that $k_i \neq k_{i+j}$ then $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ is irreducible as a $k_{\mathcal{E}}^+[\varphi]$ -module.*

(b) *If (λ, \mathcal{J}) satisfies (73) then $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ is irreducible as a $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module.*

(c) *If (λ, \mathcal{J}) satisfies (73) then $D(\mathcal{V})$ is an irreducible (ψ, Γ) -module over $k_{\mathcal{E}}^+$, and $\mathbf{D}(\mathcal{V})$ is an irreducible (φ, Γ) -module over $k_{\mathcal{E}}$. The integer $h = h_{d+1}/(p-1)$ is primitive and the $\text{Gal}_{\mathbb{Q}_p}$ -representation $W(\mathbf{D}(\mathcal{V})) \cong \text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta$ is irreducible.*

PROOF: (In the case $d = 1$ the argument was given in Theorem 5.1 of [6].) We use the formulae in Proposition 8.4. Statement (a) follows immediately from Proposition 6.2.

For statement (b) we first claim that for any $1 \leq j \leq d$ violating the hypothesis in (a), i.e. such that for all $0 \leq i \leq d$ we have $k_i = k_{i+j}$, we have $\lambda^{[i]} \circ \tau \neq \lambda^{[i+j]} \circ \tau$ for all $0 \leq i \leq d$.

Indeed, $k_i = k_{i+j}$ for all $0 \leq i \leq d$ implies $\mathcal{J}^{[i]} = \mathcal{J}^{[i+j]}$ for all $0 \leq i \leq d$. Thus, by hypothesis (73) we have $\lambda^{[i]} \neq \lambda^{[i+j]}$ for all $0 \leq i \leq d$. Now $k_i = k_{i+j}$ for all $0 \leq i \leq d$ says that

the characters $\lambda^{[i]}$ and $\lambda^{[i+j]}$ of \overline{T} differ at most by some character invariant under conjugation by the full group W . But then we must have $\lambda^{[i]} \circ \tau \neq \lambda^{[i+j]} \circ \tau$ and the claim is proven.

Next, since $\gamma(x) \in \Gamma$ for $x \in \mathbb{F}_p^\times$ acts by multiplication with $\lambda^{[i]}(\tau(x))$ on $k.e_i$ (formula (80)) it follows that the hypotheses of Proposition 6.2 are fulfilled in order to deduce the irreducibility of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ as a $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module.

Statement (c): By Pontryagin duality theory, the natural map $H_0(\overline{\mathfrak{X}}_+, \mathcal{V}) \rightarrow (H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^*)^* = D(\mathcal{V})^*$ is an isomorphism of (topological) k -vector spaces. It is checked that it is also an isomorphism of $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -modules. The same check shows that the irreducibility of the $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ implies the irreducibility of the (ψ, Γ) -module $D(\mathcal{V}) = H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^*$, which in its turn implies the irreducibility of the (φ, Γ) -module $\mathbf{D}(\mathcal{V})$ (cf. e.g. [14] Proposition 3.3.25). The irreducibility of $\mathbf{D}(\mathcal{V})$ implies that of $\text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta$ (Theorem 8.5), and hence the primitivity of h because only primitive h give rise to irreducible $\text{ind}(\omega_{d+1}^h)$. \square

Theorem 8.7. *For any $0 \leq h < (p^{d+1} - 1)/(p - 1)$, any $0 \leq s \leq p - 2$ and any $\beta \in (k^{\text{alg}})^\times$ with $\beta^{d+1} \in k^\times$ there are λ, \mathcal{J} and b as before such that we have an isomorphism of irreducible k -linear $\text{Gal}_{\mathbb{Q}_p}$ -representations $W(\mathbf{D}(\mathcal{V})) \cong \text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta$ for $\mathcal{V} = \Theta_* \mathcal{V}_{M[\lambda, \mathcal{J}, b]}$.*

If h is primitive then (λ, \mathcal{J}) satisfies (73).

PROOF: Write $h(p - 1) = i_0 + pi_1 + \dots + p^d i_d$ with $0 \leq i_j \leq p - 1$, then put $k_j = p - 1 - i_{d+1-j}$ for $1 \leq j \leq d + 1$ and $k_0 = k_{d+1}$. As $p - 1$ divides $h(p - 1)$ and hence $\sum_{j=0}^d i_j$, it also divides $\sum_{j=0}^d k_j$. As \overline{T} is generated by the images of τ and all the h_{s_j} , subject to the relation $\prod_{j=0}^d h_{s_j} = 1$, it therefore follows that there exists a unique character $\lambda : \overline{T} \rightarrow k^\times$ with $\lambda^{[i]}(h_{s_d}(x)) = \lambda(h_{s_{i-1}}(x)) = x^{k_i}$ and with $\lambda(\tau(x)) = x^{-s}$ for all $x \in \mathbb{F}_p^\times$, all $0 \leq i \leq d$. Let $\mathcal{J} = \{s_i \in S \mid k_{i+1} = p - 1\}$. Then $\mathcal{J} \subset S_\lambda$. Let $b = \delta(\lambda, \mathcal{J})\beta^{d+1} \in k^\times$.

That $W(\mathbf{D}(\mathcal{V})) \cong \text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta$ for this triple $(\lambda, \mathcal{J}, b)$ follows from Theorem 8.5. If h is primitive then $\text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta$ is irreducible, hence $\mathbf{D} = \mathbf{D}(\mathcal{V})$ is irreducible. It follows that D^\sharp , the unique (by [4] par. II.4 and II.5) non degenerate surjective (ψ, Γ) -module over $k_{\mathcal{E}}^+$ giving rise to \mathbf{D} , is irreducible. [Given a non-zero (ψ, Γ) -submodule D_1^\sharp of D^\sharp we obtain a non-zero $k_{\mathcal{E}}$ -sub vector space $D_1^\sharp \otimes_{k_{\mathcal{E}}^+} k_{\mathcal{E}}$ of \mathbf{D} stable under $\varphi_{\mathbf{D}}$ and Γ . Thus $D_1^\sharp \otimes_{k_{\mathcal{E}}^+} k_{\mathcal{E}} = \mathbf{D}$ by Proposition II.3.5 of [4], hence $D_1^\sharp = D_1^\sharp \otimes_{k_{\mathcal{E}}^+} k_{\mathcal{E}} \cap D^\sharp = D^\sharp$. See also [14] Proposition 3.3.26.] Thus $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})^*$ and hence $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ are irreducible. The irreducibility of the $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ implies the irreducibility of the $\mathcal{H}(G, I_0)_k$ -module $M[\lambda, \mathcal{J}, b]$. Indeed, a proper $\mathcal{H}(G, I_0)_k$ -sub module of $M[\lambda, \mathcal{J}, b]$ would induce a proper sub coefficient system of \mathcal{V} , and Theorem 4.3 applied to these two coefficient systems would then induce a proper $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -sub module of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$. Hence (λ, \mathcal{J}) satisfies (73). \square

Remark: Property (73) for the pair (λ, \mathcal{J}) in Theorem 8.7, and conversely, the primitivity of h in Theorem 8.5 are proven here very indirectly. It looks quite cumbersome trying to give direct, purely combinatorial proofs.

Remark: The Hecke operator $T_{u^{d+1}}$ for $u^{d+1} \in \{p \cdot \text{id}, p^{-1} \cdot \text{id}\}$ acts on $M[\lambda, \mathcal{J}, b]$ through the scalar b . The determinant of the action of the geometric Frobenius on $W(\mathbf{D}(\Theta_* \mathcal{V}_{M[\lambda, \mathcal{J}, b]}))$ is $\delta(\lambda, \mathcal{J})^{-1}b$. If $d = 1$ then $\delta(\lambda, \mathcal{J}) = 1$ in k^\times so that this determinant is equal to b . If $d > 1$ then $\delta(\lambda, \mathcal{J})$ is not necessarily equal to 1 in k^\times and is not even independent on λ .

Consider the composed functor

$$(84) \quad M \mapsto W(\mathbf{D}(\Theta_* \mathcal{V}_M)).$$

from the category $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0)_k)$ of finite dimensional $\mathcal{H}(G, I_0)_k$ -modules into the category of $\text{Gal}_{\mathbb{Q}_p}$ -representations over k .

Theorem 8.8. *The functor (84) induces a bijection between*

- (a) *the set of isomorphism classes of absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -modules of dimension $d + 1$ and*
- (b) *the set of isomorphism classes of smooth irreducible representations of $\text{Gal}_{\mathbb{Q}_p}$ over k of dimension $d + 1$.*

PROOF: By [2] Corollary 2.1.5, the discussion preceeding [2] Lemma 2.2.1 and the beginning of [2] section 3.1, any smooth irreducible representation of $\text{Gal}_{\mathbb{Q}_p}$ over k of dimension $d + 1$ is of the form $\text{ind}(\omega_{d+1}^h) \otimes \omega^s \mu_\beta$ with $1 \leq h \leq (p^{d+1} - 1)/(p - 1) - 1$. Therefore it is isomorphic with $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ for an absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -module M of dimension $d + 1$, by Theorem 8.7.

By Proposition 8.3 any absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -module M of dimension $d+1$ is of the form $M[\lambda, \mathcal{J}, b]$ with (λ, \mathcal{J}) satisfying (73). It remains to show that the isomorphism class of the simple $\mathcal{H}(G, I_0)_k$ -module $M[\lambda, \mathcal{J}, b]$ can be recovered from the isomorphism class of $\mathbf{D}(\mathcal{V})$ for $\mathcal{V} = \Theta_* \mathcal{V}_{M[\lambda, \mathcal{J}, b]}$.

Step 1: The isomorphism class of $M[\lambda, \mathcal{J}, b]$ can be recovered from the isomorphism class of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ as a $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module.

Recall the w_i , formula (76). Let $\tilde{w} = \min\{w_i \mid 0 \leq i \leq d\}$. For $w \in \mathbb{Z}_{\geq 0}$ let $F_w = \ker(t) \cap \ker(t^{w+1}\varphi^{d+1})$. By formula (79) we may recover \tilde{w} as $\tilde{w} = \min\{w \in \mathbb{Z}_{\geq 0} \mid F_w \neq 0\}$. Let $I = \{i \mid w_i = \tilde{w}\} = \{i \mid F_{w_i} = F_{\tilde{w}}\}$. Let $\overline{T}_0 = \overline{T} \cap \text{SL}_{d+1}(\mathbb{F}_p)$. For $i \in I$ we claim that we can recover $\lambda^{[i]}|_{\overline{T}_0}$ and that this is the same for all $i \in I$. Indeed, \overline{T}_0 is generated by the $u^{-m}h_{s_d}(x)u^m$ for $1 \leq m \leq d + 1$ and $x \in \mathbb{F}_p$, we have $\lambda^{[i]}(u^{-m}h_{s_d}(x)u^m) = \lambda^{[i+m]}(h_{s_d}(x)) = x^{k_{i+m}}$ and k_{i+m} can be read off from \tilde{w} as the coefficient of p^{d+1-m} , by formula (76).

Next, pick some eigenvector $e \in F_{\tilde{w}}$ for the action of Γ on $F_{\tilde{w}}$. By formula (80) there is some $i \in I$ such that the action of Γ on $k.e$ is given by $\lambda^{[i]} \circ \tau$. With the above this allows us to recover $\lambda^{[i]}$ because the domain \overline{T} of $\lambda^{[i]}$ is generated by \overline{T}_0 and the image of τ . As a result we see that we can recover λ up to conjugation by a power of u . Next, up to cyclic permutation we can recover the k_i as the digits in the expansion of \tilde{w} in base p , as already remarked. Since these determine the $\mathcal{J}^{[i]}$ we see that together with what has been said we can recover the pairs $(\lambda^{[i]}, \mathcal{J}^{[i]})$ up to cyclic permutation, or what is the same, we can recover the pair (λ, \mathcal{J}) up to

conjugation by a power of u . Finally, knowing the k_i (up to cyclic permutation) allows us to recover $\prod_{i=0}^d k_i!$. Moreover, knowing λ (up to conjugation by a power of u) allows us to recover $\lambda(-\text{id}) = \prod_{i=0}^d \lambda^{[i]}(\tau(-1))$; namely, this is the product of the eigenvalues of $\tau(-1)$ acting on $\ker(t)$ through the action of Γ (formula (80)). Hence we may recover $\delta(\lambda, \mathcal{J})$. But then we may recover b from formula (79) which holds true with w_i replaced by \check{w} and with e_i replaced by any $e \in F_{\check{w}}$.

We are done because the isomorphism class of the $\mathcal{H}(G, I_0)_k$ -module $M[\lambda, \mathcal{J}, b]$ is given by b and the pair (λ, \mathcal{J}) , the latter taken up to conjugation by powers of u .

Step 2: The isomorphism class of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})$ can be recovered from $\mathbf{D}(\mathcal{V})$.

The (φ, Γ) -module $\mathbf{D}(\mathcal{V})$ is irreducible (cf. e.g. [14] Proposition 3.3.25). Therefore (and because $\mathbf{D}(\mathcal{V})$ is of dimension > 1) $\mathbf{D}(\mathcal{V})$ contains a unique compatible (ψ, Γ) -submodule over $k_{\mathcal{E}}^+$ on which the ψ -operator is surjective ([4] par. II.4 and II.5). By uniqueness, this must be $D(\mathcal{V})$. Thus, the (ψ, Γ) -module $D(\mathcal{V})$ can be recovered from $\mathbf{D}(\mathcal{V})$. Since $H_0(\bar{\mathcal{X}}_+, \mathcal{V}) \rightarrow (H_0(\bar{\mathcal{X}}_+, \mathcal{V})^*)^* = D(\mathcal{V})^*$ is an isomorphism of $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -modules we are done. \square

Remark: A numerical version of Theorem 8.8 was proven in [15] Theorem 5: there it was shown that for fixed b , the number of absolutely simple supersingular $\mathcal{H}(G, I_0)_k$ -modules of the form $M[\lambda, \mathcal{J}, b]$ is the same as the number of smooth irreducible representations of $\text{Gal}_{\mathbb{Q}_p}$ over k of dimension $d+1$ with a fixed determinant of the Frobenius.

Remark: $\mathcal{H}(G, I_0)_k$ -modules of the form $M[\lambda, \mathcal{J}, b]$ with (λ, \mathcal{J}) violating (73) contain supersingular $\mathcal{H}(G, I_0)_k$ -modules M of dimensions l which are proper divisors of $d+1$ (and all supersingular $\mathcal{H}(G, I_0)_k$ -modules arise in this way, by [11]). The associated $\text{Gal}_{\mathbb{Q}_p}$ -representations $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ are also of dimension l , irreducible for simple M . One may either detect them inside $W(\mathbf{D}(\Theta_* \mathcal{V}_{M[\lambda, \mathcal{J}, b]}))$ (using Theorem 8.5 and the exactness of our functor). Alternatively one may observe that M is structured in the same fashion as $M[\lambda, \mathcal{J}, b]$ (but of 'perimeter' l instead of $d+1$) (simplicity for M can be characterized by a non-periodicity property analogous to (73)), hence one can compute $W(\mathbf{D}(\Theta_* \mathcal{V}_M))$ and test its irreducibility exactly along the lines we did with $W(\mathbf{D}(\Theta_* \mathcal{V}_{M[\lambda, \mathcal{J}, b]}))$.

8.2 Filtrations on the Weyl group

Let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ be the length function with respect to the set of Coxeter generators $\{s_1, \dots, s_d\}$ of W . Setting $\bar{u} = s_d \cdots s_1$ we have $s_i = \bar{u}^{d-i} s_d \bar{u}^{i-d}$ for $1 \leq i \leq d$.

Let $W^{s_d} = \{w \in W \mid \ell(ws_d) > \ell(w)\}$. Suppose that we are given a map $\sigma : W^{s_d} \rightarrow \{-1, 0, 1\}$. In the following, for $w \in W$ and $i \in \{-1, -0, 1\}$ we write $\sigma(w) = i$ as a shorthand for $[w \in W^{s_d}$ and $\sigma(w) = i]$.

Definition: (a) For a subset W' of W we define a self mapping

$$(\cdot)_+^{W'} : W \longrightarrow W, \quad w \mapsto w_+^{W'}$$

by the formula

$$(85) \quad w_+^{W'} = \begin{cases} w\bar{u}^{-1} & : \sigma(w\bar{u}^{-1}) = 0 \text{ or } \sigma(w\bar{u}^{-1}s_d) = 0 \\ & : [\sigma(w\bar{u}^{-1}) = -1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = 1] \text{ and } w\bar{u}^{-1}s_d\bar{u} \notin W' \\ w\bar{u}^{-1}s_d & : [\sigma(w\bar{u}^{-1}) = 1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = -1] \text{ and } w\bar{u}^{-1}s_d\bar{u} \in W' \\ & : [\sigma(w\bar{u}^{-1}) = 1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = -1] \text{ and } w\bar{u}^{-1}s_d\bar{u} \notin W' \\ & : [\sigma(w\bar{u}^{-1}) = -1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = 1] \text{ and } w\bar{u}^{-1}s_d\bar{u} \in W'. \end{cases}$$

(b) A σ -admissible filtration of W is a filtration by subsets

$$(86) \quad \emptyset = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_q = W$$

of W (for some $q \in \mathbb{N}$) such that for each $1 \leq i \leq q$ the map $(\cdot)_+^{W_{i-1}}$ respects both $W - W_{i-1}$ and $W_i - W_{i-1}$ and is bijective on the latter (i.e. restricts to a permutation of $W_i - W_{i-1}$).

Proposition 8.9. *σ -admissible filtrations exist for any σ .*

PROOF: Suppose we are given a sequence of subsets

$$\emptyset = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_{i-1} \subset W$$

(some $i \geq 1$) such that for all $j < i$ and all $x \in W_j - W_{j-1}$ there is a $y \in W_j - W_{j-1}$ with $x = y_+^{W_{j-1}}$.

Claim: For any $w \in W - W_{i-1}$ we have $w_+^{W_{i-1}} \in W - W_{i-1}$.

Assume that, on the contrary, $w_+^{W_{i-1}} \in W_{i-1}$, i.e. $w_+^{W_{i-1}} \in W_j - W_{j-1}$ for some $j < i$. Thus $w_+^{W_{i-1}} = v_+^{W_{j-1}}$ for some $v \in W_j - W_{j-1}$. As $w \notin W_{i-1}$ we have $w \neq v$ and this forces that either

$$(87) \quad w\bar{u}^{-1} = w_+^{W_{i-1}} = v_+^{W_{j-1}} = v\bar{u}^{-1}s_d$$

or

$$(88) \quad w\bar{u}^{-1}s_d = w_+^{W_{i-1}} = v_+^{W_{j-1}} = v\bar{u}^{-1}.$$

and moreover that neither $\sigma(w\bar{u}^{-1}) = 0$ nor $\sigma(w\bar{u}^{-1}s_d) = 0$.

Suppose first that $[\sigma(w\bar{u}^{-1}s_d) = -1 \text{ or } \sigma(w\bar{u}^{-1}) = 1]$ and we are in case (87). Then $[\sigma(v\bar{u}^{-1}) = -1 \text{ or } \sigma(v\bar{u}^{-1}s_d) = 1]$ and by the definition of $v_+^{W_{j-1}}$ we get $v\bar{u}^{-1}s_d\bar{u} \in W_{j-1}$, i.e. $w \in W_{j-1}$, contradiction.

Suppose next that $[\sigma(w\bar{u}^{-1}s_d) = 1 \text{ or } \sigma(w\bar{u}^{-1}) = -1]$ and we are in case (88). Then $[\sigma(v\bar{u}^{-1}) = 1 \text{ or } \sigma(v\bar{u}^{-1}s_d) = -1]$ and by the definition of $v_+^{W_{j-1}}$ we get $v\bar{u}^{-1}s_d\bar{u} \in W_{j-1}$, i.e. $w \in W_{j-1}$, contradiction.

Now suppose that $[\sigma(w\bar{u}^{-1}s_d) = 1 \text{ or } \sigma(w\bar{u}^{-1}) = -1]$ and we are in case (87). Then $w\bar{u}^{-1}s_d\bar{u} \notin W_{i-1}$ by the definition of $w_+^{W_{i-1}}$, i.e. $v \notin W_{i-1}$, contradiction.

Finally, suppose that $[\sigma(w\bar{u}^{-1}s_d) = -1 \text{ or } \sigma(w\bar{u}^{-1}) = 1]$ and we are in case (88). Then $w\bar{u}^{-1}s_d\bar{u} \notin W_{i-1}$ by the definition of $w_+^{W_{i-1}}$, i.e. $v \notin W_{i-1}$, contradiction.

The claim is proven. It implies that the map $(\cdot)_+^{W_{i-1}}$ respects $W - W_{i-1}$. Let $W_i - W_{i-1}$ be the maximal subset of $W - W_{i-1}$ on which $(\cdot)_+^{W_{i-1}}$ is bijective; as $W - W_{i-1}$ is a finite set, $W_i - W_{i-1}$ is non-empty. Put $W_i = (W_i - W_{i-1}) \cup W_{i-1}$.

It is clear that the sequence $\emptyset = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_i \subset W$ again satisfies the above condition. Proceeding inductively we thus obtain an admissible filtration of W . \square

Remark: The specific filtration constructed in the proof of Proposition 8.9 may be regarded as the 'socle' filtration.

Examples: (a) If $\sigma = \mathbf{0}$, i.e. if $\sigma(w) = 0$ for any $w \in W^{s_d}$, then $\emptyset = W_0 \subset W_1 = W$ is a σ -admissible filtration: the permutation $(\cdot)_+^\emptyset$ on W is right multiplication by \bar{u}^{-1} .

(b) In Theorem 8.11 below we describe an explicit σ -admissible filtration (in fact: the above 'socle' filtration) for $\sigma = \mathbf{1}$, i.e. when $\sigma(w) = 1$ for any $w \in W^{s_d}$.

(c) Assume $d = 2$ so that $W = \{1, s_1, s_2, s_1s_2, s_2s_1, w_0\}$ with $w_0 = s_1s_2s_1 = s_2s_1s_2$, and $W^{s_d} = W^{s_2} = \{1, s_1, s_2s_1\}$.

(c1) Assume $\sigma(s_2s_1) = \sigma(s_1) = 1$ and $\sigma(1) = 0$. Then $(\cdot)_+^\emptyset$ acts on $W_1 = \{s_1s_2, w_0\}$ as $w_0 \mapsto s_1s_2 \mapsto w_0$ while $(\cdot)_+^{W_1}$ acts on $W - W_1$ as $s_2s_1 \mapsto 1 \mapsto s_1 \mapsto s_2 \mapsto s_2s_1$.

(c2) Assume $\sigma(s_2s_1) = 0$ and $\sigma(s_1) = \sigma(1) = 1$. Then $(\cdot)_+^\emptyset$ acts on $W_1 = \{s_2, s_2s_1, s_1s_2, w_0\}$ as $w_0 \mapsto s_1s_2 \mapsto s_2s_1 \mapsto s_2 \mapsto w_0$ while $(\cdot)_+^{W_1}$ acts on $W - W_1$ as $s_1 \mapsto 1 \mapsto s_1$.

We assume that $d > 1$. Let $\langle s_1, \dots, s_{d-1} \rangle$ denote the subgroup of W generated by the elements s_1, \dots, s_{d-1} , and similarly define $\mathcal{R} = \langle s_1, \dots, s_{d-2} \rangle$. Put $W_0 = \emptyset$ and $W_{d+1} = W$, and for $1 \leq i \leq d$ define the union of cosets

$$W_i = \bigcup_{1 \leq j \leq i} s_j \cdots s_d \langle s_1, \dots, s_{d-1} \rangle.$$

Lemma 8.10. (a) For any $w \in W_i \cap W^{s_d}$ we have $w\bar{u} \in W_{i-1}$.

(b) For any $w \in (W - W_{i-1}) \cap W^{s_d}s_d$ we have $w\bar{u} \notin W_i$.

(c) For any $w \in W_i$ we have $|\{0 \leq j \leq d-1; w(\bar{u}^{-1}s_d)^j \in W^{s_d}\}| = i-1$.

(d) The set \mathcal{R} is a set of representatives in $\langle s_1, \dots, s_{d-1} \rangle$ for the right cosets of the cyclic subgroup $(\bar{u}^{-1}s_d)^{\mathbb{Z}} = \{(\bar{u}^{-1}s_d)^j \mid j = 0, \dots, d-1\}$ of $\langle s_1, \dots, s_{d-1} \rangle$, i.e. $\langle s_1, \dots, s_{d-1} \rangle = \coprod_{v \in \mathcal{R}} v(\bar{u}^{-1}s_d)^{\mathbb{Z}}$.

PROOF: Identify W with the group of permutations of the set $\{0, \dots, d\}$, in such a way that $s_j = (j-1, j)$ (transposition) for $1 \leq j \leq d$. Then $W_i = \{w \in W \mid w(d) \leq i\}$ and $W^{s_d} = \{w \in W \mid w(d-1) < w(d)\}$, moreover $\bar{u}(j) = j-1$ for $1 \leq j \leq d$ and $\bar{u}(0) = d$. Statements (a), (b) and (c) are now easily read off. Statement (d) is clear. \square

For $0 \leq j \leq d-1$, for $1 \leq i \leq d+1$ and for $v \in \mathcal{R}$ let

$$w_j(i, v) = s_i \cdots s_d v(\bar{u}^{-1}s_d)^j \in W$$

and let $w_d(i, v) = w_0(i, v)$. Let $W_{i,v} = \{w_0(i, v), \dots, w_{d-1}(i, v)\}$.

Theorem 8.11. *If $\sigma = \mathbf{1}$ then the filtration $\emptyset = W_0 \subset \dots \subset W_{d+1} = W$ defined above is σ -admissible. More precisely, for any $1 \leq i \leq d+1$ we have*

$$(89) \quad W_i - W_{i-1} = \coprod_{v \in \mathcal{R}} W_{i,v}$$

and $w_j(i, v)_+^{W_{i-1}} = w_{j+1}(i, v)$ for all $0 \leq j \leq d-1$. We have

$$(90) \quad |\{0 \leq j \leq d-1; w_j(i, v) \in W^{s_d}\}| = i-1.$$

PROOF: Lemma 8.10 (d) implies the decomposition (89). To see that $w_j(i, v)_+^{W_{i-1}} = w_{j+1}(i, v)$ assume first that we are in the case where $w_j(i, v)\bar{u}^{-1}s_d \in W^{s_d}$. Then, as $w_j(i, v)\bar{u}^{-1}s_d \in W_i$, Lemma 8.10 (a) tells us $w_j(i, v)\bar{u}^{-1}s_d\bar{u} \in W_{i-1}$, and the hypothesis $\sigma = \mathbf{1}$ therefore gives $w_j(i, v)_+^{W_{i-1}} = w_{j+1}(i, v)$. If on the other hand $w_j(i, v)\bar{u}^{-1} \in W^{s_d}$ then, as $w_j(i, v)\bar{u}^{-1}s_d \notin W_{i-1}$, Lemma 8.10 (b) tells us $w_j(i, v)\bar{u}^{-1}s_d\bar{u} \notin W_{i-1}$, and again the hypothesis $\sigma = \mathbf{1}$ gives $w_j(i, v)_+^{W_{i-1}} = w_{j+1}(i, v)$. Finally, formula (90) follows from Lemma 8.10 (c). \square

8.3 Reduced standard $\mathcal{H}(G, I_0)_k$ -modules

The justification for defining reduced standard modules over $\mathcal{H}(G, I_0)_k$ as we do it below is given in [9].

$\bar{u} = s_d \cdots s_1$ is the image of $u \in N(T)$ in $W = N(T)/T$. Our specific choices in (71) resp. (72) allow us to regard W as the subgroup of $N(T)$ generated by s_1, \dots, s_d .

Definition: We say that an $\mathcal{H}(G, I_0)_k$ -module M is a *reduced standard module* (or: is of *of W -type*) if it is of the following form $M = M(\theta, \sigma, \epsilon_\bullet)$. First, a k -vector space basis of M is the set of formal symbols g_w for $w \in W$. The $\mathcal{H}(G, I_0)_k$ -action on M is characterized by a character $\theta : \bar{T} \rightarrow k^\times$ (which we also read as a character of $T \cap I$ by inflation), a map $\sigma : W^{s_d} \rightarrow \{-1, 0, 1\}$ and a set $\epsilon_\bullet = \{\epsilon_w\}_{w \in W}$ of units $\epsilon_w \in k^\times$. Namely, for $w \in W$ we define $\kappa_w = \kappa_w(\theta) = \theta(w n_{s_d} s_d w^{-1}) \in \{\pm 1\}$. Then the following formulae are required for $t \in \bar{T}$ and $w \in W$:

$$\begin{aligned} T_t(g_w) &= \theta(wt^{-1}w^{-1})g_w, \\ T_{u^{-1}}(g_w) &= \epsilon_w g_{w\bar{u}^{-1}}, \\ T_{s_d}(g_w) &= \begin{cases} g_{ws_d} & : \quad [\sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}w^{-1}) \neq \mathbf{1}] \text{ or } \sigma(w) = 1 \\ -\kappa_w g_w & : \quad \sigma(ws_d) \in \{0, 1\} \text{ and } \theta(wh_{s_d}w^{-1}) = \mathbf{1} \\ g_{ws_d} - \kappa_w g_w & : \quad \sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}w^{-1}) = \mathbf{1} \\ 0 & : \quad \text{all other cases} \end{cases} \end{aligned}$$

Here the conditions involving $\theta(wh_{s_d}w^{-1}) = \theta(wh_{s_d}(\cdot)w^{-1})$ and $\mathbf{1}$ compare the homomorphism $\mathbb{F}_p^\times \rightarrow k^\times$, $x \mapsto \theta(wh_{s_d}(x)w^{-1})$ with the constant homomorphism $x \mapsto \mathbf{1}(x) = 1$.

Fix a reduced standard $\mathcal{H}(G, I_0)_k$ -module $M = M(\theta, \sigma, \epsilon_\bullet)$ and let $\mathcal{V} = \Theta_* \mathcal{V}_{M(\theta, \sigma, \epsilon_\bullet)}$. Choose a σ -admissible filtration $\emptyset = W_0 \subset W_1 \subset \dots \subset W_q = W$. For $1 \leq i \leq q$ decompose $W_i - W_{i-1}$ into its $(\cdot)_+^{W_{i-1}}$ -orbits, i.e. write

$$W_i - W_{i-1} = \coprod_{v \in \mathcal{R}_i} \{w_0(i, v), w_1(i, v), \dots, w_{t_v-1}(i, v)\}$$

where t_v is the cardinality of the $(\cdot)_+^{W_{i-1}}$ -orbit with index $v \in \mathcal{R}_i$, such that $w_j(i, v)_+^{W_{i-1}} = w_{j+1}(i, v)$ for all $0 \leq j \leq t_v - 1$; here we write $w_{t_v}(i, v) = w_0(i, v)$.

For $v \in \mathcal{R}_i$ let $N_{i,v}$ be the $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -submodule of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ generated by the elements $g_w \in M \subset H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ for all $w \in W_{i-1} \cup \{w_0(i, v), w_1(i, v), \dots, w_{t_v-1}(i, v)\}$. Let $N_i = \sum_{v \in \mathcal{R}_i} N_{i,v}$, i.e. the $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -submodule of $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ generated by the elements $g_w \in M \subset H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ for all $w \in W_i$.

Theorem 8.12. *Assume that there is no $v \in W^{s_d}$ with $\sigma(v) = -1$ and $\theta(vh_{s_d}v^{-1}) = \mathbf{1}$. Then*

$$\emptyset = N_0 \subset N_1 \subset \dots \subset N_q = H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$$

is a filtration by $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -submodules, its subquotients admit direct sum decompositions

$$N_i/N_{i-1} = \bigoplus_{v \in \mathcal{R}_i} N_{i,v}/N_{i-1}$$

with $N_{i,v}/N_{i-1}$ standard cyclic of perimeter t_v .

The $\text{Gal}_{\mathbb{Q}_p}$ -representation $W(\mathbf{D}(\mathcal{V}))$ admits a filtration $W(\mathbf{D}(\mathcal{V})) = V_0 \supset V_1 \supset \dots \supset V_q = 0$ and direct sum decompositions $V_{i-1}/V_i = \bigoplus_{v \in \mathcal{R}_i} \overline{V}_{i-1,v}$ such that $\overline{V}_{i-1,v}$ is the $\text{Gal}_{\mathbb{Q}_p}$ -representation corresponding to the $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module $N_{i,v}/N_{i-1}$.

The proof of Theorem 8.12 is based on the following Lemma 8.13.

Let $w \in W$. If $\theta(wh_{s_d}^{-1}w^{-1}) = \mathbf{1}$ we put $\lfloor k(w) \rfloor = 0$ and $\lceil k(w) \rceil = p - 1$ (but do not define $k(w)$). If $\theta(wh_{s_d}^{-1}w^{-1}) \neq \mathbf{1}$ we define $1 \leq k(w) \leq p - 2$ by $\theta(wh_{s_d}^{-1}(x)w^{-1}) = x^{-k(w)}$ for all $x \in \mathbb{F}_p^\times$, and we put $\lfloor k(w) \rfloor = \lceil k(w) \rceil = k(w)$.

Lemma 8.13. *Let $v \in W^{s_d}$. In $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ we have the following identities.*

If $\sigma(v) = 1$ then

$$(91) \quad t^{p-1}n_{s_d}^{-1}g_v = \kappa_v g_{vs_d}$$

$$(92) \quad t^{\lceil k(vs_d) \rceil} n_{s_d}^{-1} g_{vs_d} = \lceil k(vs_d) \rceil! g_{vs_d}$$

$$(93) \quad n_{s_d}^{-1} \kappa_v g_{vs_d} - g_v \in \sum_{i \geq 0} k \cdot t^i s_d g_v$$

$$(94) \quad t^{p-1-\lceil k(vs_d) \rceil} n_{s_d}^{-1} g_v - (p-1-\lceil k(vs_d) \rceil)! g_v \in \sum_{i \geq 0} k \cdot t^i s_d g_{vs_d}.$$

If $\sigma(v) = 0$ then

$$(95) \quad t^{\lfloor k(v) \rfloor} n_{s_d}^{-1} g_v = \lfloor k(v) \rfloor! g_v \quad \text{and} \quad t^{\lceil k(vs_d) \rceil} n_{s_d}^{-1} g_{vs_d} = \lceil k(vs_d) \rceil! g_{vs_d}.$$

If $\sigma(v) = -1$ and $\theta(vh_{s_d}v^{-1}) \neq \mathbf{1}$ then

$$(96) \quad t^{p-1}n_{s_d}^{-1}g_{vs_d} = \kappa_{vs_d}g_v$$

$$(97) \quad t^{\lceil k(v) \rceil}n_{s_d}^{-1}g_v = \lceil k(v) \rceil!g_v$$

$$(98) \quad n_{s_d}^{-1}\kappa_{vs_d}g_v - g_{vs_d} \in \sum_{i \geq 0} k.t^i s_d g_{vs_d}$$

$$(99) \quad t^{p-1-\lceil k(v) \rceil}n_{s_d}^{-1}g_{vs_d} - (p-1-\lceil k(v) \rceil)!g_{vs_d} \in \sum_{i \geq 0} k.t^i s_d g_v.$$

PROOF: We view M as an $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -module by means of the embedding (70). As in the proof of Proposition 8.4 we use the corresponding embedding

$$(100) \quad (\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} M \cong \mathcal{V}(\mathfrak{v}_0) \hookrightarrow H_0(\overline{\mathfrak{X}}_+, \mathcal{V}).$$

Regarded as an $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -module, M is the direct sum, indexed by all $v \in W^{s_d}$, of the two dimensional $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -modules $\langle g_v, g_{vs_d} \rangle$. If $\sigma(v) = 0$ this summand splits up further as

$$\chi_{\lfloor k(v) \rfloor} \bigoplus \chi_{\lceil k(vs_d) \rceil} \cong \langle g_v, g_{vs_d} \rangle$$

with the $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -modules $\chi_{\lfloor k(v) \rfloor}$, $\chi_{\lceil k(vs_d) \rceil}$ considered in Lemma 2.5. Namely, send $e \in \chi_{\lfloor k(v) \rfloor}$ to g_v and send $e \in \chi_{\lceil k(vs_d) \rceil}$ to g_{vs_d} . Thus, if $\sigma(v) = 0$ we conclude with Lemma 2.5. Next, if $\sigma(v) = 1$ we have an isomorphism of $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -modules

$$M_{\lceil k(vs_d) \rceil} \cong \langle g_v, g_{vs_d} \rangle$$

sending $e \in M_{\lceil k(vs_d) \rceil}$ to g_v and sending $f \in M_{\lceil k(vs_d) \rceil}$ to $T_{n_{s_d}}g_v = T_{s_d}T_{n_{s_d}s_d}g_v = \kappa_v g_{vs_d}$. Thus, in this case we conclude with Lemma 2.6. Finally, if $\sigma(v) = -1$ we have an isomorphism of $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -modules

$$M_{\lfloor k(v) \rfloor} \cong \langle g_v, g_{vs_d} \rangle$$

sending $e \in M_{\lfloor k(v) \rfloor}$ to g_{vs_d} and sending $f \in M_{\lfloor k(v) \rfloor}$ to $T_{n_{s_d}}g_{vs_d} = T_{s_d}T_{n_{s_d}s_d}g_{vs_d}$. The latter equals $\kappa_{vs_d}g_v$ as we assume $\theta(vh_{s_d}v^{-1}) \neq \mathbf{1}$ (it equals $\kappa_{vs_d}g_v - g_{vs_d}$ if $\theta(vh_{s_d}v^{-1}) = \mathbf{1}$). Thus, also in this case we conclude with Lemma 2.6. \square

PROOF OF THEOREM 8.12: As in Proposition 8.4 we see that $\ker(t|_{H_0(\overline{\mathfrak{X}}_+, \mathcal{V})})$ generates $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ as a $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module and that $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ is a torsion $k_{\mathcal{E}}^+$ -module.

Step 1: We claim that for $1 \leq i \leq q$, for $v \in \mathcal{R}_i$ and for $0 \leq j \leq t_v - 1$ there are $n = n(i, v, j) \in \mathbb{Z}_{\geq 0}$ and $\varrho = \varrho(i, v, j) \in k^\times$ with

$$(101) \quad t^n \varphi g_{w_{j-1}(i, v)} = \varrho g_{w_j(i, v)} \text{ modulo } N_{i-1}.$$

To prove this we begin by computing

$$\varphi g_{w_{j-1}(i, v)} = s_d u g_{w_{j-1}(i, v)} = s_d T_{u^{-1}}(g_{w_{j-1}(i, v)}) = \epsilon_{w_j(i, v)} s_d g_{w_{j-1}(i, v)} \overline{u}^{-1}.$$

Now $s_d g_{w_{j-1}(i,v)\bar{u}^{-1}}$ is related to $n_{s_d}^{-1} g_{w_{j-1}(i,v)\bar{u}^{-1}}$ by a factor ± 1 , and we have $w_j(i,v) = w_{j-1}(i,v)_+^{W_{i-1}}$. Thus we need to show that for $w = w_{j-1}(i,v)$ there are $n \in \mathbb{Z}_{\geq 0}$ and $\varrho' \in k^\times$ with

$$t^n n_{s_d}^{-1} g_{w\bar{u}^{-1}} = \varrho' g_{w_+^{W_{i-1}}} \quad \text{modulo } N_{i-1}.$$

To do this, we turn back to the definition of $w_+^{W_{i-1}}$:

In the cases where $\sigma(w\bar{u}^{-1}) = 0$ or $\sigma(w\bar{u}^{-1}s_d) = 0$ this follows from formula (95).

In the cases where $[\sigma(w\bar{u}^{-1}) = -1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = 1]$ and $w\bar{u}^{-1}s_d\bar{u} \notin W_{i-1}$ this follows from formula (92) resp. (97).

In the cases where $[\sigma(w\bar{u}^{-1}) = 1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = -1]$ and $w\bar{u}^{-1}s_d\bar{u} \in W_{i-1}$ this follows from formula (94) resp. (99) since $\sum_{i \geq 0} k.t^i s_d g_{w\bar{u}^{-1}s_d} = \sum_{i \geq 0} k.t^i \varphi g_{w\bar{u}^{-1}s_d\bar{u}}$ is contained in N_{i-1} .

In the cases where $[\sigma(w\bar{u}^{-1}) = 1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = -1]$ and $w\bar{u}^{-1}s_d\bar{u} \notin W_{i-1}$ this follows from formula (91) resp. (96).

In the cases where $[\sigma(w\bar{u}^{-1}) = -1 \text{ or } \sigma(w\bar{u}^{-1}s_d) = 1]$ and $w\bar{u}^{-1}s_d\bar{u} \in W_{i-1}$ this follows from formula (93) resp. (98) since $\sum_{i \geq 0} k.t^i s_d g_{w\bar{u}^{-1}s_d} = \sum_{i \geq 0} k.t^i \varphi g_{w\bar{u}^{-1}s_d\bar{u}}$ is contained in N_{i-1} .

Step 2: We claim that for any $i \geq 1$ the subquotient N_i/N_{i-1} is a direct sum, indexed by $v \in \mathcal{R}_i$, of standard cyclic $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -modules $N_{i,v}/N_{i-1}$ of perimeter t_v , and that N_i is a direct summand of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})$ as a $k_{\mathcal{E}}^+$ -module. We proceed by induction on i .

By induction hypothesis, N_{i-1} is a direct summand of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})$ as a $k_{\mathcal{E}}^+$ -module. Therefore, and since $M = \ker(t|_{H_0(\bar{\mathcal{X}}_+, \mathcal{V})})$ by Theorem 4.3, the classes of the $g_{w_j(i,v)}$ (for $v \in \mathcal{R}_i$ and $0 \leq j \leq t_v - 1$) form a k -basis of $\ker(t|_{N_i/N_{i-1}})$. Together with step 1 we deduce our claim concerning N_i/N_{i-1} and the $N_{i,v}/N_{i-1}$. Next, if $N_{i,v}/N_{i-1}$ is t -divisible then with N_{i-1} also $N_{i,v}$ is a direct summand of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})$ as a $k_{\mathcal{E}}^+$ -module. If however $N_{i,v}/N_{i-1}$ is not t -divisible then it must be entirely contained in $\ker(t|_{H_0(\bar{\mathcal{X}}_+, \mathcal{V})/N_{i-1}})$, cf. Proposition 6.2. This means that the numbers $n(i,v,j)$ appearing in step 1 are zero for all $0 \leq j \leq t_v - 1$. In view of the formulae in Lemma 8.13, the case by case distinction in step 1 in the passage from $w_j(i,v)$ to $w_{j+1}(i,v) = w_j(i,v)_+^{W_{i-1}}$ then reveals that for any $0 \leq j \leq t_v - 1$ we have $w_j(i,v)\bar{u}^{-1}s_d\bar{u} \in W_{i-1}$ or $\sigma(w_j(i,v)\bar{u}^{-1}) = 0$ or $\sigma(w_j(i,v)\bar{u}^{-1}s_d) = 0$. It follows that for all $w \in W - W_i$ and all $l \geq i$ we have $w_+^{W_l} = w_+^{W_l - \{w_0(i,v), \dots, w_{t_v-1}(i,v)\}}$. Therefore, and as $N_{i,v}/N_{i-1} \subset \ker(t|_{H_0(\bar{\mathcal{X}}_+, \mathcal{V})/N_{i-1}})$, step 1 shows that the $k_{\mathcal{E}}^+[\varphi]$ -submodule of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})/N_{i-1}$ generated by the g_w for $w \in W - \{w_0(i,v), \dots, w_{t_v-1}(i,v)\}$ is a $k_{\mathcal{E}}^+[\varphi]$ -module complement of $N_{i,v}/N_{i-1}$. In particular, $N_{i,v}/N_{i-1}$ is a direct summand of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})/N_{i-1}$ as a $k_{\mathcal{E}}^+$ -module, hence $N_{i,v}$ is a direct summand of $H_0(\bar{\mathcal{X}}_+, \mathcal{V})$. \square

Remark: We leave to the reader the necessary modifications of the proof of Theorem 8.12 in the general case, i.e. where we allow $v \in W^{s_d}$ with $\sigma(v) = -1$ and $\theta(vh_{s_d}v^{-1}) = \mathbf{1}$. Following Lemma 2.6 one first must refine the formulae (96), (97), (98), (99): If $\sigma(v) = -1$ let us put $\mathbf{F}_v = 1$ if $\theta(vh_{s_d}v^{-1}) = \mathbf{1}$ and $\mathbf{F}_v = 0$ if $\theta(vh_{s_d}v^{-1}) \neq \mathbf{1}$. Then formulae (96), (97), (98), (99) remain true for general $v \in W^{s_d}$ with $\sigma(v) = -1$ if each occurrence of $\kappa_{vs_d}g_v$ is replaced by $\kappa_{vs_d}g_v - \mathbf{F}_v g_{vs_d}$. In addition, by formulae (9), (10), if $\sigma(v) = -1$ and $\theta(vh_{s_d}v^{-1}) = \mathbf{1}$ we have

the formulae $n_{s_d}^{-1}g_v = g_v$ and $t^{p-1}n_{s_d}^{-1}g_{vs_d} + g_{vs_d} \in \sum_{i \geq 0} k.t^i s_d g_v$.

Remark: Apparently, the above proof of Theorem 8.12 allows us to precisely compute the parameters $n(i, v, j) \in \mathbb{Z}_{\geq 0}$ and $\varrho(i, v, j) \in k^\times$, hence the standard cyclic $k_\mathcal{E}^+[\varphi, \Gamma]$ -modules $N_{i,v}/N_{i-1}$, hence their associated $\text{Gal}_{\mathbb{Q}_p}$ -representations. In the following Theorem 8.14 we do this in the modular principal series case $\sigma = \mathbf{1}$.

Let $\mathcal{R} \subset W$ and $w_j(i, v) \in W$ be as in Theorem 8.11.

Theorem 8.14. *Suppose that $\sigma = \mathbf{1}$. Then $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ admits a $k_\mathcal{E}^+[\varphi, \Gamma]$ -module filtration $0 = N_0 \subset N_1 \subset \dots \subset N_{d+1} = H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ together with direct sum decompositions*

$$(102) \quad N_i/N_{i-1} = \bigoplus_{v \in \mathcal{R}} N_{i,v}/N_{i-1}$$

such that $N_{i,v}/N_{i-1}$ for $1 \leq i \leq d+1$ and $v \in \mathcal{R}$ is a standard cyclic $k_\mathcal{E}^+[\varphi, \Gamma]$ -module of perimeter d . More precisely we have: For fixed i and $v \in \mathcal{R}$ put $w_j = w_j(i, v)$ for $0 \leq j \leq d$; then there is a k -basis e_0, \dots, e_{d-1} of $\ker(t|_{N_{i,v}/N_{i-1}})$ such that, setting $e_d = e_0$, for $0 \leq j \leq d-1$ we have

$$(103) \quad t^{p-1}\varphi e_j = \epsilon_{w_j} e_{j+1} \quad \text{if } w_{j+1} \notin W^{s_d},$$

$$(104) \quad \varphi e_j = \epsilon_{w_j} e_{j+1} \quad \text{if } w_{j+1} \in W^{s_d},$$

$$(105) \quad \gamma(x)e_j = \theta(w_j \tau(x)w_j^{-1})e_j \quad \text{for all } x \in \mathbb{F}_p^\times.$$

The distribution between the cases (103) and (104) is given by formula (90). In particular, if $1 \leq i \leq d$ then the $\text{Gal}_{\mathbb{Q}_p}$ -representation associated with $N_{i,v}/N_{i-1}$ has dimension d , whereas if $i = d+1$ it has dimension 0.

PROOF: This follows from Theorem 8.11 and Theorem 8.12. To see formula (105) recall that $\gamma(x)$ acts as $\tau(x)$, and that $\tau(x)$ acts as the Hecke operator $T_{\tau(x)^{-1}}$. For the formulae (103) and (104) we must inspect the above proof of formula (101); namely, it is enough to prove that for all $w \in W$ we have

$$(106) \quad t^{p-1}\varphi g_w = \epsilon_w g_{w\bar{u}^{-1}s_d} \quad \text{if } w\bar{u}^{-1} \in W^{s_d},$$

$$(107) \quad \varphi g_w - \epsilon_w g_{w\bar{u}^{-1}s_d} \in \sum_{n \geq 0} k.t^n \varphi g_{w\bar{u}^{-1}s_d \bar{u}} \quad \text{if } w\bar{u}^{-1} \notin W^{s_d}$$

in $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$. As before we begin with

$$\varphi g_w = s_d u g_w = s_d T_{u^{-1}}(g_w) = \epsilon_w s_d g_{w\bar{u}^{-1}}.$$

Suppose first that $w\bar{u}^{-1} \in W^{s_d}$. Define $1 \leq r \leq p-1$ by the condition

$$\theta(w\bar{u}^{-1}h_{s_d}^{-1}(x)\bar{u}w^{-1}) = x^r$$

for all $x \in \mathbb{F}_p^\times$. Then, since $\sigma(w\bar{u}^{-1}) = 1$, we find that, as an $\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k$ -module, $\langle g_{w\bar{u}^{-1}s_d}, g_{w\bar{u}^{-1}} \rangle$ is isomorphic with M_r as considered in Lemma 2.6 — take $e = g_{w\bar{u}^{-1}}$ there. We compute

$$\begin{aligned} t^{p-1}s_d g_{w\bar{u}^{-1}} &= t^{p-1}s_d n_{s_d} n_{s_d}^{-1} g_{w\bar{u}^{-1}} \\ &\stackrel{(i)}{=} s_d n_{s_d} t^{p-1} n_{s_d}^{-1} g_{w\bar{u}^{-1}} \\ &\stackrel{(ii)}{=} s_d n_{s_d} T_{n_{s_d}} g_{w\bar{u}^{-1}} \\ &= s_d n_{s_d} T_{s_d n_{s_d}} T_{s_d} g_{w\bar{u}^{-1}} \\ &= g_{w\bar{u}^{-1}s_d} \end{aligned}$$

where (i) uses $t^{p-1}s_d n_{s_d} = s_d n_{s_d} t^{p-1}$ while (ii) uses formula (5).

Now suppose that $w\bar{u}^{-1} \notin W^{s_d}$. Define $1 \leq r \leq p-1$ by

$$\theta(w\bar{u}^{-1}s_d h_{s_d}^{-1}(x) s_d \bar{u} w^{-1}) = x^r$$

for all $x \in \mathbb{F}_p^\times$. Then, since $\sigma(w\bar{u}^{-1}s_d) = 1$, we find that $\langle g_{w\bar{u}^{-1}s_d}, g_{w\bar{u}^{-1}} \rangle$ is isomorphic with M_r as considered in Lemma 2.6 — this time take $e = g_{w\bar{u}^{-1}s_d}$ there. We compute

$$\begin{aligned} s_d g_{w\bar{u}} &= s_d T_{s_d}(g_{w\bar{u}^{-1}s_d}) \\ &= s_d T_{n_{s_d}^{-1}s_d} T_{n_{s_d}}(g_{w\bar{u}^{-1}s_d}) \\ &= n_{s_d} T_{n_{s_d}}(g_{w\bar{u}^{-1}s_d}) \\ &\equiv g_{w\bar{u}^{-1}s_d} \end{aligned}$$

where the last congruence uses formula (7) and is to be understood as an identity modulo

$$\sum_{n \geq 0} k.t^n n_{s_d} g_{w\bar{u}^{-1}s_d} = \sum_{n \geq 0} k.t^n s_d g_{w\bar{u}^{-1}s_d} = \sum_{n \geq 0} k.t^n \varphi g_{w\bar{u}^{-1}s_d \bar{u}}.$$

□

Further examples: (a) The easiest case is the generic case $\sigma = \mathbf{0}$, i.e. $\sigma(w) = 0$ for all $w \in W^{s_d}$. ('Most' reduced standard $\mathcal{H}(G, I_0)_k$ -modules arising by reduction from a locally unitary principal series representation are of this sort, cf. [9].) To describe it, fix a set of representatives \mathcal{Q} in W for the right cosets of the cyclic group $\bar{u}^{\mathbb{Z}} = \{\bar{u}^j \mid j = 0, \dots, d\}$, i.e. such that $W = \coprod_{v \in \mathcal{Q}} v\bar{u}^{\mathbb{Z}}$. For example, one may take $\mathcal{Q} = \langle s_1, \dots, s_{d-1} \rangle$. Then $H_0(\bar{\mathfrak{X}}_+, \mathcal{V})$ admits a direct sum decomposition

$$H_0(\bar{\mathfrak{X}}_+, \mathcal{V}) = \oplus_{v \in \mathcal{Q}} N_{1,v}$$

such that $N_{1,v}$ for each $v \in \mathcal{Q}$ is standard cyclic of perimeter $d+1$. More precisely, it can be described as follows. The set with $d+1$ elements $\{g_v, g_{v\bar{u}}, \dots, g_{v\bar{u}^d}\}$ is a k -basis of $\ker(t|_{N_{1,v}})$. It generates $N_{1,v}$ as a $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module. In $H_0(\bar{\mathfrak{X}}_+, \mathcal{V})$ we have the identities (the proof proceeds as in Theorem 8.14, but is easier)

$$\begin{aligned} t^{\lceil k(v\bar{u}^{j-1}) \rceil} \varphi g_{v\bar{u}^j} &= \epsilon_{v\bar{u}^j} \kappa_{v\bar{u}^{j-1}} \lceil k(v\bar{u}^{j-1}) \rceil! g_{v\bar{u}^{j-1}} && \text{if } v\bar{u}^{j-1} \notin W^{s_d}, \\ t^{\lfloor k(v\bar{u}^{j-1}) \rfloor} \varphi g_{v\bar{u}^j} &= \epsilon_{v\bar{u}^j} \kappa_{v\bar{u}^{j-1}} \lfloor k(v\bar{u}^{j-1}) \rfloor! g_{v\bar{u}^{j-1}} && \text{if } v\bar{u}^{j-1} \in W^{s_d}, \\ \gamma(a) g_{v\bar{u}^j} &= \theta(v\bar{u}^j \tau(a) \bar{u}^{-j} v^{-1}) g_{v\bar{u}^j} && \text{for all } a \in \mathbb{Z}_p. \end{aligned}$$

(b) Assume $d = 2$. Theorem 8.12 and our examples in section 8.2 show that, for suitable choices of σ , the $k_{\mathcal{E}}^+[\varphi, \Gamma]$ -module $H_0(\overline{\mathfrak{X}}_+, \mathcal{V})$ admits a two-step filtration with standard cyclic subquotients of perimeters 2 and 4 (resp. 4 and 2). At least for generic θ , the corresponding $\mathrm{Gal}_{\mathbb{Q}_p}$ -representation admits a two-step filtration with irreducible subquotients of dimensions 2 and 4 (resp. 4 and 2).

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