From pro-p Iwahori-Hecke modules to (φ,Γ)-modules II

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Abstract

Let $\mathfrak{o}$ be the ring of integers in a finite extension field of $\mathbb{Q}_p$, let k be its residue field. Let G be a split reductive group over $\mathbb{Q}_p$, let $\mathcal{H}(G,I_0)$ be its pro-p-Iwahori Hecke $\mathfrak{o}$-algebra. In [3] we introduced a general principle how to assign to a certain additionally chosen datum $(C^{(\bullet)},\phi,\tau)$ an exact functor $M\mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ from finite length $\mathcal{H}(G,I_0)$-modules to (φ^r,Γ)-modules. In the present paper we concretely work out such data $(C^{(\bullet)},\phi,\tau)$ for the classical matrix groups. We show that the corresponding functor identifies the set of (quasi) supersingular $\mathcal{H}(G,I_0)\otimes_{\mathfrak{o}}k$-modules with the set of (φ^r,Γ)-modules satisfying a certain symmetry condition.

Contents

1 Introduction

Let $\mathfrak{o}$ be the ring of integers in a finite extension field of $\mathbb{Q}_p$, let k be its residue field. Let G be a split reductive group over $\mathbb{Q}_p$, let T be a maximal split torus in G, let

I_0 be a pro-p-Iwahori subgroup fixing a chamber C in the T-stable apartment of the semi simple Bruhat Tits building of G. Let $\mathcal{H}(G, I_0)$ be the pro-p-Iwahori Hecke $\mathfrak{o}$-algebra. Let $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ denote the category of $\mathcal{H}(G, I_0)$-modules of finite $\mathfrak{o}$-length. From a certain additional datum $(C^{(\bullet)}, \phi, \tau)$ we constructed in [3] an exact functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of étale (φ^r, Γ)-modules (with $r \in \mathbb{N}$ depending on ϕ). For $G = \mathrm{GL}_2(\mathbb{Q}_p)$, when precomposed with the functor of taking I_0-invariants, this yields the functor from smooth $\mathfrak{o}$-torsion representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (or at least from those generated by their I_0-invariants) to étale (φ, Γ)-modules which plays a crucial role in Colmez' construction of a p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In [3] we studied in detail the functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ when $G = \mathrm{GL}_{d+1}(\mathbb{Q}_p)$ for $d \geq 1$. The purpose of the present paper is to explain how the general construction of [3] can be installed concretely for other G's.

Recall that $C^{(\bullet)} = (C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots)$ is a minimal gallery, starting at C, in the T-stable apartment, that $\phi \in N(T)$ is 'period' of $C^{(\bullet)}$ and that τ is a homomorphism from $\mathbb{Z}_p^\times$ to T, compatible with ϕ in a suitable sense. The above $r \in \mathbb{N}$ is just the length of ϕ. It turns out that τ must be a co minuscule fundamental coweight (at least if the underlying root system is simple). Conversely, any co minuscule fundamental coweight τ can be included into a datum $(C^{(\bullet)}, \phi, \tau)$, in such a way that some power of τ is a power of ϕ.

While for $G = \mathrm{GL}_{d+1}(\mathbb{Q}_p)$ we gave explicit choices of $(C^{(\bullet)}, \phi, \tau)$ with $r = 1$ in [3] (in fact there are essentially just two such choices), we did not discuss the existence of $(C^{(\bullet)}, \phi, \tau)$ for other G's. This discussion is the main contribution of the present paper. More specifically, we work out 'priviledged' choices $(C^{(\bullet)}, \phi, \tau)$ for the classical matrix groups (of type B, C, D, and also A again), as well as for G of type E_6, E_7. We mostly consider G with connected center Z. Our choices of $(C^{(\bullet)}, \phi, \tau)$ are such that $\phi \in N(T)$ projects modulo ZT_0 (where T_0 denotes the maximal bounded subgroup of T) to the affine Weyl group (viewed as a subgroup of $N(T)/ZT_0$). In particular, up to modifications by elements of Z these ϕ can also be included into data $(C^{(\bullet)}, \phi, \tau)$ for the other G's with the same underlying root system, not necessarily with connected center. We indicate these modifications along the way.

Notice that the $\phi \in N(T)$ considered in [3] for $G = \mathrm{GL}_{d+1}(\mathbb{Q}_p)$ does *not* project to the affine Weyl group, only its $(d+1)$-st power, as considered here, has this property. But since the discussion is essentially the same, our treatment of the case A here is very brief.

As an application, in either case we work out the behaviour of the functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ on those $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$-modules which we call 'quasi supersingular'. Roughly speaking, these are induced from characters of the pro-p-Iwahori Hecke algebra of the corresponding simply connected group. At least conjecturally (i.e. extrapolating one of the main results from [6] from $G = \mathrm{GL}_{d+1}$ to arbitrary G's), the set of quasi supersingular $\mathcal{H}(G,I_0)_k$-modules contains the set of all irreducible supersingular $\mathcal{H}(G,I_0)_k$-modules (and the very few quasi supersingular $\mathcal{H}(G,I_0)_k$-modules which are not irreducible supersingular are easily identified). We show that our functor induces a bijection between the set of (isomorphism classes of) quasi supersingular $\mathcal{H}(G,I_0)_k$-modules and the set of (isomorphism classes of) certain 'symmetric' étale (φ^r,Γ)-modules over $k_{\mathcal{E}} = k((t))$. These are defined as direct sums of one dimensional étale (φ^r,Γ)-modules which satisfy certain symmetry conditions (depending on the root system underlying G). Their $k_{\mathcal{E}}$-dimension is the k-dimension of the corresponding quasi supersingular $\mathcal{H}(G,I_0)_k$-module.

Of course, the potential interest in étale (φ^r,Γ)-modules lies in their relation with $\mathrm{Gal}_{\mathbb{Q}_p}$-representations. For any $r \in \mathbb{N}$ there is an exact functor from the category of étale (φ^r,Γ)-modules to the category of étale (φ,Γ)-modules (it multiplies the rank by the factor r), and by means of Fontaine's functor, the latter one is equivalent with the category of $\mathrm{Gal}_{\mathbb{Q}_p}$-representations.

In [3] we also explained that a datum $(C^{(\bullet)},\phi)$ alone, i.e. without a τ as above, can be used to define an exact functor $M \mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G,I_0))$ to the category of étale (φ^r,Γ_0)-modules, where Γ_0 denotes the maximal pro-p-subgroup of $\Gamma \cong \mathbb{Z}_p^\times$. Such data $(C^{(\bullet)},\phi)$ are not tied to co minuscule coweights and exist in abundance. One may ask for such $(C^{(\bullet)},\phi)$ with small length r of ϕ. Without further discussing them we give such $(C^{(\bullet)},\phi)$ with r equal to the semisimple rank of G, for G of type C, B and D.

The outline is as follows. In section 2 we explain the functor from étale (φ^r,Γ)-modules to étale (φ,Γ)-modules, and we introduce the 'symmetric' étale (φ^r,Γ)-modules mentioned above, for each of the root systems C, B, D and A. In section 3, Lemma 3.1, we discuss the relation between the data $(C^{(\bullet)},\phi,\tau)$ and co minuscule fundamental coweights. Our discussions of classical matrix groups G in section 4 are just concrete incarnations of Lemma 3.1, although in neither of these cases there is a need to make formal reference to Lemma 3.1. On the other hand, in our discussion of the cases E_6 and E_7 in section 5 we do invoke Lemma 3.1. We tried to synchronize our discussions of the various matrix groups. As a consequence, arguments repeat themselves, and we do not write them out again and again. In subsection 5.3 we consider the groups G with underlying root systems G_2, F_4 or E_8. As these do not admit (co)minuscule (co)weights, there are no data $(C^{(\bullet)},\phi,\tau)$ available as needed for a functor producing (φ^r,Γ)-modules. We thus discuss the question if for a suitable choice of $(C^{(\bullet)},\phi)$ the étale (φ^r,Γ_0)-modules in the image of the corresponding functor in fact extend to (φ^r,Γ)-modules. In the appendix we record calculations relevant for the cases E_6 and E_7, carried out with the help of the computer algebra system *sage*.

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2 (φ^r,Γ)-modules

We often regard elements of $\mathbb{F}_p^\times$ as elements of $\mathbb{Z}_p^\times$ by means of the Teichmüller lifting. In $\mathrm{SL}_2(\mathbb{Z}_p)$ we define the subgroups

$$\Gamma = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} 1+p\mathbb{Z}_p & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathfrak{N}_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

and the elements

$$\varphi = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad h(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \qquad \gamma(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{Z}_p^\times$.

Let $\mathcal{O}_{\mathcal{E}}^+ = \mathfrak{o}[[\mathfrak{N}_0]]$ denote the completed group ring of $\mathfrak{N}_0$ over $\mathfrak{o}$. Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of the localization of $\mathcal{O}_{\mathcal{E}}^+$ with respect to the complement of $\pi_K\mathcal{O}_{\mathcal{E}}^+$, where $\pi_K \in \mathfrak{o}$ is a uniformizer. In the completed group ring $k_{\mathcal{E}}^+ = k[[\mathfrak{N}_0]]$ we put $t = [\nu]-1$. Let $k_{\mathcal{E}} = \mathrm{Frac}(k_{\mathcal{E}}^+) = \mathcal{O}_{\mathcal{E}}^+ \otimes_{\mathfrak{o}} k$. For definitions and notational conventions concerning étale φ^r and étale (φ^r,Γ)-modules we refer to [3].

Let $r \in \mathbb{N}$. Let $\mathbf{D} = (\mathbf{D},\varphi_{\mathbf{D}}^r)$ be an étale φ^r-module over $\mathcal{O}_{\mathcal{E}}$. For $0 \le i \le r-1$ let $\mathbf{D}^{(i)} = \mathbf{D}$ be a copy of $\mathbf{D}$. For $1 \le i \le r-1$ define $\varphi_{\widetilde{\mathbf{D}}} : \mathbf{D}^{(i)} \to \mathbf{D}^{(i-1)}$ to be the identity map on $\mathbf{D}$, and define $\varphi_{\widetilde{\mathbf{D}}} : \mathbf{D}^{(0)} \to \mathbf{D}^{(r-1)}$ to be the structure map $\varphi_{\mathbf{D}}^r$ on $\mathbf{D}$. Together we obtain a $\mathbb{Z}_p$-linear endomorphism $\varphi_{\widetilde{\mathbf{D}}}$ on

$$\widetilde{\mathbf{D}} = \bigoplus_{i=0}^{r-1} \mathbf{D}^{(i)}.$$

Define an $\mathcal{O}_{\mathcal{E}}$-action on $\widetilde{\mathbf{D}}$ by the formula

$$(1) \qquad x \cdot ((d_i)_{0\le i\le r-1}) = (\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x)d_i)_{0\le i\le r-1}.$$

Lemma 2.1. *The endomorphism $\varphi_{\widetilde{\mathbf{D}}}$ of $\widetilde{\mathbf{D}}$ is semilinear with respect to the $\mathcal{O}_{\mathcal{E}}$-action (1), hence it defines on $\widetilde{\mathbf{D}}$ the structure of an étale φ-module over $\mathcal{O}_{\mathcal{E}}$.*

Proof:

$$\begin{aligned}
\varphi_{\widetilde{\mathbf{D}}}(x\cdot((d_i)_i)) &= \varphi_{\widetilde{\mathbf{D}}}((\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x)d_i)_i) \\
&= ((\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x)d_{i+1})_{0\le i\le r-2}, (\varphi_{\mathbf{D}}^r(x\cdot d_0))_{r-1}) \\
&= ((\varphi_{\mathcal{O}_{\mathcal{E}}}^i(x)d_{i+1})_{0\le i\le r-2}, (\varphi_{\mathcal{O}_{\mathcal{E}}}^r(x)\varphi_{\mathbf{D}}^r(d_0))_{r-1}) \\
&= \varphi_{\mathcal{O}_{\mathcal{E}}}(x)((d_{i+1})_{0\le i\le r-2}, (\varphi_{\mathbf{D}}^r(d_0))_{r-1}) \\
&= \varphi_{\mathcal{O}_{\mathcal{E}}}(x)\varphi_{\widetilde{\mathbf{D}}}((d_i)_i).
\end{aligned}$$

□

Let Γ' be an open subgroup of Γ, let $\mathbf{D}$ be an étale (φ^r, Γ')-module over $\mathcal{O}_{\mathcal{E}}$. Define an action of Γ' on $\widetilde{\mathbf{D}}$ by

$$\gamma \cdot ((d_i)_{0 \le i \le r-1}) = (\gamma \cdot d_i)_{0 \le i \le r-1}.$$

Lemma 2.2. *The Γ'-action on $\widetilde{\mathbf{D}}$ commutes with $\varphi_{\widetilde{\mathbf{D}}}$ and is semilinear with respect to the $\mathcal{O}_{\mathcal{E}}$-action (1), hence we obtain on $\widetilde{\mathbf{D}}$ the structure of an étale (φ, Γ')-module over $\mathcal{O}_{\mathcal{E}}$. We thus obtain an exact functor from the category of étale (φ^r, Γ')-modules to the category of étale (φ, Γ')-modules over $\mathcal{O}_{\mathcal{E}}$.*

PROOF: This is immediate from the respective properties of the Γ'-action on $\mathbf{D}$. □

Lemma 2.3. *(a) Let $\mathbf{D}$ be a one-dimensional étale (φ^r, Γ)-module over $k_{\mathcal{E}}$. There exists a basis element g for $\mathbf{D}$, uniquely determined integers $0 \le s(\mathbf{D}) \le p-2$ and $1 \le n(\mathbf{D}) \le p^r - 1$ and a uniquely determined scalar $\xi(\mathbf{D}) \in k^\times$ such that*

$$\begin{aligned} \varphi^r g &= \xi(\mathbf{D}) t^{n(\mathbf{D})+1-p^r} g \\ \gamma(x) g - x^{s(\mathbf{D})} g & \qquad \in t \cdot k_{\mathcal{E}}^+ \cdot g \end{aligned}$$

for all $x \in \mathbb{Z}_p^\times$. Thus, one may define $0 \le k_i(\mathbf{D}) \le p-1$ by $n(\mathbf{D}) = \sum_{i=0}^{r-1} k_i(\mathbf{D}) p^i$. One has $n \equiv 0$ modulo $(p-1)$.

(b) For any given integers $0 \le s \le p-2$ and $1 \le n \le p^r - 1$ with $n \equiv 0$ modulo $(p-1)$ and any scalar $\xi \in k^\times$ there is a uniquely determined (up to isomorphism) one-dimensional étale (φ^r, Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ with $s = s(\mathbf{D})$ and $n = n(\mathbf{D})$ and $\xi = \xi(\mathbf{D})$.

PROOF: (a) Begin with an arbitrary basis element g_0 for $\mathbf{D}$; then $\varphi^r g_0 = F g_0$ for some unit $F \in k((t)) = k_{\mathcal{E}}$. After multiplying g_0 with a suitable power of t we may assume $F = \xi t^m (1 + t^{n_0} F_0)$ for some $0 \ge m \ge 2 - p^r$, some $\xi \in k^\times$, some $n_0 > 0$, and some $F_0 \in k[[t]]$ (use $t^{p^r} \varphi^r = \varphi^r t$). For $g_1 = (1 + t^{n_0} F_0) g_0$ we then get $\varphi^r g_1 = \xi t^m (1 + t^{n_1} F_1) g_1$ for some $n_1 > n_0 > 0$ and some $F_1 \in k[[t]]$. We may continue in this way; by completeness we get $g = g_\infty \in \mathbf{D}$ such that $\varphi^r g = \xi t^m g$. It is clear that $\xi(\mathbf{D}) = \xi$ and $n(\mathbf{D}) = p^r - 1 + m$ are well defined. Next, to see that there is some $s(\mathbf{D})$ as required we only need to see that $\gamma(x) g = F_x g$ for some unit $F_x \in k[[t]]$. But this follows from the fact that $\gamma(x)^{p-1}$ is topologically nilpotent (and acts by an automorphism on $\mathbf{D}$). It follows from Lemma 6.3 in [3] that $n(\mathbf{D}) \equiv 0$ modulo $p-1$.

(b) Put $D = k[[t]] = k_{\mathcal{E}}^+$ and $D^* = \mathrm{Hom}_k^{\mathrm{ct}}(k_{\mathcal{E}}^+, k)$ and define $\ell_0 \in D^*$ by $\ell_0(\sum_{i \ge 0} a_i t^i) = a_0$. Endow D^* with an action of $k_{\mathcal{E}}^+$ by putting $(\alpha \cdot \ell)(x) = \ell(\alpha \cdot x)$ for $\alpha \in k_{\mathcal{E}}^+$, $\ell \in D^*$ and

$x \in D$. We claim that this action uniquely extends to an action by $k_{\mathcal{E}}^{+}[\varphi^r, \Gamma]$ satisfying

$$t^n \varphi^r \ell_0 = \xi^{-1} \ell_0 \tag{2}$$

$$\gamma(x)\ell_0 = x^{-s}\ell_0 \quad \text{for all } x \in \mathbb{Z}_p^\times. \tag{3}$$

Indeed, as $t^{p^r}\varphi^r = \varphi^r t$, formula (2) defines a unique extension to $k_{\mathcal{E}}^{+}[\varphi^r]$. Next, ℓ_0 then generates D^* as a $k_{\mathcal{E}}^{+}[\varphi^r]$-module, and this shows that an extension to $k_{\mathcal{E}}^{+}[\varphi^r, \Gamma]$ satisfying formula (3) must be unique. To see that it does indeed exist, we check the compatibility of formulae (2) and (3):

$$\gamma(x)t^n\varphi^r\ell_0 \overset{(i)}{=} x^n t^n \gamma(x)\varphi^r \ell_0 = x^{n-s}t^n\varphi^r\ell_0 = \xi^{-1}x^{n-s}\ell_0 \overset{(ii)}{=} \gamma(x)\xi^{-1}\ell_0.$$

Here (i) follows from $\gamma(x)t^n\gamma(x)^{-1} - x^n t^n \in t^{n+1}k_{\mathcal{E}}^{+}$ (see [3], proof of Lemma 6.3), whereas (ii) follows from our hypothesis $n \equiv 0$ modulo $(p-1)$. Now passing to the dual $D \cong (D^*)^*$ of D^* yields a non degenerate (ψ^r, Γ)-module over $k_{\mathcal{E}}^{+}$ with an associated étale (φ^r, Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ with $s = s(\mathbf{D})$ and $n = n(\mathbf{D})$ and $\xi = \xi(\mathbf{D})$; this is explained in [3] Lemma 6.4. This dualization argument also proves the uniqueness of $\mathbf{D}$. □

Definition: We say that an étale (φ^r, Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ is C-symmetric if it admits a direct sum decomposition $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ with one-dimensional étale (φ^r, Γ)-modules $\mathbf{D}_1$, $\mathbf{D}_2$ satisfying the following conditions (1), (2C) and (3C):
(1) $k_i(\mathbf{D}_1) = k_{r-1-i}(\mathbf{D}_2)$ for all $0 \leq i \leq r-1$
(2C) $\xi(\mathbf{D}_1) = \xi(\mathbf{D}_2)$
(3C) $s(\mathbf{D}_2) - s(\mathbf{D}_1) \equiv \sum_{i=0}^{r-1} i k_i(\mathbf{D}_1)$ modulo $(p-1)$

Definition: We say that an étale (φ^r, Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ is B-symmetric if r is odd and if $\mathbf{D}$ admits a direct sum decomposition $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ with one-dimensional étale (φ^r, Γ)-modules $\mathbf{D}_1$, $\mathbf{D}_2$ satisfying the following conditions (1), (2B) and (3B):
(1) $k_i(\mathbf{D}_1) = k_{r-1-i}(\mathbf{D}_2)$ for all $0 \leq i \leq r-1$
(2B) For both $\mathbf{D} = \mathbf{D}_1$ and $\mathbf{D} = \mathbf{D}_2$ we have $\xi(\mathbf{D}) = \prod_{i=0}^{r-1}(k_i(\mathbf{D})!)^{-1}$ and $k_i(\mathbf{D}) = k_{r-1-i}(\mathbf{D})$ for all $1 \leq i \leq \frac{r+1}{2}$
(3B) $s(\mathbf{D}_2) - s(\mathbf{D}_1) \equiv k_0(\mathbf{D}_1) - k_{r-1}(\mathbf{D}_1)$ modulo $(p-1)$

Lemma 2.4. *The conjunction of the conditions (1), (2C) and (3C) (resp. (1), (2B) and (3B)) is symmetric in* $\mathbf{D}_1$ *and* $\mathbf{D}_2$.

Proof: That each one of the conditions (1), (2C) and (2B) is symmetric even individually is obvious. Now $n(\mathbf{D}) \equiv 0$ modulo $p-1$ implies $\sum_{i=0}^{r-1} k_i(\mathbf{D}_1) \equiv 0$ modulo $p-1$. Therefore $s(\mathbf{D}_2) - s(\mathbf{D}_1) \equiv \sum_{i=0}^{r-1} i k_i(\mathbf{D}_1)$ and $k_i(\mathbf{D}_1) = k_{r-1-i}(\mathbf{D}_2)$ for all i (condition (1)) together imply $s(\mathbf{D}_1) - s(\mathbf{D}_2) \equiv \sum_{i=0}^{r-1} i k_i(\mathbf{D}_2)$. Thus condition (3C) is symmetric, assuming condition (1). Similarly, condition (3B) is symmetric, assuming condition (1). $\square$

Definition: (i) Let $\widetilde{\mathfrak{S}}_C(r)$ denote the set of triples (n,s,ξ) with integers $1\leq n\leq p^r-1$ and $0\leq s\leq p-2$ and scalars $\xi\in k^\times$ such that $n\equiv 0$ modulo $(p-1)$. Let $\mathfrak{S}_C(r)$ denote the quotient of $\widetilde{\mathfrak{S}}_C(r)$ by the involution

$$(\sum_{i=0}^{r-1}k_ip^i,s,\xi)\mapsto(\sum_{i=0}^{r-1}k_{r-i-1}p^i,s+\sum_{i=0}^{r-1}ik_i,\xi).$$

(Here and in the following, in the second component we mean the representative modulo $p-1$ belonging to $[0,p-2]$.)

(ii) Let r be odd and let $\widetilde{\mathfrak{S}}_B(r)$ denote the set of pairs (n,s) with integers $1\leq n=\sum_{i=0}^{r-1}k_ip^i\leq p^r-1$ and $0\leq s\leq p-2$ such that $n\equiv 0$ modulo $(p-1)$ and such that $k_i=k_{r-1-i}$ for all $1\leq i\leq\frac{r+1}{2}$. Let $\mathfrak{S}_B(r)$ denote the quotient of $\widetilde{\mathfrak{S}}_B(r)$ by the involution

$$(\sum_{i=0}^{r-1}k_ip^i,s)\mapsto(\sum_{i=0}^{r-1}k_{r-i-1}p^i,s+k_0-k_{r-1}).$$

Lemma 2.5. *(i) Sending $\mathbf{D}=\mathbf{D}_1\oplus\mathbf{D}_2$ to $(n(\mathbf{D}_1),s(\mathbf{D}_1),\xi(\mathbf{D}_1))$ induces a bijection between the set of isomorphism classes of C-symmetric étale (φ^r,Γ)-modules and $\mathfrak{S}_C(r)$.*

(ii) Sending $\mathbf{D}=\mathbf{D}_1\oplus\mathbf{D}_2$ to $(n(\mathbf{D}_1),s(\mathbf{D}_1))$ induces a bijection between the set of isomorphism classes of B-symmetric étale (φ^r,Γ)-modules and $\mathfrak{S}_B(r)$.

Proof: This follows from Lemma 2.3. $\square$

Definition: Let r be even. Let $\widetilde{\mathfrak{S}}_D(r)$ denote the set of triples (n,s,ξ) with integers $1\leq n=\sum_{i=0}^{r-1}k_ip^i\leq p^r-1$ and $0\leq s\leq p-2$ and scalars $\xi\in k^\times$ such that $n\equiv 0$ modulo $(p-1)$ and such that $k_i=k_{i+\frac{r}{2}}$ for all $1\leq i\leq\frac{r}{2}-2$. We consider the following permutations ι_0 and ι_1 of $\widetilde{\mathfrak{S}}_D(r)$. The value of ι_0 at $(\sum_{i=0}^{r-1}k_ip^i,s,\xi)$ is

$$(k_{\frac{r}{2}}+\sum_{i=1}^{\frac{r}{2}-2}k_ip^i+k_{r-1}p^{\frac{r}{2}-1}+k_0p^{\frac{r}{2}}+\sum_{i=\frac{r}{2}+1}^{r-2}k_ip^i+k_{\frac{r}{2}-1}p^{r-1},s+\sum_{i=0}^{\frac{r}{2}-1}k_i,\xi).$$

The value of ι_1 at $(\sum_{i=0}^{r-1}k_ip^i,s,\xi)$ is

$$(\sum_{i=1}^{r-1}k_{r-i-1}p^i,s+\frac{r-2}{4}(k_{\frac{r}{2}}+k_0)+\sum_{i=2}^{\frac{r}{2}-1}(i-1)k_{\frac{r}{2}-i},\xi)$$

if r is odd, whereas if r is even the value is

$$(k_{\frac{r}{2}-1}+\sum_{i=1}^{\frac{r}{2}-2}k_{r-i-1}p^i+k_{\frac{r}{2}}p^{\frac{r}{2}-1}+k_{r-1}p^{\frac{r}{2}}+\sum_{i=\frac{r}{2}+1}^{r-2}k_{r-i-1}p^i+k_0p^{r-1},s+(\frac{r}{4}-1)k_{\frac{r}{2}}+\frac{r}{4}k_0+\sum_{i=2}^{\frac{r}{2}-1}(i-1)k_{\frac{r}{2}-i}p^i,\xi).$$

It is straightforward to check that $\iota_0^2 = \mathrm{id}$ and $\iota_0\iota_1 = \iota_1\iota_0$, and moreover that $\iota_1^2 = \mathrm{id}$ if r is odd, but $\iota_1^2 = \iota_0$ if r is even. In either case, the subgroup $\langle \iota_0, \iota_1 \rangle$ of $\mathrm{Aut}(\widetilde{\mathfrak{S}}_D(r))$ generated by ι_0 and ι_1 is commutative and contains 4 elements. We let $\mathfrak{S}_D(r)$ denote the quotient of $\widetilde{\mathfrak{S}}_D(r)$ by the action of $\langle \iota_0, \iota_1 \rangle$.

Definition: Let r be even. We say that an étale (φ^r, Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ is D-symmetric if it admits a direct sum decomposition $\mathbf{D} = \mathbf{D}_{11} \oplus \mathbf{D}_{12} \oplus \mathbf{D}_{21} \oplus \mathbf{D}_{22}$ with one-dimensional étale (φ^r, Γ)-modules $\mathbf{D}_{11}$, $\mathbf{D}_{12}$, $\mathbf{D}_{21}$, $\mathbf{D}_{22}$ satisfying the following conditions:

(1) For all $1 \leq i \leq \frac{r}{2} - 2$ and all $1 \leq s, t \leq 2$ we have $k_i(\mathbf{D}_{st}) = k_{\frac{r}{2}+i}(\mathbf{D}_{st})$

(2) For all $1 \leq i \leq \frac{r}{2} - 2$ we have $k_i(\mathbf{D}_{11}) = k_i(\mathbf{D}_{12})$ and $k_i(\mathbf{D}_{21}) = k_i(\mathbf{D}_{22})$

(3)

$$k_0(\mathbf{D}_{11}) = k_{\frac{r}{2}}(\mathbf{D}_{12}), \quad k_{\frac{r}{2}}(\mathbf{D}_{11}) = k_0(\mathbf{D}_{12}), \quad k_{\frac{r}{2}-1}(\mathbf{D}_{11}) = k_{r-1}(\mathbf{D}_{12}), \quad k_{r-1}(\mathbf{D}_{11}) = k_{\frac{r}{2}-1}(\mathbf{D}_{12})$$

(4)

$$k_i(\mathbf{D}_{11}) = k_{r-i-1}(\mathbf{D}_{21}) \quad \text{and} \quad k_i(\mathbf{D}_{12}) = k_{r-i-1}(\mathbf{D}_{22})$$

if $i \in [0, r-1]$ and $\frac{r}{2}$ is odd, or if $i \in [1, \frac{r}{2} - 2] \cup [\frac{r}{2} + 1, r - 2]$ and $\frac{r}{2}$ is even. Moreover, if $\frac{r}{2}$ is even then

$$k_0(\mathbf{D}_{11}) = k_{r-1}(\mathbf{D}_{21}), \quad k_{\frac{r}{2}-1}(\mathbf{D}_{11}) = k_0(\mathbf{D}_{21}), \quad k_{\frac{r}{2}}(\mathbf{D}_{11}) = k_{\frac{r}{2}-1}(\mathbf{D}_{21}), \quad k_{r-1}(\mathbf{D}_{11}) = k_{\frac{r}{2}}(\mathbf{D}_{21})$$

(5) $\xi(\mathbf{D}_{11}) = \xi(\mathbf{D}_{12}) = \xi(\mathbf{D}_{21}) = \xi(\mathbf{D}_{22})$

(6) Modulo $(p-1)$ we have

$$s(\mathbf{D}_{12}) - s(\mathbf{D}_{11}) \equiv \sum_{i=0}^{\frac{r}{2}-1} k_i(\mathbf{D}_{11})$$

$$s(\mathbf{D}_{22}) - s(\mathbf{D}_{21}) \equiv \begin{cases} \sum_{i=0}^{\frac{r}{2}-1} k_i(\mathbf{D}_{11}) & : \ \frac{r}{2} \text{ is odd} \\ k_{\frac{r}{2}}(\mathbf{D}_{11}) - k_0(\mathbf{D}_{11}) + \sum_{i=0}^{\frac{r}{2}-1} k_i(\mathbf{D}_{11}) & : \ \frac{r}{2} \text{ is even} \end{cases}$$

$$s(\mathbf{D}_{21}) - s(\mathbf{D}_{11}) \equiv \begin{cases} \frac{r-2}{4}(k_{\frac{r}{2}}(\mathbf{D}_{11}) + k_0(\mathbf{D}_{11})) + \sum_{i=2}^{\frac{r}{2}-1}(i-1)k_{\frac{r}{2}-i}(\mathbf{D}_{11}) & : \ \frac{r}{2} \text{ is odd} \\ (\frac{r}{4} - 1)k_{\frac{r}{2}}(\mathbf{D}_{11}) + \frac{r}{4}k_0(\mathbf{D}_{11}) + \sum_{i=2}^{\frac{r}{2}-1}(i-1)k_{\frac{r}{2}-i}(\mathbf{D}_{11}) & : \ \frac{r}{2} \text{ is even} \end{cases}$$

Lemma 2.6. *Sending $\mathbf{D} = \mathbf{D}_{11} \oplus \mathbf{D}_{12} \oplus \mathbf{D}_{21} \oplus \mathbf{D}_{22}$ to $(n(\mathbf{D}_{11}), s(\mathbf{D}_{11}), \xi(\mathbf{D}_{11}))$ induces a bijection between the set of isomorphism classes of D-symmetric étale (φ^r, Γ)-modules and $\mathfrak{S}_D(r)$.*

PROOF: Again we use Lemma 2.3. For a one-dimensional étale (φ^r, Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ put $\alpha(\mathbf{D}) = (n(\mathbf{D}), s(\mathbf{D}), \xi(\mathbf{D}))$; this is an element of $\widetilde{\mathfrak{S}}_D(r)$. If $\mathbf{D} = \mathbf{D}_{11} \oplus \mathbf{D}_{12} \oplus \mathbf{D}_{21} \oplus \mathbf{D}_{22}$ is D-symmetric as above, then it is straighforward to check $\iota_0(\alpha(\mathbf{D}_{11})) = \alpha(\mathbf{D}_{12})$, $\iota_0(\alpha(\mathbf{D}_{21})) = \alpha(\mathbf{D}_{22})$, $\iota_1(\alpha(\mathbf{D}_{11})) = \alpha(\mathbf{D}_{21})$ and $\iota_0(\alpha(\mathbf{D}_{12})) = \alpha(\mathbf{D}_{22})$. It follows that the above map is well defined and bijective. $\square$

Definition: We say that an étale (φ^r,Γ)-module $\mathbf{D}$ over $k_{\mathcal{E}}$ is A-symmetric if $\mathbf{D}$ admits a direct sum decomposition $\mathbf{D}=\oplus_{i=0}^{r-1}\mathbf{D}_i$ with one-dimensional étale (φ^r,Γ)-modules $\mathbf{D}_i$ satisfying the following conditions for all i, j (where we understand the sub index in $k_?$ as the unique representative in $[0,r-1]$ modulo r):

$$k_i(\mathbf{D}_j)=k_{i-j}(\mathbf{D}_0),\qquad \xi(\mathbf{D}_j)=\xi(\mathbf{D}_0),\qquad s(\mathbf{D}_0)-s(\mathbf{D}_j)\equiv\sum_{i=1}^{j}k_{-i}(\mathbf{D}_0)\text{ modulo }(p-1)$$

3 Semiinfinite chamber galleries and functor D

3.1 Power multiplicative elements in the extended affine Weyl group

Let G be the group of $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$. Fix a maximal $\mathbb{Q}_p$-split torus T in G, let $N(T)$ be its normalizer in G. Let Φ denote the set of roots of T. For $\alpha\in\Phi$ let N_α be the corresponding root subgroup in G. Choose a positive system Φ^+ in Φ, let $\Delta\subset\Phi^+$ be the set of simple roots. Let $N=\prod_{\alpha\in\Phi^+}N_\alpha$.

Let X denote the semi simple Bruhat-Tits building of G, let A denote its apartment corresponding to T. Our notational and terminological convention is that the facets of A or X are *closed* in X (i.e. *contain* all their faces (the lower dimensional facets at their boundary)). A chamber is a facet of codimension 0. For a chamber D in A let I_D be the Iwahori subgroup in G fixing D.

Fix a special vertex x_0 in A, let K be the corresponding hyperspecial maximal compact open subgroup in G. Let $T_0=T\cap K$ and $N_0=N\cap K$. We have the isomorphism $T/T_0\cong X_*(T)$ sending $\xi\in X_*(T)$ to the class of $\xi(p)\in T$. Let $I\subset K$ be the Iwahori subgroup determined by Φ^+. [If red : $K\to\overline{K}$ denotes the reduction map onto the reductive (over $\mathbb{F}_p$) quotient $\overline{K}$ of K, then $I=\mathrm{red}^{-1}(\mathrm{red}(T_0N_0))$.] Let $C\subset A$ be the chamber fixed by I.

We are interested in semiinfinite chamber galleries

$$C^{(0)},C^{(1)},C^{(2)},C^{(3)},\ldots \tag{4}$$

in A such that $C=C^{(0)}$ (and thus $I=I_{C^{(0)}}$) and such that, setting

$$N_0^{(i)}=I_{C^{(i)}}\cap N=\prod_{\alpha\in\Phi^+}I_{C^{(i)}}\cap N_\alpha,$$

we have $N_0 = N_0^{(0)}$ and

$$N_0^{(0)} \supset N_0^{(1)} \supset N_0^{(2)} \supset N_0^{(3)} \supset \dots \qquad \text{with } [N_0^{(i)} : N_0^{(i+1)}] = p \text{ for all } i \geq 0. \tag{5}$$

In this situation there is a unique sequence $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \dots$ in Φ^+ such that, setting

$$e[i, \alpha] = |\{0 \leq j \leq i-1 \,|\, \alpha = \alpha^{(j)}\}|$$

for $i \geq 0$ and $\alpha \in \Phi^+$, we have

$$N_0^{(i)} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{e[i,\alpha]}}.$$

Geometrically, $C^{(i+1)}$ and $C^{(i)}$ share a common facet of codimension 1 contained in a wall which belongs to the translation class of walls corresponding to $\alpha^{(i)}$.

Suppose that the center Z of G is connected. Then G/Z is a semisimple group of adjoint type with maximal torus $\check{T} = T/Z$. Let $\check{T}_0 = T_0/(T_0 \cap Z) \subset \check{T}$. The extended affine Weyl group $\widehat{W} = N(\check{T})/\check{T}_0$ can be identified with the semidirect product between the finite Weyl group $W = N(\check{T})/\check{T} = N(T)/T$ and $X_*(\check{T})$. We identify $A = X_*(\check{T}) \otimes \mathbb{R}$ such that $x_0 \in A$ corresponds to the origin in the $\mathbb{R}$-vector space $X_*(\check{T}) \otimes \mathbb{R}$. We then regard $\widehat{W}$ as acting on A through affine transformations. We regard $\Delta \subset X^*(T)$ as a subset of $X^*(\check{T})$. We usually enumerate the elements of Δ as $\alpha_1, \dots, \alpha_d$, and we enumerate the corresponding simple reflection $s_\alpha \in W$ for $\alpha \in \Delta$ as $s_1, \dots, s_d$ with $s_i = s_{\alpha_i}$. Assume that the root system Φ is irreducible and let $\alpha_0 \in \Phi$ be the *negative* of the highest root. Let s_{α_0} be the corresponding reflection in the finite Weyl group W; define the affine reflection $s_0 = t_{\alpha_0^\vee} \circ s_{\alpha_0} \in \widehat{W}$, where $t_{\alpha_0^\vee}$ denotes the translation by the coroot $\alpha_0^\vee \in A$ of α_0. The affine Weyl group W_{aff} is the subgroup of $\widehat{W}$ generated by $s_0, s_1, \dots, s_d$; in fact it is a Coxeter group with these Coxeter generators. The corresponding length function ℓ on W_{aff} extends to $\widehat{W}$.

Let $X_*(\check{T})_+$ denote the set of dominant coweights. [Let $T_+ = \{t \in T \,|\, tN_0t^{-1} \subset N_0\}$, then $X_*(\check{T})_+$ is the image of T_+ under the map $T_+ \subset T \to T/T_0 \cong X_*(T) \to X_*(\check{T})$.] The monoid $X_*(\check{T})_+$ is free and has a unique basis ∇, the set of fundamental coweights. The cone (vector chamber) in A with origin in x_0 which is spanned by all the $-\xi$ for $\xi \in \nabla$ contains C, and C is precisely the 'top' chamber of this cone. The reflections $s_0, s_1, \dots, s_d$ are precisely the reflections in the affine hyperplanes (walls) of A which contain a codimension-1-face of C.

Let us say that $w \in \widehat{W}$ is power multiplicative if we have $\ell(w^m) = m \cdot \ell(w)$ for all $m \geq 0$. Of course, any element in the image of $T \to N(\check{T}) \to \widehat{W} = N(\check{T})/\check{T}_0$ is power multiplicative.

Suppose we are given a fundamental coweight $\tau \in \nabla$ and some non trivial element $\phi \in \widehat{W}$ satisfying the following conditions:

(a) ϕ is power multiplicative,

(b) τ is co minuscule, i.e. we have $\langle \alpha, \tau \rangle \in \{0, 1\}$ for all $\alpha \in \Phi^+$,

(c) viewing τ via the embedding $X_*(\check{T}) \subset \widehat{W}$ as an element of $\widehat{W}$, we have

$$\phi^{\mathbb{N}} \cap \tau^{\mathbb{N}} \neq \emptyset. \tag{6}$$

Lemma 3.1. *Let ϕ and τ be as above. Write $\phi = \phi' v$ with $\phi' \in W_{\rm aff}$ and $v \in \widehat{W}$ with $vC = C$. Choose a reduced expresssion*

$$\phi' = s_{\beta(1)} \cdots s_{\beta(r)}$$

of ϕ' with some function $\beta : \{1, \ldots, r\} \to \{0, \ldots, d\}$ (with $r = \ell(\phi) = \ell(\phi')$) and put

$$C^{(ar+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < r$. Lift $\tau \in \nabla \subset X_*(\check{T})$ to some element of $X_*(T)$ and denote again by τ the corresponding homomorphism $\mathbb{Z}_p^\times \to T_0$. Then we have:

(i) The sequence

$$C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots$$

satisfies hypothesis (5). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \geq 0$.

(ii) For any $j \geq 0$ we have $\alpha^{(j)} \circ \tau = {\rm id}_{\mathbb{Z}_p^\times}$.

(iii) For any lifting $\phi \in N(T)$ of $\phi \in \widehat{W}$ we have $\tau(a)\phi = \phi\tau(a)$ in $N(T)$, for all $a \in \mathbb{Z}_p^\times$.

PROOF: (i) As τ is co minuscule, it is in particular a dominant coweight. Therefore it follows from hypothesis (6) that also some power of ϕ is a dominant coweight. As ϕ is power multiplicative, this implies statement (i).

(ii) As ϕ is power multiplicative, hypothesis (6) implies that for any $m \in \mathbb{N}$ for which ϕ^m belongs to $X_*(T)$ we have

$$\{\alpha^{(j)} \,|\, j \geq 0\} = \{\alpha \in \Phi^+ \,|\, \langle \alpha, \phi^m \rangle \neq 0\} = \{\alpha \in \Phi^+ \,|\, \langle \alpha, \tau \rangle \neq 0\}$$

and as τ is co minuscule this is the set

$$\{\alpha \in \Phi^+ \,|\, \langle \alpha, \tau \rangle = 1\} = \{\alpha \in \Phi^+ \,|\, \langle \alpha \circ \tau \rangle = {\rm id}_{\mathbb{Z}_p^\times}\}.$$

(iii) By hypothesis (6) we have $\tau^m = \phi^n$ for some $m, n \in \mathbb{N}$. We deduce $\tau^m = \phi\tau^m\phi^{-1} = (\phi\tau\phi^{-1})^m$ and hence also $\tau = \phi\tau\phi^{-1}$ as τ and $\phi\tau\phi^{-1}$ belong to the free abelian group $X_*(T)$. Thus $\tau\phi = \phi\tau$ in $\widehat{W}$ which implies claim (iii). $\square$

Remark: For a given co minuscule fundamental coweight $\tau \in \nabla$ some positive power τ^m of τ belongs to $X_*(\check{T})$, and so $\phi = \tau^m$ satisfies the assumptions of Lemma 3.1. However, for our purposes it is of interest to find ϕ (as in Lemma 3.1, possibly also required to project to $W_{\rm aff}$) of small length; the minimal positive power of τ belonging to $X_*(\check{T})$ is usually not optimal in this sense.

3.2 Functor D

By I_0 we denote the pro-p-Iwahori subgroup contained in I. We often read $\overline{T} = T_0/T_0 \cap I_0$ as a subgroup of T_0 by means of the Teichmüller character. Conversely, we read characters of $\overline{T}$ also as characters of T_0 (and do not introduce another name for these inflations).

Let $\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ denote the $\mathfrak{o}$-module of $\mathfrak{o}$-valued compactly supported functions f on G such that $f(ig) = f(g)$ for all $g \in G$, all $i \in I_0$. It is a G-representation by means of $(g'f)(g) = f(gg')$ for $g, g' \in G$. Let

$$\mathcal{H}(G, I_0) = \mathrm{End}_{\mathfrak{o}[G]}(\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{\rm op}$$

denote the corresponding pro-p-Iwahori Hecke algebra with coefficients in $\mathfrak{o}$. For a subset H of G let χ_H denote the characteristic function of H. For $g \in G$ let $T_g \in \mathcal{H}(G, I_0)$ denote the Hecke operator corresponding to the double coset $I_0 g I_0$. It sends $f : G \to \mathfrak{o}$ to

$$T_g(f) : G \longrightarrow \mathfrak{o}, \qquad h \mapsto \sum_{x \in I_0 \backslash G} \chi_{I_0 g I_0}(hx^{-1}) f(x).$$

Let $\mathrm{Mod}^{\rm fin}(\mathcal{H}(G, I_0))$ denote the category of $\mathcal{H}(G, I_0)$-modules which as $\mathfrak{o}$-modules are of finite length. We write $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$. Given liftings $\dot{s} \in N(T)$ of all $s \in S = \{s_i \,|\, 0 \le i \le d\}$ we let $\mathcal{H}(G, I_0)_{{\rm aff},k}$ denote the k-subalgebra of $\mathcal{H}(G, I_0)_k$ generated by the $T_{\dot{s}}$ for all $s \in S$ and the T_t for $t \in \overline{T}$.

Suppose we are given a reduced expression

$$\phi = \epsilon \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(r)} \tag{7}$$

(some function $\beta : \{1, \ldots, r = \ell(\phi)\} \to \{0, \ldots, d\}$, some $\epsilon \in Z$) of a power multiplicative element $\phi \in N(T)$, some power of which maps to a dominant coweight in $N(T)/ZT_0$. Put

$$C^{(ar+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\ge 0}$ with $0 \le b < r$. Then, by power multiplicativity of ϕ, the sequence (4) thus defined satisfies property (5). Therefore we may use it to place ourselves into the setting (and notations) of [3], as follows.

We define the half tree Y whose edges are the N_0-orbits of the $C^{(i)} \cap C^{(i+1)}$ and whose vertices are the N_0-orbits of the $C^{(i)}$. We choose an isomorphism $\Theta : Y \cong \mathfrak{X}_+$ with the $\lfloor \mathfrak{N}_0, \varphi, \Gamma \rfloor$-equivariant half sub tree $\mathfrak{X}_+$ of the Bruhat Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$, satisfying the requirements of Theorem 3.1 of loc.cit.. It sends the edge $C^{(i)} \cap C^{(i+1)}$ (resp. the vertex $C^{(i)}$) of Y to the edge $\mathfrak{e}_{i+1}$ (resp. the vertex $\mathfrak{v}_i$) of $\mathfrak{X}_+$. The half tree $\overline{\mathfrak{X}}_+$ is obtained from $\mathfrak{X}_+$ by removing the 'loose' edge $\mathfrak{e}_0$.

To an $\mathcal{H}(G, I_0)$-module M we associate the G-equivariant (partial) coefficient system $\mathcal{V}_M^X$ on X. Briefly, its value at the chamber C is $\mathcal{V}_M^X(C) = M$. The transition maps $\mathcal{V}_M^X(D) \to \mathcal{V}_M^X(F)$ for chambers (codimension-0-facets) D and codimension-1-facets F with $D \subset F$ are injective, and $\mathcal{V}_M^X(F)$ for any such F is the sum of the images of the $\mathcal{V}_M^X(D) \to \mathcal{V}_M^X(F)$ for all D with $D \subset F$.

The pushforward $\Theta_* \mathcal{V}_M$ of the restriction of $\mathcal{V}_M^X$ to Y carries a natural $\lfloor \mathfrak{N}_0, \varphi^r, \Gamma_0 \rfloor$-action. Taking global sections, dualizing and tensoring with $\mathcal{O}_{\mathcal{E}}$ leads to the exact functor

$$M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M) \tag{8}$$

from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^r, Γ_0)-modules over $\mathcal{O}_{\mathcal{E}}$, where $r = \ell(\Phi)$. If in addition we are given a homomorphism $\tau : \mathbb{Z}_p^\times \to T_0$ satisfying the conclusions of Lemma 3.1 (with respect to ϕ), then this functor in fact takes values in the category of (φ^r, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

For $0 \leq i \leq r-1$ we put

$$y_i = \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(i+1)} \dot{s}_{\beta(i)}^{-1} \cdots \dot{s}_{\beta(1)}^{-1}.$$

Lemma 3.2. *(a) For any $0 \leq i \leq r-1$ we have $y_i = y_{i-1} \cdots y_0 \dot{s}_{\beta(i+1)} y_0 \cdots y_{i-1}$. We have $\phi = \epsilon y_{r-1} \cdots y_0$.*

(b) For any $0 \leq i \leq r-1$ we have: y_i is the affine reflection in the wall passing through $C^{(i)} \cap C^{(i+1)}$.

PROOF: To see (b) observe that y_i indeed *is* a reflection, and that it sends $C^{(i)}$ to $C^{(i+1)}$. $\square$

Notations: Let us introduce some more notations which will be employed uniformly in all the separate cases to be discussed.

For $\alpha \in \Phi$ we denote by $\alpha^\vee$ the associated coroot. For any $\alpha \in \Phi$ there is a corresponding homomorphism of algebraic groups $\iota_\alpha : \mathrm{SL}_2(\mathbb{Q}_p) \to G$ as described in [5], Ch.II, section 1.3. The element $\iota_\alpha(\nu)$ belongs to $I \cap N_\alpha$ and generates it as a topological group. For $x \in \mathbb{F}_p^\times \subset \mathbb{Z}_p^\times$ (via the Teichmüller character) we have $\alpha^\vee(x) = \iota_\alpha(h(x)) \in T$.

For a character $\lambda : \overline{T} \to k^\times$ let S_λ be the subset of S consisting of all s_i such that $\lambda(\alpha_i^\vee(x)) = 1$ for all $x \in \mathbb{F}_p^\times$. Given λ and a subset $\mathcal{J}$ of S_λ there is a uniquely determined character

$$\chi_{\lambda,\mathcal{J}} : \mathcal{H}(G, I_0)_{\mathrm{aff},k} \longrightarrow k$$

which sends T_t to $\lambda(t^{-1})$ for $t \in \overline{T}$, which sends $T_{\dot{s}}$ to 0 for $s \in S - \mathcal{J}$ and which sends $T_{\dot{s}}$ to -1 for $s \in \mathcal{J}$ (see [7] Proposition 2). Moreover, for $0 \leq i \leq d$ we define a number $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ such that

$$\lambda(\alpha_i^\vee(x)) = x^{k_i} \qquad \text{for all } x \in \mathbb{F}_p^\times, \tag{9}$$

as follows. If $\lambda \circ \alpha_i^\vee$ is not the constant character $\mathbf{1}$ then k_i is already uniquely determined by formula (9). Next notice that $\lambda \circ \alpha_i^\vee = \mathbf{1}$ is equivalent with $s_i \in S_\lambda$. If $\lambda \circ \alpha_i^\vee = \mathbf{1}$ and $s_i \in \mathcal{J}$ we put $k_i = p-1$, if $\lambda \circ \alpha_i^\vee = \mathbf{1}$ and $s_i \notin \mathcal{J}$ we put $k_i = 0$.

4 Classical matrix groups

For $m \in \mathbb{N}$ let $E_m \in \mathrm{GL}_m$ denote the identity matrix and let E_d^* denotes the standard antidiagonal element in GL_d (i.e. the permutation matrix of maximal length). Let

$$\widehat{S}_m = \begin{pmatrix} & E_m \\ -E_m & \end{pmatrix}, \qquad S_m = \begin{pmatrix} & E_m \\ E_m & \end{pmatrix}.$$

4.1 Affine root system $\tilde{C}_d$

Assume $d \geq 2$. Here W_{aff} is the Coxeter group with Coxeter generators $s_0, s_1, \ldots, s_d$ (thus $s_i^2 = 1$ for all i) and relations

$$(s_0 s_1)^4 = (s_{d-1} s_d)^4 = 1 \qquad \text{and} \qquad (s_{i-1} s_i)^3 = 1 \quad \text{for } 2 \leq i \leq d-1 \tag{10}$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i \neq j$. In the extended affine Weyl group $\widehat{W}$ we find (cf. [4]) an element u of length 0 with

$$u^2 = 1 \qquad \text{and} \qquad u s_i u = s_{d-i} \quad \text{for } 0 \leq i \leq d. \tag{11}$$

($\widehat{W}$ is the semidirect product of its two-element subgroup $W_\Omega = \{1, u\}$ with W_{aff}.) Consider the general symplectic group

$$G = \mathrm{GSp}_{2d}(\mathbb{Q}_p) = \{A \in \mathrm{GL}_{2d}(\mathbb{Q}_p) \,|\, {}^T A \widehat{S}_d A = \kappa(A) \widehat{S}_d \text{ for some } \kappa(A) \in \mathbb{Q}_p^\times\}.$$

Let T denote the maximal torus consisting of all diagonal matrices in G. For $1 \leq i \leq d$ let

$$e_i : T \cap \mathrm{SL}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times, \quad A = \mathrm{diag}(x_1, \ldots, x_{2d}) \mapsto x_i.$$

For $1 \leq i, j \leq d$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ we thus obtain characters (using additive notation as usual) $\epsilon_1 e_i + \epsilon_2 e_j : T \cap \mathrm{SL}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$. We extend these latter ones to T by setting

$$\epsilon_1 e_i + \epsilon_2 e_j : T \longrightarrow \mathbb{Q}_p^\times, \quad A = \mathrm{diag}(x_1, \ldots, x_{2d}) \mapsto x_i^{\epsilon_1} x_j^{\epsilon_2} \kappa(A)^{\frac{-\epsilon_1 - \epsilon_2}{2}}.$$

For $i = j$ and $\epsilon = \epsilon_1 = \epsilon_2$ we simply write $\epsilon 2e_i$. Then $\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\}$ is the root system of G with respect to T. It is of type C_d.

For $\alpha \in \Phi$ let N_α^0 be the subgroup of the corresponding root subgroup N_α of G all of which elements belong to $\mathrm{GL}_{2d}(\mathbb{Z}_p)$.

We choose the positive system $\Phi^+ = \{e_i \pm e_j \mid i < j\} \cup \{2e_i \mid 1 \leq i \leq d\}$ with corresponding set of simple roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = 2e_d\}$. The negative of the highest root is $\alpha_0 = -2e_1$. For $0 \leq i \leq d$ let $s_i = s_{\alpha_i}$ be the reflection corresponding to α_i.

Remark: For $0 \leq i \leq d$ we have the following explicit formula for $\alpha_i^\vee = (\alpha_i)^\vee$:

$$\alpha_i^\vee(x) = \begin{cases} \mathrm{diag}(x^{-1}, E_{d-1}, x, E_{d-1}) & : \quad i = 0 \\ \mathrm{diag}(E_{i-1}, x, x^{-1}, E_{d-i-1}, E_{i-1}, x^{-1}, x, E_{d-i-1}) & : \quad 1 \leq i \leq d-1 \\ \mathrm{diag}(E_{d-1}, x, E_{d-1}, x^{-1}) & : \quad i = d \end{cases} \tag{12}$$

Let I_0 denote the pro-p-Iwahori subgroup generated by the N_α^0 for all $\alpha \in \Phi^+$, by the $(N_\alpha^0)^p$ for all $\alpha \in \Phi^- = \Phi - \Phi^+$, and by the maximal pro-p-subgroup of T_0. Let I denote the Iwahori subgroup of G containing I_0. Let N_0 be the subgroup of G generated by all N_α^0 for $\alpha \in \Phi^+$.

For $1 \leq i \leq d-1$ define the block diagonal matrix

$$\dot{s}_i = \mathrm{diag}(E_{i-1}, \widehat{S}_1, E_{d-i-1}, E_{i-1}, \widehat{S}_1, E_{d-i-1})$$

and furthermore

$$\dot{s}_d = \begin{pmatrix} E_{d-1} & & & \\ & & & 1 \\ & & E_{d-1} & \\ & -1 & & \end{pmatrix}, \qquad \dot{s}_0 = \begin{pmatrix} & & -p^{-1} & \\ & E_{d-1} & & \\ p & & & \\ & & & E_{d-1} \end{pmatrix}.$$

Then $\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_{d-1}, \dot{s}_d$ belong to G (in fact even to the symplectic group $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$) and normalize T. Their images $s_0, s_1, \ldots, s_{d-1}, s_d$ in $N(T)/ZT_0$ are Coxeter generators of $W_{\mathrm{aff}} \subset N(T)/ZT_0 = \widehat{W}$ satisfying the relations (10). Put

$$\dot{u} = \begin{pmatrix} & E_d^* \\ pE_d^* & \end{pmatrix}.$$

Then $\dot{u}$ belongs to $N(T)$ and normalizes I and I_0. The image u of $\dot{u}$ in $N(T)/ZT_0$ satisfies the formulae (11). In $N(T)$ we consider the element

$$\phi = (p \cdot \mathrm{id})\dot{s}_d \dot{s}_{d-1} \cdots \dot{s}_1 \dot{s}_0.$$

We may rewrite this as $\phi = (p \cdot \mathrm{id})\dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(d+1)}$ where we put $\beta(i) = d+1-i$ for $1 \le i \le d+1$. For $a, b \in \mathbb{Z}_{\ge 0}$ with $0 \le b < d+1$ we put

$$C^{(a(d+1)+b)} = \phi^a s_d \cdots s_{d-b+1} C = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C.$$

Define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \qquad x \mapsto \mathrm{diag}(xE_d, E_d).$$

Lemma 4.1. *We have $\phi^d \in T$ and $\phi^d N_0 \phi^{-d} \subset N_0$. The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$ satisfies hypothesis (5). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \ge 0$.*

(b) For all $j \ge 0$ we have $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}$.

(c) We have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$.

Proof: (a) A matrix computation shows $\phi^d = \mathrm{diag}(p^{d+1}E_d, p^{d-1}E_d) \in T$. Using this we find

$$\phi^d N_0 \phi^{-d} = \prod_{\alpha \in \Phi^+} \phi^d (N_0 \cap N_\alpha) \phi^{-d} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : \quad \alpha = e_i + e_j \text{ with } 1 \le i < j \le d \\ 2 & : \quad \alpha = 2e_i \text{ with } 1 \le i \le d \\ 0 & : \quad \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^d N_0 \phi^{-d} \subset N_0$ and $[N_0 : \phi^d N_0 \phi^{-d}] = p^{d(d+1)}$. This implies that the length of $\phi^m \in \widehat{W}$ is at least $(d+1)m$, for all $m \ge 0$. On the other hand this length is at most $(d+1)m$ because the image of ϕ in $\widehat{W}$ is a product of $d+1$ Coxeter generators. Thus ϕ^m has length $(d+1)m$ and ϕ is power multiplicative. We also see from this that

$$[N_0 : \phi^d N_0 \phi^{-d}] = [N_0 : (N_0 \cap \phi^d N_0 \phi^{-d})] = [I_0 : (I_0 \cap \phi^d I_0 \phi^{-d})]$$

(because $[I_0 : (I_0 \cap \phi^d I_0 \phi^{-d})]$ is the length of ϕ^d, as ϕ is power multiplicative). We get $I_0 = N_0 \cdot (I_0 \cap \phi^d I_0 \phi^{-d})$ and that hypothesis (5) holds true.

(b) As $\phi^d \in T$ we have $\{\alpha^{(j)} \,|\, j \ge 0\} = \{\alpha \in \Phi^+ \,|\, m_\alpha \ne 0\}$. This implies (b).

(c) Another matrix computation. $\square$

As explained in subsection 3.2 we now obtain a functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{d+1}, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$. As in explained in [3], to compute it we need to understand the intermediate objects $H_0(\overline{\mathfrak{X}}_+,\Theta_*\mathcal{V}_M)$, acted on by $\lfloor\mathfrak{N}_0,\varphi^{d+1},\Gamma\rfloor$.

Let $\mathcal{H}(G,I_0)'_{\mathrm{aff},k}$ denote the k-sub algebra of $\mathcal{H}(G,I_0)_k$ generated by $\mathcal{H}(G,I_0)_{\mathrm{aff},k}$ together with $T_{p\cdot\mathrm{id}}=T_{\dot{u}^2}$ and $T_{p\cdot\mathrm{id}}^{-1}=T_{p^{-1}\cdot\mathrm{id}}$.

Suppose we are given a character $\lambda:\overline{T}\to k^\times$, a subset $\mathcal{J}\subset S_\lambda$ and some $b\in k^\times$. Define the numbers $0\le k_i=k_i(\lambda,\mathcal{J})\le p-1$ as in subsection 3.2. The character $\chi_{\lambda,\mathcal{J}}$ of $\mathcal{H}(G,I_0)_{\mathrm{aff},k}$ extends uniquely to a character

$$\chi_{\lambda,\mathcal{J},b}:\mathcal{H}(G,I_0)'_{\mathrm{aff},k}\longrightarrow k$$

which sends $T_{p\cdot\mathrm{id}}$ to b (see the proof of [7] Proposition 3). Define the $\mathcal{H}(G,I_0)_k$-module

$$M=M[\lambda,\mathcal{J},b]=\mathcal{H}(G,I_0)_k\otimes_{\mathcal{H}(G,I_0)'_{\mathrm{aff},k}}k.e$$

where $k.e$ denotes the one dimensional k-vector space on the basis element e, endowed with the action of $\mathcal{H}(G,I_0)'_{\mathrm{aff},k}$ by the character $\chi_{\lambda,\mathcal{J},b}$. As a k-vector space, M has dimension 2, a k-basis is e,f where we write $e=1\otimes e$ and $f=T_{\dot{u}}\otimes e$.

Definition: We call an $\mathcal{H}(G,I_0)_k$-module *quasi supersingular* if it is isomorphic with $M[\lambda,\mathcal{J},b]$ for some $\lambda,\mathcal{J},b$ such that $k_i>0$ for at least one i.

For $0\le j\le d$ put $\widetilde{j}=d-j$. Letting $\widetilde{\beta}=\widetilde{(.)}\circ\beta$ we then have

$$\dot{u}\phi\dot{u}^{-1}=(p\cdot\mathrm{id})\dot{s}_{\widetilde{\beta}(1)}\cdots\dot{s}_{\widetilde{\beta}(d+1)}.$$

Put $n_e=\sum_{i=0}^d k_{d-i}p^i=\sum_{i=0}^d k_{\beta(i+1)}p^i$ and $n_f=\sum_{i=0}^d k_ip^i=\sum_{i=0}^d k_{\widetilde{\beta}(i+1)}p^i$. Put $\varrho=\prod_{i=0}^d(k_i!)=\prod_{i=0}^d(k_{\beta(i+1)}!)=\prod_{i=0}^d(k_{\widetilde{\beta}(i+1)}!)$. Let $0\le s_e,s_f\le p-2$ be such that $\lambda(\tau(x))=x^{-s_e}$ and $\lambda(\dot{u}\tau(x)\dot{u}^{-1})=x^{-s_f}$ for all $x\in\mathbb{F}_p^\times$.

Lemma 4.2. *The assigment $M[\lambda,\mathcal{J},b]\mapsto(n_e,s_e,b\varrho^{-1})$ induces a bijection between the set of isomorphism classes of quasi supersingular $\mathcal{H}(G,I_0)_k$-modules and $\mathfrak{S}_C(d+1)$.*

Proof: We have $\prod_{i=0}^d\alpha_i^\vee(x)=1$ for all $x\in\mathbb{F}_p^\times$ (as can be seen e.g. from formula (12)). This implies

$$\sum_{i=0}^d k_i\equiv n_e\equiv n_f\equiv 0\mod(p-1).\tag{13}$$

One can deduce from [7] Proposition 3 that for two sets of data $\lambda,\mathcal{J},b$ and $\lambda',\mathcal{J}',b'$ the $\mathcal{H}(G,I_0)_k$-modules $M[\lambda,\mathcal{J},b]$ and $M[\lambda',\mathcal{J}',b']$ are isomorphic if and only if $b=b'$ and the pair $(\lambda,\mathcal{J})$ is conjugate with the pair $(\lambda',\mathcal{J}')$ by means of a power of $\dot{u}$, i.e. by means of $\dot{u}^0=1$ or $\dot{u}^1=\dot{u}$. Conjugating $(\lambda,\mathcal{J})$ by $\dot{u}$ has the effect of substituting k_{d-i} with

k_i, for any i. The datum of the character λ is equivalent with the datum of s_e together with all the k_i taken modulo $(p-1)$ since the images of τ and all $\alpha_i^\vee$ together generate $\overline{T}$. Knowing the set $\mathcal{J}$ is then equivalent with knowing the numbers k_i themselves (not just modulo $(p-1)$). Thus, our mapping is well defined and bijective. $\square$

Let $0 \leq j \leq d$ and recall the homomorphism $\iota_{\alpha_j} : \mathrm{SL}_2(\mathbb{Q}_p) \to G$. Let $t_j = \iota_{\alpha_j}([\nu])-1 \in k[[\iota_{\alpha_j}\mathfrak{N}_0]] \subset k[[N_0]]$. Let F_j denote the codimension-1-face of C contained in the (affine) reflection hyperplane (in $A \subset X$) for s_j.

Lemma 4.3. *In $\mathcal{V}_M^X(F_j)$ we have $t_j^{k_j}\dot{s}_j e = k_j! e$ and $t_j^{k_{d-j}}\dot{s}_j f = k_{d-j}! f$ for all $0 \leq j \leq d$.*

Proof: This reduces to a computation in $\mathcal{V}_M^X(F_j)$, viewed as an $\mathrm{SL}_2(\mathbb{F}_p)$-representation. Namely, the analog of Lemma 8.2 of [3] holds verbatim in the present context as well (compare with Proposition 5.1 of [3]); the computation thus follows from Lemma 2.5 in [3]. (Compare with the proof of Proposition 8.4 of [3].) Notice that as the Hecke operator T_t for $t \in \overline{T}$ acts on $k.e$ through $\lambda(t^{-1})$, it acts on $k.f = k.T_{\dot{u}}e$ through $\lambda(\dot{u}t^{-1}\dot{u}^{-1})$ (the same computation as in formula (18) below), and that formula (9) implies $\lambda(\dot{u}\alpha_j^\vee(x)\dot{u}^{-1}) = x^{k_{d-j}}$. $\square$

Lemma 4.4. *In $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ we have*

$$t^{n_e}\varphi^{d+1}e = \varrho b^{-1}e, \tag{14}$$

$$t^{n_f}\varphi^{d+1}f = \varrho b^{-1}f, \tag{15}$$

$$\gamma(x)e = x^{-s_e}e, \tag{16}$$

$$\gamma(x)f = x^{-s_f}f \tag{17}$$

for $x \in \mathbb{F}_p^\times$. The action of Γ_0 on $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ is trivial on the subspace M.

Proof: We use the notations and the statements of Lemma 3.2, observing $\beta(i+1) = d-i$. For $0 \leq i \leq d$ we have $y_{i-1}\cdots y_0 F_{d-i} = C^{(i)} \cap C^{(i+1)}$ and $y_{i-1}\cdots y_0 C = C^{(i)}$. Thus $y_{i-1}\cdots y_0$ defines an isomorphism

$$\mathcal{V}_M^X(F_{d-i}) \cong \mathcal{V}_M^X(C^{(i)} \cap C^{(i+1)}) = \Theta_*\mathcal{V}_M(\mathfrak{v}_i),$$

restricting to an isomorphism $\mathcal{V}_M^X(C) \cong \mathcal{V}_M^X(C^{(i)}) = \Theta_*\mathcal{V}_M(\mathfrak{e}_i)$. Under this isomorphism, the action of t_{d-i}, resp. of $\dot{s}_{d-i}$, on $\mathcal{V}_M^X(F_{d-i})$ becomes the action of $[\nu]^{p^i}-1$, resp. of y_i, on $\Theta_*\mathcal{V}_M(\mathfrak{v}_i)$. Now as we are in characteristic p we have $t^{p^i} = ([\nu]-1)^{p^i} = [\nu]^{p^i}-1$. Applying this to the element e, resp. f, of $\mathcal{V}_M^X(C) \subset \mathcal{V}_M^X(F_{d-i})$, Lemma 4.3 tells us

$$(t^{p^i})^{k_{d-i}}y_i\cdots y_0 e = k_{d-i}! y_{i-1}\cdots y_0 e \quad \text{resp.} \quad (t^{p^i})^{k_i}y_i\cdots y_0 f = k_i! y_{i-1}\cdots y_0 f.$$

We compose these formulae for all $0 \leq i \leq d$ and finally recall that the central element $p \cdot \mathrm{id}$ acts on M through the Hecke operator $T_{p^{-1}\cdot \mathrm{id}}$, i.e. by b^{-1}. We get formulae (15) and (14).

Next recall that the action of $\gamma(x)$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ is given by that of $\tau(x) \in T$, i.e. by the Hecke operator $T_{\tau(x)^{-1}}$. We thus compute

$$\gamma(x)e = T_{\tau(x)^{-1}}e = \lambda(\tau(x))e,$$

(18)

$$\gamma(x)f = T_{\tau(x)^{-1}}T_{\dot{u}}e = T_{\dot{u}\tau(x)^{-1}}e = T_{\dot{u}}T_{\dot{u}\tau(x)^{-1}\dot{u}^{-1}}e = T_{\dot{u}}\lambda(\dot{u}\tau(x)\dot{u}^{-1})e = \lambda(\dot{u}\tau(x)\dot{u}^{-1})f$$

and obtain formulae (16) and (17). $\square$

Corollary 4.5. *The étale (φ^{d+1}, Γ)-module $\mathbf{D}(\Theta_* \mathcal{V}_M)$ over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ admits a $k_{\mathcal{E}}$-basis g_e, g_f such that*

$$\begin{aligned}
\varphi^{d+1} g_e &= b\varrho^{-1} t^{n_e+1-p^{d+1}} g_e \\
\varphi^{d+1} g_f &= b\varrho^{-1} t^{n_f+1-p^{d+1}} g_f \\
\gamma(x) g_e - x^{s_e} g_e &\in t \cdot k_{\mathcal{E}}^{+} \cdot g_e \\
\gamma(x) g_f - x^{s_f} g_f &\in t \cdot k_{\mathcal{E}}^{+} \cdot g_f.
\end{aligned}$$

PROOF: This follows from Lemma 4.4 as explained in [3] Lemma 6.4. $\square$

Corollary 4.6. *The functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ induces a bijection between*

(a) the set of isomorphism classes of quasi supersingular $\mathcal{H}(G, I_0)_k$-modules, and

(b) the set of isomorphism classes of C-symmetric étale (φ^{d+1}, Γ)-modules over $k_{\mathcal{E}}$.

PROOF: For $x \in \mathbb{F}_p^{\times}$ we have

$$\tau(x) \cdot \dot{u}\tau^{-1}(x)\dot{u}^{-1} = \mathrm{diag}(xE_d, x^{-1}E_d) = (\sum_{i=0}^{d}(i+1)\alpha_i^{\vee})(x)$$

in $\overline{T}$. Applying λ and observing $\sum_{i=0}^{d} k_i \equiv 0$ modulo $(p-1)$ we get

$$x^{s_f - s_e} = \lambda((\sum_{i=0}^{d}(i+1)\alpha_i^{\vee})(x)) = x^{\sum_{i=0}^{d} ik_i}$$

and hence $s_f - s_e \equiv \sum_{i=0}^{d} ik_i$ modulo $(p-1)$. Together with Corollary (4.5) we see that $\mathbf{D}(\Theta_* \mathcal{V}_M)$ is a C-symmetric étale (φ^{d+1}, Γ)-module over $k_{\mathcal{E}}$. Now we conclude with Lemmata 2.5 and 4.2. $\square$

Remark: Consider the subgroup $G' = \mathrm{Sp}_{2d}(\mathbb{Q}_p)$ of G. If we replace the above τ by $\tau : \mathbb{Z}_p^\times \longrightarrow T_0$, $x \mapsto \mathrm{diag}(xE_d, x^{-1}E_d)$ and if we replace the above ϕ by $\phi = \dot{s}_d\dot{s}_{d-1}\cdots\dot{s}_1\dot{s}_0$ then everything in fact happens inside G'. We then have $\alpha^{(j)} \circ \tau = \mathrm{id}^2_{\mathbb{Z}_p^\times}$ for all $j \geq 0$. Let $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ denote the category of finite-$\mathfrak{o}$-length $\mathcal{H}(G', G' \cap I_0)$-modules on which $\tau(-1)$ (i.e. $T_{\tau(-1)} = T_{\tau(-1)^{-1}}$) acts trivially. For $M \in \mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ we obtain an action of $\lfloor \mathfrak{N}_0, \varphi^{d+1}, \Gamma^2 \rfloor$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$, where $\Gamma^2 = \{\gamma^2 \,|\, \gamma \in \Gamma\} \subset \Gamma$. Correspondingly, following [3] (as a slight variation from what we explained in subsection 3.2), we obtain a functor from $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ to the category of $(\varphi^{d+1}, \Gamma^2)$-modules over $\mathcal{O}_{\mathcal{E}}$.

Remark: In the case $d = 2$ one may also work with $\phi = (p \cdot \mathrm{id})\dot{s}_2\dot{s}_1\dot{s}_2\dot{u}$. Its square is the square of the $\phi = (p \cdot \mathrm{id})\dot{s}_2\dot{s}_1\dot{s}_0$ used above.

Remark: We discuss a choice of $(C^{(\bullet)}, \phi)$ with $\ell(\phi) = d$ (but leading only to (φ^d, Γ_0)-modules, not to (φ^d, Γ)-modules). In $N(T)$ we consider the element $\phi = \dot{s}_1\dot{s}_2\cdots\dot{s}_{d-1}\dot{s}_d\dot{u}$. For $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < d$ put $C^{(ad+b)} = \phi^a s_1 \cdots s_b C$. A matrix computation shows $\phi^2 = \mathrm{diag}(p^2, pE_{d-1}, 1, pE_{d-1})$. Using this we find

$$\phi^2 N_0 \phi^{-2} = \prod_{\alpha \in \Phi^+} \phi^2(N_0 \cap N_\alpha)\phi^{-2} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : \quad \alpha = e_i + e_j \text{ with } i < j < d \\ 1 & : \quad \alpha = e_i + e_d \text{ with } i < d \\ 1 & : \quad \alpha = e_i - e_d \text{ with } i < d \\ 0 & : \quad \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^2 N_0 \phi^{-2} \subset N_0$ and $[N_0 : \phi^2 N_0 \phi^{-2}] = p^{2d}$. This implies that the length of $\phi^m \in \widehat{W}$ is at least dm, for all $m \geq 0$. On the other hand this length is at most dm because ϕ is a product of d simple reflections and of an element of length 0. Thus ϕ^m has length dm. Therefore the sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots$ satisfies hypothesis (5).

4.2 Affine root system $\tilde{B}_d$

Assume $d \geq 3$. Here W_{aff} is the Coxeter group with Coxeter generators $s_0, s_1, \ldots, s_d$ (thus $s_i^2 = 1$ for all i) and relations

(19) $\quad (s_d s_{d-1})^4 = 1 \quad$ and $\quad (s_2 s_0)^3 = (s_{i-1}s_i)^3 = 1 \quad$ for $2 \leq i \leq d-1$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i \neq j$. In the extended affine Weyl group $\widehat{W}$ we find (cf. [4]) an element u of length 0 with

(20) $\quad u^2 = 1 \quad$ and $\quad us_0u = s_1 \quad$ and $\quad us_iu = s_i \quad$ for $2 \leq i \leq d$.

($\widehat{W}$ is the semidirect product of its two-element subgroup $W_\Omega = \{1, u\}$ with W_{aff}.) Let

$$\widetilde{O}_d = \begin{pmatrix} S_d & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{2d+1}(\mathbb{Q}_p)$$

and consider the special orthogonal group

$$G = \mathrm{SO}_{2d+1}(\mathbb{Q}_p) = \{A \in \mathrm{SL}_{2d+1}(\mathbb{Q}_p) \,|\, {}^T A \widetilde{O}_d A = \widetilde{O}_d\}.$$

Let T denote the maximal torus consisting of all diagonal matrices in G. For $1 \le i \le d$ let

$$e_i : T \longrightarrow \mathbb{Q}_p^\times, \quad \mathrm{diag}(x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}, 1) \mapsto x_i.$$

Then (in *additive* notation) $\Phi = \{\pm e_i \pm e_j \,|\, i \ne j\} \cup \{\pm e_i\}$ is the root system of G with respect to T. It is of type B_d. For $\alpha \in \Phi$ let N_α^0 be the subgroup of the corresponding root subgroup N_α of G all of which elements belong to $\mathrm{SL}_{2d+1}(\mathbb{Z}_p)$.

We choose the positive system $\Phi^+ = \{e_i \pm e_j \,|\, i < j\} \cup \{e_i \,|\, 1 \le i \le d\}$ with corresponding set of simple roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = e_d\}$. The negative of the highest root is $\alpha_0 = -e_1 - e_2$. For $0 \le i \le d$ let $s_i = s_{\alpha_i}$ be the reflection corresponding to α_i.

Remark: For roots $\alpha \in \Phi$ of the form $\alpha = \pm e_i \pm e_j$ the homomorphism $\iota_\alpha : \mathrm{SL}_2 \to \mathrm{SO}_{2d+1}$ is injective. For roots $\alpha \in \Phi$ of the form $\alpha = \pm e_i$ the homomorphism $\iota_\alpha : \mathrm{SL}_2 \to \mathrm{SO}_{2d+1}$ induces an embedding $\mathrm{PSL}_2 \to \mathrm{SO}_{2d+1}$.

Remark: For $0 \le i \le d$ we have the following explicit formula for $\alpha_i^\vee = (\alpha_i)^\vee$:

$$\alpha_i^\vee(x) = \begin{cases} \mathrm{diag}(x^{-1}, x^{-1}, E_{d-2}, x, x, E_{d-2}, 1) & : \quad i = 0 \\ \mathrm{diag}(E_{i-1}, x, x^{-1}, E_{d-i-1}, E_{i-1}, x^{-1}, x, E_{d-i-1}, 1) & : \quad 1 \le i \le d-1 \\ \mathrm{diag}(E_{d-1}, x^2, E_{d-1}, x^{-2}, 1) & : \quad i = d \end{cases} \tag{21}$$

Let I_0 denote the pro-p-Iwahori subgroup generated by the N_α^0 for all $\alpha \in \Phi^+$, by the $(N_\alpha^0)^p$ for all $\alpha \in \Phi^- = \Phi - \Phi^+$, and by the maximal pro-p-subgroup of T_0. Let I denote the Iwahori subgroup of G containing I_0. Let N_0 be the subgroup of G generated by all N_α^0 for $\alpha \in \Phi^+$.

For $1 \le i \le d-1$ define the block diagonal matrix

$$\dot{s}_i = \mathrm{diag}(E_{i-1}, S_1, E_{d-i-1}, E_{i-1}, S_1, E_{d-i-1}, 1)$$

and furthermore

$$\dot{s}_d = \begin{pmatrix} E_{d-1} & & & & \\ & & & 1 & \\ & & E_{d-1} & & \\ & 1 & & & \\ & & & & -1 \end{pmatrix}.$$

Define

$$\dot{u} = \begin{pmatrix} & & p^{-1} & & \\ & E_{d-1} & & & \\ p & & & & \\ & & & E_{d-1} & \\ & & & & -1 \end{pmatrix}$$

and $\dot{s}_0 = \dot{u}\dot{s}_1\dot{u}$. Then $\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_{d-1}, \dot{s}_d$ belong to G and normalize T. Their images $s_0, s_1, \ldots, s_{d-1}, s_d$ in $N(T)/T_0$ are Coxeter generators of $W_{\rm aff} \subset N(T)/T_0$ satisfying the relations (19). The element $\dot{u}$ of $N(T)$ normalizes I and I_0. The image u of $\dot{u}$ in $N(T)/T_0 = \widehat{W}$ satisfies the formulae (20). In $N(T)$ we consider the element

$$\phi = \dot{s}_1\dot{s}_2 \cdots \dot{s}_{d-1}\dot{s}_d\dot{s}_{d-1} \cdots \dot{s}_2\dot{s}_0. \tag{22}$$

We may rewrite this as $\phi = \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(2d-1)}$ where we put $\beta(i) = i$ for $1 \le i \le d$ and $\beta(i) = 2d - i$ for $d \le i \le 2d-2$ and $\beta(2d-1) = 0$. We put

$$C^{(a(2d-1)+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\ge 0}$ with $0 \le b < 2d-1$. Define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \qquad x \mapsto \mathrm{diag}(x, E_{d-1}, x^{-1}, E_{d-1}, 1).$$

Lemma 4.7. *We have $\phi^2 \in T$ and $\phi^2 N_0 \phi^{-2} \subset N_0$. The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots$ satisfies hypothesis (5). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \ge 0$.*

(b) For any $j \ge 0$ we have $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}$.

(c) We have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$.

Proof: (a) A matrix computation shows $\phi^2 = \mathrm{diag}(p^2, E_{d-1}, p^{-2}, E_{d-1}, 1)$. Using this we find

$$\phi^2 N_0 \phi^{-2} = \prod_{\alpha \in \Phi^+} \phi^2 (N_0 \cap N_\alpha) \phi^{-2} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : \quad \alpha = e_1 - e_i \text{ with } 1 < i \\ 2 & : \quad \alpha = e_1 + e_i \text{ with } 1 < i \\ 2 & : \quad \alpha = e_1 \\ 0 & : \quad \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^2 N_0 \phi^{-2} \subset N_0$ and $[N_0 : \phi^2 N_0 \phi^{-2}] = p^{2(2d-1)}$. This implies that the length of $\phi^m \in \widehat{W}$ is at least $(2d-1)m$, for all $m \ge 0$. On the other hand this length is at most $(2d-1)m$ because the image of ϕ in $\widehat{W}$ is a product of $2d-1$ Coxeter generators. Thus ϕ^m has length $(2d-1)m$. We obtain that hypothesis (5) holds true, by the same reasoning as in Lemma 4.1.

(b) As $\phi^2 \in T$ we have $\{\alpha^{(j)}\,|\,j \geq 0\} = \{\alpha \in \Phi^+\,|\,m_\alpha \neq 0\}$. This implies (b).

(c) Another matrix computation. $\square$

As explained in subsection 3.2 we now obtain a functor from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{2d-1}, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

Suppose we are given a character $\lambda : \overline{T} \to k^\times$ and a subset $\mathcal{J} \subset S_\lambda$. Define the numbers $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ as in subsection 3.2. Define the $\mathcal{H}(G, I_0)_k$-module

$$M = M[\lambda, \mathcal{J}] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G,I_0)_{\mathrm{aff},k}} k.e$$

where $k.e$ denotes the one dimensional k-vector space on the basis element e, endowed with the action of $\mathcal{H}(G, I_0)_{\mathrm{aff},k}$ by the character $\chi_{\lambda,\mathcal{J}}$. As a k-vector space, M has dimension 2, a k-basis is e, f where we write $e = 1 \otimes e$ and $f = T_{\dot{u}} \otimes e$.

Definition: We call an $\mathcal{H}(G, I_0)_k$-module *quasi supersingular* if it is isomorphic with $M[\lambda, \mathcal{J}]$ for some $\lambda, \mathcal{J}$ such that $k_i > 0$ for at least one i.

For $2 \leq j \leq d$ we put $\widetilde{j} = j$, furthermore we put $\widetilde{0} = 1$ and $\widetilde{1} = 0$. Letting $\widetilde{\beta} = \widetilde{(.)} \circ \beta$ we then have

$$\dot{u}\phi\dot{u}^{-1} = \dot{s}_{\widetilde{\beta}(1)} \cdots \dot{s}_{\widetilde{\beta}(2d-1)}.$$

Put $n_e = \sum_{i=0}^{2d-2} k_{\beta(i+1)} p^i$ and $n_f = \sum_{i=0}^{2d-2} k_{\widetilde{\beta}i+1} p^i$. Put $\varrho = k_0!k_1!k_d! \prod_{i=2}^{d-1}(k_i!)^2 = \prod_{i=0}^{2d-2}(k_{\beta(i+1)}!) = \prod_{i=0}^{2d-2}(k_{\widetilde{\beta}(i+1)}!)$. Let $0 \leq s_e, s_f \leq p-2$ be such that $\lambda(\tau(x)) = x^{-s_e}$ and $\lambda(\dot{u}\tau(x)\dot{u}^{-1}) = x^{-s_f}$ for all $x \in \mathbb{F}_p^\times$.

Lemma 4.8. *The assigment $M[\lambda, \mathcal{J}] \mapsto (n_e, s_e)$ induces a bijection between the set of isomorphism classes of quasi supersingular $\mathcal{H}(G, I_0)_k$-modules and $\mathfrak{S}_B(2d-1)$.*

Proof: We have $\alpha_0^\vee(x)\alpha_1^\vee(x)\alpha_d^\vee(x) \prod_{i=2}^{d-1}(\alpha_i^\vee)^2(x) = 1$ for all $x \in \mathbb{F}_p^\times$ (as can be seen e.g. from formula (21)). This implies

$$k_0 + k_1 + k_d + 2\sum_{i=2}^{d-1} k_i \equiv n_e \equiv n_f \equiv 0 \mod (p-1). \tag{23}$$

We further proceed exactly as in the proof of Lemma 4.2. $\square$

Lemma 4.9. *In $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ we have*

$$t^{n_e}\varphi^{2d-1}e = \varrho e, \tag{24}$$

$$t^{n_f}\varphi^{2d-1}f = \varrho f, \tag{25}$$

$$\gamma(x)e = x^{-s_e}e, \tag{26}$$

$$\gamma(x)f = x^{-s_f}f \tag{27}$$

for $x \in \mathbb{F}_p^\times$. The action of Γ_0 on $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ is trivial on the subspace M.

Proof: As in Lemma 4.4. □

Corollary 4.10. *The étale (φ^{2d-1}, Γ)-module over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ admits a $k_{\mathcal{E}}$-basis g_e, g_f such that*

$$\varphi^{2d-1}g_e = \varrho^{-1}t^{n_e+1-p^{2d-1}}g_e$$
$$\varphi^{2d-1}g_f = \varrho^{-1}t^{n_f+1-p^{2d-1}}g_f$$
$$\gamma(x)g_e - x^{s_e}g_e \in t \cdot k_{\mathcal{E}}^+ \cdot g_e$$
$$\gamma(x)g_f - x^{s_f}g_f \in t \cdot k_{\mathcal{E}}^+ \cdot g_f.$$

Proof: This follows from Lemma 4.9 as explained in [3] Lemma 6.4. □

Corollary 4.11. *The functor $M \mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ induces a bijection between*
(a) the set of isomorphism classes of quasi supersingular $\mathcal{H}(G, I_0)_k$-modules, and
(b) the set of isomorphism classes of B-symmetric étale (φ^{2d-1}, Γ)-modules $\mathbf{D}$ over $k_{\mathcal{E}}$.

Proof: For $x \in \mathbb{F}_p^\times$ we compute

$$\tau(x) \cdot \dot{u}\tau^{-1}(x)\dot{u}^{-1} = \mathrm{diag}(x^2, E_{d-1}, x^{-2}, E_{d-1}, 1) = (\alpha_1^\vee - \alpha_0^\vee)(x)$$

in $\overline{T}$. Application of λ gives $x^{s_f - s_e} = x^{k_1 - k_0}$ and hence $s_f - s_e \equiv k_1 - k_0 = k_{\beta(1)} - k_{\beta(2d-1)}$ modulo $(p-1)$. The required symmetry in the p-adic digits of n_e, n_f is due to the corresponding symmetry of the function β. Thus, $\mathbf{D}(\Theta_*\mathcal{V}_M)$ is a B-symmetric étale (φ^{2d-1}, Γ)-module. Now we conclude with Lemmata 4.8 and 2.5. □

Remark: We discuss a choice of $(C^{(\bullet)}, \phi)$ with $\ell(\phi) = d$ (but leading only to (φ^d, Γ_0)-modules, not to (φ^d, Γ)-modules). In $N(T)$ consider the element $\phi = \dot{s}_d \cdots \dot{s}_1\dot{u}$. For $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < d$ we put $C^{(ad+b)} = \phi^a s_d \cdots s_{d-b+1}C$. A matrix computation shows $\phi^d = \mathrm{diag}(pE_d, p^{-1}E_d, 1)$. Using this we find

$$\phi^d N_0 \phi^{-d} = \prod_{\alpha \in \Phi^+} \phi^d(N_0 \cap N_\alpha)\phi^{-d} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : \quad \alpha = e_i + e_j \text{ with } i < j \\ 1 & : \quad \alpha = e_i \\ 0 & : \quad \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^d N_0 \phi^{-d} \subset N_0$ and $[N_0 : \phi^d N_0 \phi^{-d}] = p^{d^2}$. This implies that the length of $\phi^m \in \widehat{W}$ is at least dm, for all $m \geq 0$. On the other hand this length is at most dm because ϕ is a product of d simple reflections and of an element of length 0. Thus ϕ^m has length dm. Therefore the sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots$ satisfies hypothesis (5).

Remark: $\mathrm{SO}_{2d+1} \cong \mathrm{PGSpin}_{2d+1}$ is semisimple, of adjoint type. Like SO_{2d+1} also GSpin_{2d+1} has connected center; its derived group is isomorphic with Spin_{2d+1}, the simply connected double covering of SO_{2d+1}. Using the concrete description of GSpin_{2d+1} given e.g. in [1], section 2, it is straightforward to extend our constructions from SO_{2d+1} to GSpin_{2d+1}. (Like in our treatment of the group GSp_{2d}, the non trivial center of GSpin_{2d+1} allows us to twist our ϕ by a suitable central element — in this way, the action of the center on an $\mathcal{H}_k$-module defines a twist of the φ^{2d-1}-action on the corresponding (φ^{2d-1}, Γ)-module.)

4.3 Affine root system $\tilde{D}_d$

Assume $d \geq 4$. Here W_{aff} is the Coxeter group with Coxeter generators $s_0, s_1, \ldots, s_d$ (thus $s_i^2 = 1$ for all i) and relations

$$(s_{d-2}s_d)^3 = (s_2 s_0)^3 = (s_{i-1}s_i)^3 = 1 \quad \text{for } 2 \leq i \leq d-1 \tag{28}$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i \neq j$. In the extended affine Weyl group $\widehat{W}$ we find (cf. [4]) an element u of length 0 with

$$u^2 = 1 \qquad \text{and} \qquad us_0u = s_1, \quad us_1u = s_0, \quad us_{d-1}u = s_d, \quad us_du = s_{d-1}$$
$$us_iu = s_i \quad \text{for } 2 \leq i \leq d-2. \tag{29}$$

($\widehat{W}$ is the semidirect product of a four-element subgroup W_Ω with W_{aff}, in such a way that u is an element of order 2 in W_Ω.) Consider the general orthogonal group

$$\mathrm{GO}_{2d}(\mathbb{Q}_p) = \{A \in \mathrm{GL}_{2d}(\mathbb{Q}_p) \,|\, {}^T A S_d A = \kappa(A) S_d \text{ for some } \kappa(A) \in \mathbb{Q}_p^\times\}.$$

It contains the special orthogonal group

$$\mathrm{SO}_{2d}(\mathbb{Q}_p) = \{A \in \mathrm{SL}_{2d}(\mathbb{Q}_p) \,|\, {}^T A S_d A = S_d\}.$$

Let $G = \mathrm{GSO}_{2d}(\mathbb{Q}_p)$ be the connected component of $\mathrm{GO}_{2d}(\mathbb{Q}_p)$. It has connected center and is of index 2 in $\mathrm{GO}_{2d}(\mathbb{Q}_p)$. Explicitly, G is the subgroup generated by $\mathrm{SO}_{2d}(\mathbb{Q}_p)$ and by all $\mathrm{diag}(xE_d, E_d)$ with $x \in \mathbb{Q}_p^\times$.

Let T be the maximal torus consisting of all diagonal matrices in G. For $1 \leq i \leq d$ let

$$e_i : T \cap \mathrm{SL}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times, \quad A = \mathrm{diag}(x_1, \ldots, x_{2d}) \mapsto x_i.$$

For $1 \leq i, j \leq d$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ we thus obtain characters (using additive notation as usual) $\epsilon_1 e_i + \epsilon_2 e_j : T \cap \mathrm{SL}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$. We extend these latter ones to T by setting

$$\epsilon_1 e_i + \epsilon_2 e_j : T \longrightarrow \mathbb{Q}_p^\times, \quad A = \mathrm{diag}(x_1, \ldots, x_{2d}) \mapsto x_i^{\epsilon_1} x_j^{\epsilon_2} \kappa(A)^{\frac{-\epsilon_1 - \epsilon_2}{2}}.$$

Then $\Phi = \{\pm e_i \pm e_j \,|\, i \neq j\}$ is the root system of G with respect to T. It is of type D_d.

For $\alpha \in \Phi$ let N_α^0 be the subgroup of the corresponding root subgroup N_α of G all of which elements belong to $\mathrm{SL}_{2d}(\mathbb{Z}_p)$.

Choose the positive system $\Phi^+ = \{e_i \pm e_j \,|\, i < j\}$ with corresponding set of simple roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = e_{d-1} + e_d\}$. The negative of the highest root is $\alpha_0 = -e_1 - e_2$. For $0 \leq i \leq d$ let $s_i = s_{\alpha_i}$ be the reflection corresponding to α_i.

Remark: For $0 \leq i \leq d$ we have the following explicit formula for $\alpha_i^\vee = (\alpha_i)^\vee$:

$$(30) \quad \alpha_i^\vee(x) = \begin{cases} \mathrm{diag}(x^{-1}, x^{-1}, E_{d-2}, x, x, E_{d-2}) & : \quad i = 0 \\ \mathrm{diag}(E_{i-1}, x, x^{-1}, E_{d-i-1}, E_{i-1}, x^{-1}, x, E_{d-i-1}) & : \quad 1 \leq i \leq d-1 \\ \mathrm{diag}(E_{d-2}, x, x, E_{d-2}, x^{-1}, x^{-1}) & : \quad i = d \end{cases}$$

Let I_0 denote the pro-p-Iwahori subgroup generated by the N_α^0 for all $\alpha \in \Phi^+$, by the $(N_\alpha^0)^p$ for all $\alpha \in \Phi^- = \Phi - \Phi^+$, and by the maximal pro-p-subgroup of T_0. Let I denote the Iwahori subgroup of G containing I_0. Let N_0 be the subgroup of G generated by all N_α^0 for $\alpha \in \Phi^+$.

For $1 \leq i \leq d-1$ define the block diagonal matrix

$$\dot{s}_i = \mathrm{diag}(E_{i-1}, S_1, E_{d-i-1}, E_{i-1}, S_1, E_{d-i-1}) = \mathrm{diag}(E_{i-1}, S_1, E_{d-2}, S_1, E_{d-i-1}).$$

Put

$$\dot{u} = \begin{pmatrix} & & & p^{-1} & & \\ & E_{d-2} & & & & \\ & & & & & 1 \\ p & & & & & \\ & & & & E_{d-2} & \\ & & 1 & & & \end{pmatrix}$$

and $\dot{s}_0 = \dot{u}\dot{s}_1\dot{u}$ and $\dot{s}_d = \dot{u}\dot{s}_{d-1}\dot{u}$. Then $\dot{s}_0, \dot{s}_1, \ldots, \dot{s}_{d-1}, \dot{s}_d$ belong to G and normalize T. Their images $s_0, s_1, \ldots, s_{d-1}, s_d$ in $N(T)/ZT_0$ are Coxeter generators of $W_{\mathrm{aff}} \subset$

$N(T)/ZT_0 = \widehat{W}$ satisfying the relations (28). The element $\dot{u}$ of $N(T)$ normalizes I and I_0. The image u of $\dot{u}$ in $N(T)/ZT_0$ satisfies the formulae (29).

In $N(T)$ we consider the element

$$\phi = (p \cdot \mathrm{id})\dot{s}_{d-1}\dot{s}_{d-2}\cdots\dot{s}_2\dot{s}_1\dot{s}_d\dot{s}_{d-2}\dot{s}_{d-3}\cdots\dot{s}_3\dot{s}_2\dot{s}_0 \qquad \text{if } d \text{ is even,}$$
$$\phi = (p^2 \cdot \mathrm{id})\dot{s}_{d-1}\dot{s}_{d-2}\cdots\dot{s}_2\dot{s}_1\dot{s}_d\dot{s}_{d-2}\dot{s}_{d-3}\cdots\dot{s}_3\dot{s}_2\dot{s}_0 \qquad \text{if } d \text{ is odd.}$$

We may rewrite this as $\phi = (p{\cdot}\mathrm{id})\dot{s}_{\beta(1)}\cdots\dot{s}_{\beta(2d-2)}$ if d is even, resp. $\phi = (p^2{\cdot}\mathrm{id})\dot{s}_{\beta(1)}\cdots\dot{s}_{\beta(2d-2)}$ if d is odd, where $\beta(i) = d - i$ for $1 \leq i \leq d-1$, $\beta(d) = d$, $\beta(i) = 2d-1-i$ for $d+1 \leq i \leq 2d-3$ and $\beta(2d-2) = 0$. We put

$$C^{(a(2d-2)+b)} = \phi^a s_{\beta(1)}\cdots s_{\beta(b)}C$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < 2d-2$. Define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \qquad x \mapsto \mathrm{diag}(xE_{d-1}, E_d, x).$$

Lemma 4.12. *We have $\phi^d \in T$ and $\phi^d N_0 \phi^{-d} \subset N_0$. The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$ satisfies hypothesis (5). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \geq 0$.*

(b) For any $j \geq 0$ we have $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}$.

(c) We have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$.

Proof: (a) A matrix computation shows $\phi^d = \mathrm{diag}(p^{2d+2}E_{d-1}, p^{2d-2}E_d, p^{2d+2})$ if d is even, and $\phi^d = \mathrm{diag}(p^{4d+4}E_{d-1}, p^{4d-4}E_d, p^{4d+4})$ if d is odd. Using this we find

$$\phi^d N_0 \phi^{-d} = \prod_{\alpha \in \Phi^+} \phi^d(N_0 \cap N_\alpha)\phi^{-d} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 4 & : \quad \alpha = e_i + e_j \text{ with } 1 \leq i < j < d \\ 4 & : \quad \alpha = e_i - e_d \text{ with } 1 \leq i < d \\ 0 & : \quad \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^d N_0 \phi^{-d} \subset N_0$ and $[N_0 : \phi^d N_0 \phi^{-d}] = p^{2d(d-1)}$. This implies that the length of $\phi^m \in \widehat{W}$ is at least $2(d-1)m$, for all $m \geq 0$. On the other hand this length is at most $2(d-1)m$ because the image of ϕ in $\widehat{W}$ is a product of $2d-2$ Coxeter generators and of an element of length 0. Thus ϕ^m has length $2(d-1)m$. We obtain that hypothesis (5) holds true, by the same reasoning as in Lemma 4.1.

(b) As $\phi^d \in T$ we have $\{\alpha^{(j)} \mid j \geq 0\} = \{\alpha \in \Phi^+ \mid m_\alpha \neq 0\}$. This implies (b).

(c) Another matrix computation. $\square$

As explained in subsection 3.2 we now obtain a functor from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G,I_0))$ to the category of (φ^{2d-2},Γ)-modules over $\mathcal{O}_{\mathcal{E}}$. Consider the elements

$$\dot{\omega}=\begin{pmatrix} & E_d^* \\ pE_d^* & \end{pmatrix},\qquad \dot{\rho}=\begin{pmatrix} & & & E_{d-1}^* \\ p & & & \\ & pE_{d-1}^* & & \\ & & 1 & \end{pmatrix}$$

of $\mathrm{GO}_{2d}(\mathbb{Q}_p)$. They normalize T and satisfy

$$\begin{aligned}\dot{\omega}\dot{u}&=\dot{u}\dot{\omega},\\ \dot{\omega}\dot{s}_i\dot{\omega}^{-1}&=\dot{s}_{d-i}\quad \text{ for } 0\leq i\leq d,\\ \dot{\rho}^2&=p\cdot\dot{u},\\ \dot{\rho}\dot{s}_i\dot{\rho}^{-1}&=\dot{s}_{d-i}\quad \text{ for } 2\leq i\leq d-2,\end{aligned}$$
$$\dot{\rho}\dot{s}_{d-1}\dot{\rho}^{-1}=\dot{s}_1,\qquad \dot{\rho}\dot{s}_d\dot{\rho}^{-1}=\dot{s}_0,\qquad \dot{\rho}\dot{s}_0\dot{\rho}^{-1}=\dot{s}_{d-1},\qquad \dot{\rho}\dot{s}_1\dot{\rho}^{-1}=\dot{s}_d.$$

The element $\dot{\omega}$ belongs to G if and only if d is even. The element $\dot{\rho}$ belongs to G if and only if d is odd.

Let $\mathcal{H}(G,I_0)'_{\mathrm{aff},k}$ denote the k-sub algebra of $\mathcal{H}(G,I_0)_k$ generated by $\mathcal{H}(G,I_0)_{\mathrm{aff},k}$ together with $T_{p\cdot\mathrm{id}}=T_{\dot{\omega}^2}$ and $T_{p\cdot\mathrm{id}}^{-1}=T_{p^{-1}\cdot\mathrm{id}}$ if d is even, resp. $T_{p^2\cdot\mathrm{id}}=T_{\dot{\rho}^4}$ and $T_{p^2\cdot\mathrm{id}}^{-1}=T_{p^{-2}\cdot\mathrm{id}}$ if d is odd.

Suppose we are given a character $\lambda:\overline{T}\to k^\times$, a subset $\mathcal{J}\subset S_\lambda$ and some $b\in k^\times$. Define the numbers $0\leq k_i=k_i(\lambda,\mathcal{J})\leq p-1$ as in subsection 3.2. The character $\chi_{\lambda,\mathcal{J}}$ of $\mathcal{H}(G,I_0)_{\mathrm{aff},k}$ extends uniquely to a character

$$\chi_{\lambda,\mathcal{J},b}:\mathcal{H}(G,I_0)'_{\mathrm{aff},k}\longrightarrow k$$

which sends $T_{p\cdot\mathrm{id}}$ to b if d is even, resp. which sends $T_{p^2\cdot\mathrm{id}}$ to b if d odd (see the proof of [7] Proposition 3). We define the $\mathcal{H}(G,I_0)_k$-module

$$M=M[\lambda,\mathcal{J},b]=\mathcal{H}(G,I_0)_k\otimes_{\mathcal{H}(G,I_0)'_{\mathrm{aff},k}}k.e$$

where $k.e$ denotes the one dimensional k-vector space on the basis element e, endowed with the action of $\mathcal{H}(G,I_0)'_{\mathrm{aff},k}$ by the character $\chi_{\lambda,\mathcal{J},b}$. As a k-vector space, M has dimension 4. A k-basis is e_0,e_1,f_0,f_1 where we write

$$e_0=1\otimes e,\quad f_0=T_{\dot{u}}\otimes e,\quad e_1=T_{\dot{\omega}}\otimes e,\quad f_1=T_{\dot{u}\dot{\omega}}\otimes e\qquad\text{if } d \text{ is even},$$

$$e_0=1\otimes e,\quad f_0=T_{\dot{u}}\otimes e,\quad e_1=T_{\dot{\rho}}\otimes e,\quad f_1=T_{\dot{u}\dot{\rho}}\otimes e\qquad\text{if } d \text{ is odd}.$$

Definition: We call an $\mathcal{H}(G,I_0)_k$-module *quasi supersingular* if it is isomorphic with $M[\lambda,\mathcal{J},b]$ for some $\lambda,\mathcal{J},b$ such that $k_i>0$ for at least one i.

For $2\le j\le d-2$ let $\widetilde{j}=j$, and furthermore let $\widetilde{d-1}=d$ and $\widetilde{d}=d-1$ and $\widetilde{1}=0$ and $\widetilde{0}=1$. Letting $\widetilde{\beta}=\widetilde{(.)}\circ\beta$ we then have

$$\dot{u}\phi\dot{u}^{-1}=(p^n\cdot\mathrm{id})\dot{s}_{\widetilde{\beta}(1)}\cdots\dot{s}_{\widetilde{\beta}(2d-2)}$$

with $n=1$ if d is even, but $n=2$ if d is odd. If d is odd we consider in addition the following two maps γ and δ from $[1,2d-2]$ to $[0,d]$. We put $\gamma(1)=1$, $\gamma(d-1)=d$, $\gamma(d)=0$ and $\gamma(2d-2)=d-1$. We put $\delta(1)=0$, $\delta(d-1)=d-1$, $\delta(d)=1$ and $\delta(2d-2)=d$. We put $\gamma(i)=\delta(i)=\beta(2d-2-i)$ for all $i\in[1,\ldots,d-2]\cup[d+1,\ldots,2d-3]$. We then have

$$\dot{\varrho}\phi\dot{\varrho}^{-1}=(p^2\cdot\mathrm{id})\dot{s}_{\gamma(1)}\cdots\dot{s}_{\gamma(2d-2)},\qquad \dot{\varrho}^{-1}\phi\dot{\varrho}=(p^2\cdot\mathrm{id})\dot{s}_{\delta(1)}\cdots\dot{s}_{\delta(2d-2)}.$$

Put

$$n_{e_0}=\sum_{i=0}^{2d-3}k_{\beta(i+1)}p^i,\qquad n_{f_0}=\sum_{i=0}^{2d-3}k_{\widetilde{\beta}(i+1)}p^i\qquad\text{for any parity of }d,$$

$$n_{e_1}=\sum_{i=0}^{2d-3}k_{\beta(2d-2-i)}p^i,\qquad n_{f_1}=\sum_{i=0}^{2d-3}k_{\widetilde{\beta}(2d-2-i)}p^i\qquad\text{if }d\text{ is even},$$

$$n_{e_1}=\sum_{i=0}^{2d-3}k_{\gamma(i+1)}p^i,\qquad n_{f_1}=\sum_{i=0}^{2d-3}k_{\delta(i+1)}p^i\qquad\text{if }d\text{ is odd}.$$

Let $0\le s_{e_0},s_{f_0},s_{e_1},s_{f_1}\le p-2$ be such that for all $x\in\mathbb{F}_p^\times$ we have

$$\lambda(\tau(x))=x^{-s_{e_0}},\qquad \lambda(\dot{u}\tau(x)\dot{u}^{-1})=x^{-s_{f_0}}\qquad\text{for any parity of }d,$$

$$\lambda(\dot{\omega}\tau(x)\dot{\omega}^{-1})=x^{-s_{e_1}},\qquad \lambda(\dot{\omega}\dot{u}\tau(x)\dot{u}^{-1}\dot{\omega}^{-1})=x^{-s_{f_1}}\qquad\text{if }d\text{ is even},$$

$$\lambda(\dot{\rho}\tau(x)\dot{\rho}^{-1})=x^{-s_{e_1}},\qquad \lambda(\dot{\rho}\dot{u}\tau(x)\dot{u}^{-1}\dot{\rho}^{-1})=x^{-s_{f_1}}\qquad\text{if }d\text{ is odd}.$$

Put $\varrho=k_0!k_1!k_{d-1}!k_d!\prod_{i=2}^{d-2}(k_i!)^2=\prod_{i=0}^{2d-3}(k_{\beta(i+1)}!)=\prod_{i=0}^{2d-3}(k_{\widetilde{\beta}(i+1)}!)$.

Lemma 4.13. *The assigment $M[\lambda,\mathcal{J},b]\mapsto(n_{e_0},s_{e_0},b\varrho^{-1})$ induces a bijection between the set of isomorphism classes of quasi supersingular $\mathcal{H}(G,I_0)_k$-modules and $\mathfrak{S}_D(2d-2)$.*

Proof: We have $\alpha_0^\vee(x)\alpha_1^\vee(x)\alpha_{d-1}^\vee(x)\alpha_d^\vee(x)\prod_{i=2}^{d-2}(\alpha_i^\vee)^2(x)=1$ for all $x\in\mathbb{F}_p^\times$ (as can be seen e.g. from formula (30)). This implies

$$k_0+k_1+k_{d-1}+k_d+2\sum_{i=2}^{d-2}k_i\equiv n_{e_0}\equiv n_{f_0}\equiv n_{e_1}\equiv n_{f_1}\equiv 0\mod(p-1).\tag{31}$$

It follows from [7] Proposition 3 that $M[\lambda, \mathcal{J}, b]$ and $M[\lambda', \mathcal{J}', b']$ are isomorphic if and only if $b = b'$ and the pair $(\lambda, \mathcal{J})$ is conjugate with the pair $(\lambda', \mathcal{J}')$ by means of $\dot{u}^n \dot{\omega}^m$ for some $n, m \in \{0, 1\}$ (if d is even), resp. by means of $\dot{u}^n \dot{\rho}^m$ for some $n, m \in \{0, 1\}$ (if d is odd). Under the map $M[\lambda, \mathcal{J}, b] \mapsto (n_{e_0}, s_{e_0}, b\varrho^{-1})$, conjugation by $\dot{u}$ corresponds to the permutation ι_0 of $\widetilde{\mathfrak{S}}_D(2d-2)$, while conjugation by $\dot{\omega}$, resp. by $\dot{\rho}$, corresponds to the permutation ι_1 of $\widetilde{\mathfrak{S}}_D(2d-2)$. We may thus proceed as in the proof of Lemma 4.2 to see that our mapping is well defined and bijective. □

Lemma 4.14. *In $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ we have*

$$\begin{aligned} t^{n_{e_j}} \varphi^{2d-2} e_j &= \varrho b^{-1} e_j, \\ t^{n_{f_j}} \varphi^{2d-2} f_j &= \varrho b^{-1} f_j, \\ \gamma(x) e_j &= x^{-s_{e_j}} e_j, \\ \gamma(x) f_j &= x^{-s_{f_j}} f_j \end{aligned}$$

for $x \in \mathbb{F}_p^\times$ and $j = 0, 1$. The action of Γ_0 on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ is trivial on the subspace M.

PROOF: As in Lemma 4.4. □

Corollary 4.15. *The étale (φ^{2d-2}, Γ)-module over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ admits a $k_{\mathcal{E}}$-basis g_{e_0}, g_{f_0}, g_{e_1}, g_{f_1} such that for both $j = 0$ and $j = 1$ we have*

$$\begin{aligned} \varphi^{2d-2} g_{e_j} &= b\varrho^{-1} t^{n_{e_j}+1-p^{2d-2}} g_{e_j} \\ \varphi^{2d-2} g_{f_j} &= b\varrho^{-1} t^{n_{f_j}+1-p^{2d-2}} g_{f_j} \\ \gamma(x)(g_{e_j}) - x^{s_{e_j}} g_{e_j} &\in t \cdot k_{\mathcal{E}}^+ \cdot g_{e_j} \\ \gamma(x)(g_{f_j}) - x^{s_{f_j}} g_{f_j} &\in t \cdot k_{\mathcal{E}}^+ \cdot g_{f_j} \end{aligned}$$

PROOF: This follows from Lemma 4.14 as explained in [3] Lemma 6.4. □

Corollary 4.16. *The functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ induces a bijection between*

(a) the set of isomorphism classes of quasi supersingular $\mathcal{H}(G, I_0)_k$-modules, and

(b) the set of isomorphism classes of D-symmetric étale (φ^{2d-2}, Γ)-modules over $k_{\mathcal{E}}$.

PROOF: We let $\mathbf{D}_{11} = \langle g_{e_0} \rangle$, $\mathbf{D}_{12} = \langle g_{f_0} \rangle$, $\mathbf{D}_{21} = \langle g_{e_1} \rangle$, $\mathbf{D}_{22} = \langle g_{f_1} \rangle$. Then $k_i(\mathbf{D}_{11}) = k_{\beta(i+1)}$ and $k_i(\mathbf{D}_{12}) = k_{\widetilde{\beta}(i+1)}$; moreover $k_i(\mathbf{D}_{21}) = k_{\beta(2d-d-i)}$ and $k_i(\mathbf{D}_{22}) = k_{\widetilde{\beta}(2d-d-i)}$ if d is even, but $k_i(\mathbf{D}_{21}) = k_{\gamma(i+1)}$ and $k_i(\mathbf{D}_{22}) = k_{\delta(i+1)}$ if d is odd.

For the condition on $s_{f_0}-s_{e_0}=s(\mathbf{D}_{12})-s(\mathbf{D}_{11})$ we compute

$$\tau(x)\cdot\dot{u}\tau^{-1}(x)\dot{u}^{-1}=\mathrm{diag}(x,E_{d-2},x^{-1},x^{-1},E_{d-2},x)=(\sum_{i=1}^{d-1}\alpha_i^\vee)(x),$$

hence application of λ gives $x^{s_{f_0}-s_{e_0}}=x^{\sum_{i=1}^{d-1}k_i}$ and hence $s_{f_0}-s_{e_0}\equiv\sum_{i=1}^{d-1}k_i=\sum_{i=0}^{d-2}k_i(\mathbf{D}_{11})$ modulo $(p-1)$. The condition on $s_{f_1}-s_{e_1}=s(\mathbf{D}_{22})-s(\mathbf{D}_{21})$ in case d is even is exactly verified like the one for $s_{f_0}-s_{e_0}$ because $\dot{\omega}\tau(x)\dot{\omega}^{-1}\cdot\dot{\omega}\dot{u}\tau^{-1}(x)\dot{u}^{-1}\dot{\omega}^{-1}=\tau(x)\cdot\dot{u}\tau^{-1}(x)\dot{u}^{-1}$. In case d is odd the computation is

$$\dot{\rho}\tau(x)\dot{\rho}^{-1}\cdot\dot{\rho}\dot{u}\tau^{-1}(x)\dot{u}^{-1}\dot{\rho}^{-1}=\mathrm{diag}(x,E_{d-2},x,x^{-1},E_{d-2},x^{-1})=(\alpha_d^\vee+\sum_{i=1}^{d-2}\alpha_i^\vee)(x),$$

hence $s_{f_1}-s_{e_1}\equiv k_d+\sum_{i=1}^{d-2}k_i=s(\mathbf{D}_{12})-s(\mathbf{D}_{11})+k_d-k_{d-1}=k_{d-1}(\mathbf{D}_{11})-k_0(\mathbf{D}_{11})+\sum_{i=0}^{d-2}k_i(\mathbf{D}_{11})$ modulo $(p-1)$.

To see the condition on $s_{e_1}-s_{e_0}=s(\mathbf{D}_{21})-s(\mathbf{D}_{11})$ in case d is even we compute

$$\begin{aligned}\tau(x)\cdot\dot{\omega}\tau^{-1}(x)\dot{\omega}^{-1}&=\mathrm{diag}(1,xE_{d-2},1,1,x^{-1}E_{d-2},1)\\&=(\frac{d-2}{2}\alpha_{d-1}^\vee+\frac{d-2}{2}\alpha_d^\vee+\sum_{i=2}^{d-2}(i-1)\alpha_i^\vee)(x),\end{aligned}\tag{32}$$

hence application of λ gives $x^{s_{e_1}-s_{e_0}}=x^{\frac{d-2}{2}k_{d-1}+\frac{d-2}{2}k_d+\sum_{i=2}^{d-2}(i-1)k_i}$ and hence $s_{e_1}-s_{e_0}\equiv\frac{d-2}{2}k_{d-1}+\frac{d-2}{2}k_d+\sum_{i=2}^{d-2}(i-1)k_i=\frac{d-2}{2}(k_{d-1}(\mathbf{D}_{11})+k_0(\mathbf{D}_{11}))+\sum_{i=2}^{d-2}(i-1)k_{d-i-1}(\mathbf{D}_{11})$ modulo $(p-1)$. If however d is odd we compute

$$\begin{aligned}\tau(x)\cdot\dot{\rho}\tau^{-1}(x)\dot{\rho}^{-1}&=\mathrm{diag}(1,xE_{d-2},x^{-1},1,x^{-1}E_{d-2},x)\\&=(\frac{d-1}{2}\alpha_{d-1}^\vee+\frac{d-3}{2}\alpha_d^\vee+\sum_{i=2}^{d-2}(i-1)\alpha_i^\vee)(x),\end{aligned}$$

hence application of λ gives $x^{s_{e_1}-s_{e_0}}=x^{\frac{d-1}{2}k_{d-1}+\frac{d-3}{2}k_d+\sum_{i=2}^{d-2}(i-1)k_i}$ and hence $s_{e_1}-s_{e_0}\equiv\frac{d-1}{2}k_{d-1}+\frac{d-3}{2}k_d+\sum_{i=2}^{d-2}(i-1)k_i=\frac{d-3}{2}k_{\frac{\tau}{2}}(\mathbf{D}_{11})+\frac{d-1}{2}k_0(\mathbf{D}_{11})+\sum_{i=2}^{d-2}(i-1)k_{d-i-1}(\mathbf{D}_{11})$ modulo $(p-1)$. Now we conclude with Lemmata 4.13 and 2.6. $\square$

Remark: Consider the subgroup $G'=\mathrm{SO}_{2d}(\mathbb{Q}_p)$ of G. If we replace the above τ by $\tau:\mathbb{Z}_p^\times\longrightarrow T_0$, $x\mapsto\mathrm{diag}(xE_{d-1},x^{-1}E_d,x)$, and if we replace the above ϕ by $\phi=\dot{s}_{d-1}\dot{s}_{d-2}\cdots\dot{s}_2\dot{s}_1\dot{s}_d\dot{s}_{d-2}\dot{s}_{d-3}\cdots\dot{s}_3\dot{s}_2\dot{s}_0$, then everything in fact happens inside G', and there is no dichotomy between d even or odd. We then have $\alpha^{(j)}\circ\tau=\mathrm{id}^2_{\mathbb{Z}_p^\times}$ for all $j\geq 0$. Let $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G',G'\cap I_0)$ denote the category of finite-$\mathfrak{o}$-length $\mathcal{H}(G',G'\cap I_0)$-modules on which $\tau(-1)$ (i.e. $T_{\tau(-1)}=T_{\tau(-1)^{-1}}$) acts trivially. For $M\in\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G',G'\cap I_0)$ we obtain an action of $\lfloor \mathfrak{N}_0, \varphi^{2d-2}, \Gamma^2 \rfloor$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$, where $\Gamma^2 = \{\gamma^2 \mid \gamma \in \Gamma\} \subset \Gamma$. Correspondingly, we obtain a functor from $\mathrm{Mod}_0^{\mathrm{fin}} \mathcal{H}(G', G' \cap I_0)$ to the category of $(\varphi^{2d-2}, \Gamma^2)$-modules over $\mathcal{O}_{\mathcal{E}}$.

Remark: Instead of the element $\phi \in N(T)$ used above we might also work with the element $\dot{s}_{d-1} \cdots \dot{s}_2 \dot{s}_1 \dot{u}$ of length $d-1$ (or products of this with elements of $p^{\mathbb{Z}} \cdot \mathrm{id}$), keeping the same $C^{(\bullet)}$. This results in a functor from $\mathcal{H}(G, I_0)$-modules to (φ^{d-1}, Γ)-modules. Up to a factor in $p^{\mathbb{Z}} \cdot \mathrm{id}$, the square of $\dot{s}_{d-1} \cdots \dot{s}_2 \dot{s}_1 \dot{u}$ is the element ϕ used above.

Remark: For the affine root system of type D_d there are three co minuscule fundamental coweights (cf. [2] chapter 8, par 7.3]). We leave it to the reader to work out $(C^{(\bullet)}, \phi)$ correspondigng to the two other co minuscule fundamental coweights. (These ϕ's will be longer.)

Remark: We discuss a choice of $(C^{(\bullet)}, \phi)$ with $\ell(\phi) = d$ (but leading only to (φ^d, Γ_0)-modules, not to (φ^d, Γ)-modules). In $N(T)$ we consider $\phi = \dot{s}_d \cdots \dot{s}_1 \dot{u}$. For $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < d$ we put $C^{(ad+b)} = \phi^a s_d \cdots s_{d-b+1} C$. A matrix computation shows $\phi^{d-1} = \mathrm{diag}(pE_{d-1}, 1, p^{-1}E_{d-1}, 1)$. Using this we find

$$\phi^{d-1} N_0 \phi^{1-d} = \prod_{\alpha \in \Phi^+} \phi^{d-1}(N_0 \cap N_\alpha)\phi^{1-d} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : \quad \alpha = e_i + e_j \text{ with } i < j < d \\ 1 & : \quad \alpha = e_i + e_d \\ 1 & : \quad \alpha = e_i - e_d \\ 0 & : \quad \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^{d-1} N_0 \phi^{1-d} \subset N_0$ and $[N_0 : \phi^{d-1} N_0 \phi^{1-d}] = p^{d(d-1)}$. This implies that the length of $\phi^m \in \widehat{W}$ is at least dm, for all $m \geq 0$. On the other hand this length is at most dm because ϕ is a product of d simple reflections and of an element of length 0. Thus ϕ^m has length dm. Therefore the sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots$ satisfies hypothesis (5).

4.4 Affine root system $\tilde{A}_d$

Assume $d \geq 1$ and consider $G = \mathrm{GL}_{d+1}(\mathbb{Q}_p)$. Let

$$\dot{u} = \begin{pmatrix} & E_d \\ p & \end{pmatrix}.$$

For $1 \leq i \leq d$ let

$$\dot{s}_i = \mathrm{diag}(E_{i-1}, S_1, E_{d-i})$$

and let $\dot{s}_0 = \dot{u}\dot{s}_1\dot{u}^{-1}$. Let T be the maximal torus consisting of diagonal matrices. Let Φ^+ be such that $N = \prod_{\alpha\in\Phi^+} N_\alpha$ is the subgroup of upper triangular unipotent matrices. Let I_0 be the subgroup consisting of elements in $\mathrm{GL}_{d+1}(\mathbb{Z}_p)$ which are upper triangular modulo p. We put

$$\phi = (p \cdot \mathrm{id})\dot{s}_d \cdots \dot{s}_0 = (p \cdot \mathrm{id})\dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(d+1)}$$

where $\beta(i) = d + 1 - i$ for $1 \le i \le d + 1$. For $a, b \in \mathbb{Z}_{\ge 0}$ with $0 \le b < d + 1$ we put

$$C^{(a(d+1)+b)} = \phi^a s_d \cdots s_{d-b+1} C = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C.$$

We define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \qquad x \mapsto \mathrm{diag}(E_d, x^{-1}).$$

The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \ldots$ satisfies hypothesis (5). The corresponding $\alpha^{(j)} \in \Phi^+$ for $j \ge 0$ satisfy $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}$, and we have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$. We thus obtain a functor from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{d+1}, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

Let $\mathcal{H}(G, I_0)'_{\mathrm{aff},k}$ denote the k-sub algebra of $\mathcal{H}(G, I_0)_k$ generated by $\mathcal{H}(G, I_0)_{\mathrm{aff},k}$ together with $T_{p\cdot\mathrm{id}} = T_{\dot{u}^{d+1}}$ and $T_{p\cdot\mathrm{id}}^{-1} = T_{p^{-1}\cdot\mathrm{id}}$.

Suppose we are given a character $\lambda : \overline{T} \to k^\times$, a subset $\mathcal{J} \subset S_\lambda$ and some $b \in k^\times$. Define the numbers $0 \le k_i = k_i(\lambda, \mathcal{J}) \le p - 1$ as in subsection 3.2. The character $\chi_{\lambda,\mathcal{J}}$ of $\mathcal{H}(G, I_0)_{\mathrm{aff},k}$ extends uniquely to a character

$$\chi_{\lambda,\mathcal{J},b} : \mathcal{H}(G, I_0)'_{\mathrm{aff},k} \longrightarrow k$$

which sends $T_{p\cdot\mathrm{id}}$ to b (see the proof of [7] Proposition 3). Define the $\mathcal{H}(G, I_0)_k$-module

$$M = M[\lambda, \mathcal{J}, b] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G,I_0)'_{\mathrm{aff},k}} k.e$$

where $k.e$ denotes the one dimensional k-vector space on the basis element e, endowed with the action of $\mathcal{H}(G, I_0)'_{\mathrm{aff},k}$ by the character $\chi_{\lambda,\mathcal{J},b}$. As a k-vector space, M has dimension $d + 1$, a k-basis is $\{e_i\}_{0\le i\le d}$ where we write $e_i = T_{\dot{u}^{-i}} \otimes e$.

Definition: We call an $\mathcal{H}(G, I_0)_k$-module *quasi supersingular* if it is isomorphic with $M[\lambda, \mathcal{J}, b]$ for some $\lambda, \mathcal{J}, b$ such that $k_i > 0$ for at least one i.

For $0 \le j \le d$ put $n_{e_j} = \sum_{i=0}^{d} k_{j-i}p^i$ (reading $j - i$ as its representative modulo $(d+1)$ in $[0, d]$) and let s_{e_j} be such that $\lambda(\dot{u}^{-j}\tau(x)\dot{u}^j) = x^{-s_{e_j}}$. Put $\varrho = \lambda(-\mathrm{id})\prod_{i=0}^{d}(k_i!)$.

Theorem 4.17. *The étale (φ^{d+1}, Γ)-module $\mathbf{D}(\Theta_*\mathcal{V}_M)$ over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ admits a $k_{\mathcal{E}}$-basis $\{g_{e_j}\}_{0\le j\le d}$ such that for all j we have*

$$\varphi^{d+1}g_{e_j} = b\varrho^{-1}t^{n_{e_j}+1-p^{d+1}}g_{e_j},$$
$$\gamma(x)g_{e_j} - x^{s_{e_j}}g_{e_j} \in t \cdot k_{\mathcal{E}}^+ \cdot g_{e_j}.$$

The functor $M \mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ induces a bijection between

(a) the set of isomorphism classes of quasi supersingular $\mathcal{H}(G, I_0)_k$-modules, and

(b) the set of isomorphism classes of A-symmetric étale (φ^{d+1}, Γ)-modules over $k_{\mathcal{E}}$.

PROOF: For the formulae describing $\mathbf{D}(\Theta_*\mathcal{V}_M)$ one may proceed exactly as in the proof of Corollary 4.5. (The only tiny additional point to be observed is that the $\dot{s}_i$ (in keeping with our choice in [3]) do not ly in the images of the ι_{α_i}; this is accounted for by the sign factor $\lambda(-\mathrm{id})$ in the definition of ϱ.) Alternatively, as our ϕ is the $(d+1)$-st power of the ϕ considered in section 8 of [3], the computations of loc. cit. may be carried over.

To see that $\mathbf{D}(\Theta_*\mathcal{V}_M)$ is A-symmetric put $\mathbf{D}_j = \langle g_{e_j} \rangle$ for $0 \leq j \leq d$ and compare the above formulae with those defining A-symmetry; e.g. we find $s_{e_0} - s_{e_j} \equiv \sum_{i=1}^{j} k_i$ modulo $(p-1)$. The bijectivity statement is then verified as before. $\square$

Remark: Application of the functor of Lemma 2.2 to any one of the direct summands $\mathbf{D}_j$ of an A-symmetric étale (φ^{d+1}, Γ)-module over $k_{\mathcal{E}}$ (i.e. of $\mathbf{D}(\Theta_*\mathcal{V}_M)$) yields an étale (φ, Γ)-module isomorphic with the one assigned to M in [3].

Remark: Consider the subgroup $G' = \mathrm{SL}_{d+1}(\mathbb{Q}_p)$ of G. If we replace the above τ by $\tau : \mathbb{Z}_p^\times \longrightarrow T_0$, $x \mapsto \mathrm{diag}(xE_d, x^{-d})$ and if we replace the above ϕ by $\phi = \dot{s}_d\dot{s}_{d-1}\cdots\dot{s}_1\dot{s}_0$ then everything in fact happens inside G'. We then have $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}^{d+1}$ for all $j \geq 0$. Let $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ denote the category of finite-$\mathfrak{o}$-length $\mathcal{H}(G', G' \cap I_0)$-modules on which the xE_{d+1} (i.e. the $T_{x^{-1}E_{d+1}}$) for all $x \in \mathbb{Z}_p^\times$ with $x^{d+1} = 1$ act trivially. (Notice that $\tau(x) = xE_{d+1}$ for such x.) For $M \in \mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ we obtain an action of $\lfloor \mathfrak{N}_0, \varphi^{d+1}, \Gamma^{d+1} \rfloor$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$, where $\Gamma^{d+1} = \{\gamma^{d+1} \,|\, \gamma \in \Gamma\} \subset \Gamma$. Correspondingly, we obtain a functor from $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ to the category of $(\varphi^{d+1}, \Gamma^{d+1})$-modules over $\mathcal{O}_{\mathcal{E}}$.

Remark: As all the fundamental coweights τ of T are co minuscule, each of them admits ϕ's for which the pair (ϕ, τ) satisfies the properties asked for in Lemma 3.1. For example, let $1 \leq g \leq d$. For $\phi = \dot{s}_g \cdot \dot{s}_{g+1} \cdots \dot{s}_d \cdot \dot{u}$ as well as for $\phi = \dot{s}_g \cdot \dot{s}_{g-1} \cdots \dot{s}_1 \cdot \dot{u}^{-1}$ there is a unique minimal gallery from C to $\phi(C)$ which admits a ϕ-periodic continuation to a gallery (4), giving rise to a functor from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^r, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$, where $r = \ell(\phi)$.

5 Exceptional groups

Let G be the group of $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$ with connected center Z. Fix a maximal $\mathbb{Q}_p$-split torus T and define Φ, $N(T)$, W, $\widehat{W}$ and W_{aff} as before.

5.1 Affine root system $\tilde{E}_6$

Assume that the root system Φ is of type E_6. Following [2] (for the indexing) we then have Coxeter generators $s_1,\ldots,s_6$ of W and $s_0,\ldots,s_6$ of W_{aff} (thus $s_i^2=1$ for all i) such that

$$(s_1s_3)^3=(s_3s_4)^3=(s_4s_5)^3=(s_5s_6)^3=(s_4s_2)^3=(s_2s_0)^3=1$$

and moreover $(s_is_j)^2=1$ for all other pairs $i\neq j$. In the extended affine Weyl group $\widehat{W}$ we find (cf. [4]) an element u of length 0 with

$$\begin{aligned} u^3=1 \qquad &\text{and} \qquad us_4u^{-1}=s_4,\\ us_3u^{-1}=s_5,\quad us_5u^{-1}&=s_2,\quad us_2u^{-1}=s_3,\\ us_1u^{-1}=s_6,\quad us_6u^{-1}&=s_0,\quad us_0u^{-1}=s_1. \end{aligned} \tag{33}$$

(Then $\widehat{W}$ is the semidirect product of the three-element subgroup $W_\Omega=\{1,u,u^2\}$ with W_{aff}.) Let $e_1,\ldots,e_8$ denote the standard basis of $\mathbb{R}^8$. We use the standard inner product $\langle .,.\rangle$ on $\mathbb{R}^8$ to view both the root system Φ as well as its dual $\Phi^\vee$ as living inside $\mathbb{R}^8$. We choose a positive system Φ^+ in Φ such that, as in [2], the simple roots are $\alpha_1=\alpha_1^\vee=\frac{1}{2}(e_1+e_8-e_2-e_3-e_4-e_5-e_6-e_7)$, $\alpha_2=\alpha_2^\vee=e_2+e_1$, $\alpha_3=\alpha_3^\vee=e_2-e_1$, $\alpha_4=\alpha_4^\vee=e_3-e_2$, $\alpha_5=\alpha_5^\vee=e_4-e_3$, $\alpha_6=\alpha_6^\vee=e_5-e_4$ while the negative of the highest root is $\alpha_0=\alpha_0^\vee=\frac{1}{2}(e_6+e_7-e_1-e_2-e_3-e_4-e_5-e_8)$. The set of positive roots is

$$\Phi_+=\{e_j\pm e_i\,|\,1\leq i<j\leq 5\}\cup\{\frac{1}{2}(-e_6-e_7+e_8+\sum_{i=1}^{5}(-1)^{\nu_i}e_i)\,|\,\sum_{i=1}^{5}\nu_i\text{ even}\}.$$

We lift u and the s_i to elements $\dot{u}$ and $\dot{s}_i$ in $N(T)$. We then put

$$\phi=\dot{s}_2\dot{s}_4\dot{s}_3\dot{s}_1\dot{u}^{-1}\in N(T).$$

We define ∇ as in section 3.2.

Proposition 5.1. *There is a $\tau\in\nabla$ such that the pair (ϕ,τ) satisfies the hypotheses of Lemma 3.1. More precisely, ϕ is power multiplicative, and for the co minuscule fundamental (co)weight $\tau=\omega_1=\frac{2}{3}(e_8-e_7-e_6)\in\nabla$ we have $\phi^{12}=\tau^3$ in W_{aff}.*

PROOF: (Here the symbol ω_1 in fact designates the *translation* by ω_1, therefore we write $\omega_1^3 = \tau^3$ (rather than $3\omega_1$) for the three fold iterate of this translation.) Consider the set of affine root hyperplanes crossed by a minimal gallery from C to $\omega_1^3 C$. Assigning to each of these affine root hyperplanes its corresponding positive root in Φ^+, each element of the subset

$$\Phi(\omega_1) = \{\frac{1}{2}(-e_6 - e_7 + e_8 + \sum_{i=1}^{5}(-1)^{\nu_i} e_i) \,|\, \sum_{i=1}^{5} \nu_i \text{ even}\}$$

of Φ^+ is hit exactly three times, whereas no other element of Φ^+ is hit. In this way, the set $\Phi(\omega_1)$ characterizes ω_1^3 as an element of W_{aff}. In particular, the length of ω_1^3 is $3 \cdot 16 = 48$. It is now enough to verify that ϕ^{12} satisfies this characterization of ω_1^3.

Alternatively, one may want to use a computer to verify $\phi^{12} = \omega_1^3$. See the appendix for how this can be done. $\square$

As explained in subsection 3.2 we now obtain a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^4, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

Similarly, we may replace ϕ by its third power ϕ^3 which (in contrast to ϕ) is an element of W_{aff} (modulo T_0). It yields a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{12}, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$. As in our treatment of the cases C, B, D and A, this functor identifies the set of quasi supersingular $\mathcal{H}(G, I_0)_k$-modules bijectively with a set of certain E-symmetric étale (φ^{12}, Γ)-modules over $k_{\mathcal{E}}$ of dimension 3. We leave the details to the reader.

Remarks: (a) Dual to the above choice of ϕ is the choice

$$\phi = \dot{s}_2 \dot{s}_4 \dot{s}_5 \dot{s}_6 \dot{u} \in N(T). \tag{34}$$

For this choice, Proposition 5.1 holds true verbatim the same way, but now with the co minuscule fundamental (co)weight $\tau = \omega_6 = \frac{1}{3}(3e_5 + e_8 - e_7 - e_6)$ with its corresponding subset (again containing 16 elements)

$$\Phi(\omega_6) = \{\frac{1}{2}(e_5 - e_6 - e_7 + e_8 + \sum_{i=1}^{4}(-1)^{\nu_i} e_i) \,|\, \sum_{i=1}^{4} \nu_i \text{ even}\} \cup \{e_5 \pm e_i \,|\, 1 \le i < 5\}$$

of Φ^+. Again see the appendix.

(b) In either case, the multiplicities of the s_i in $\phi^3 \in W_{\text{aff}}$ are the coefficients of the $\alpha_i^\vee$ in

$$\alpha_0^\vee + \alpha_1^\vee + \alpha_6^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_5^\vee + 3\alpha_4^\vee = 0.$$

5.2 Affine root system $\tilde{E}_7$

Assume that the root system Φ is of type E_7. Following [2] we then have Coxeter generators $s_1, \ldots, s_7$ of W and $s_0, \ldots, s_7$ of W_{aff} (thus $s_i^2 = 1$ for all i) such that

$$(s_0s_1)^3 = (s_1s_3)^3 = (s_3s_4)^3 = (s_4s_5)^3 = (s_5s_6)^3 = (s_6s_7)^3 = (s_4s_2)^3 = 1$$

and moreover $(s_is_j)^2 = 1$ for all other pairs $i \neq j$. In the extended affine Weyl group $\widehat{W}$ we find (cf. [4]) an element u of length 0 with

$$\begin{aligned} u^2 = 1 \qquad \text{and} \qquad & us_4u = s_4, \quad us_2u = s_2, \\ us_3u = s_5, \quad us_6u = s_1, \quad & us_7u = s_0, \\ us_5u = s_3, \quad us_1u = s_6, \quad & us_0u = s_7. \end{aligned}$$

($\widehat{W}$ is the semidirect product of the two-element subgroup $W_\Omega = \{1, u\}$ with W_{aff}.) Let $e_1, \ldots, e_8$ denote the standard basis of $\mathbb{R}^8$. We use the standard inner product $\langle ., . \rangle$ on $\mathbb{R}^8$ to view both the root system Φ as well as its dual $\Phi^\vee$ as living inside $\mathbb{R}^8$. We choose a positive system Φ^+ in Φ such that, as in [2], the simple roots are $\alpha_1 = \alpha_1^\vee = \frac{1}{2}(e_1+e_8-e_2-e_3-e_4-e_5-e_6-e_7)$, $\alpha_2 = \alpha_2^\vee = e_2+e_1$, $\alpha_3 = \alpha_3^\vee = e_2-e_1$, $\alpha_4 = \alpha_4^\vee = e_3-e_2$, $\alpha_5 = \alpha_5^\vee = e_4 - e_3$, $\alpha_6 = \alpha_6^\vee = e_5 - e_4$, $\alpha_7 = \alpha_7^\vee = e_6 - e_5$ while the negative of the highest root is $\alpha_0 = \alpha_0^\vee = e_7 - e_8$. The set of positive roots is

$$\Phi_+ = \{e_j \pm e_i \,|\, 1 \leq i < j \leq 6\} \cup \{e_8 - e_7\} \cup \{\frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{\nu_i}e_i) \,|\, \sum_{i=1}^{6} \nu_i \text{ odd}\}.$$

We lift u and the s_i to elements $\dot{u}$ and $\dot{s}_i$ in $N(T)$. We then put

$$\phi = \dot{s}_1\dot{s}_3\dot{s}_4\dot{s}_2\dot{s}_5\dot{s}_4\dot{s}_3\dot{s}_1\dot{s}_0\dot{u} \in N(T).$$

We define ∇ as in section 3.2.

Proposition 5.2. *There is a $\tau \in \nabla$ such that the pair (ϕ, τ) satisfies the hypotheses of Lemma 3.1. More precisely, ϕ is power multiplicative, and for the co minuscule fundamental (co)weight $\tau = \omega_7 = e_6 + \frac{1}{2}(e_8 - e_7) \in \nabla$ we have $\phi^6 = \tau^2$ in W_{aff}.*

PROOF: Exactly the same as for Propostion 5.1. The corresponding subset in Φ^+ is

$$\Phi(\omega_7) = \{\frac{1}{2}(e_6 + e_8 - e_7 + \sum_{i=1}^{5}(-1)^{\nu_i}e_i) \,|\, \sum_{i=1}^{5} \nu_i \text{ odd}\} \cup \{e_8 - e_7\} \cup \{e_6 \pm e_i \,|\, 1 \leq i < 6\}.$$

It contains exactly 27 elements, thus $\ell(\omega_7) = 27$. For a computer proof of $\phi^6 = \tau^2$ see the appendix. $\square$

As explained in subsection 3.2 we now obtain a functor from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G,I_0))$ to the category of (φ^9,Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

Similarly, we may replace ϕ by its square ϕ^2 which (in contrast to ϕ) is an element of W_{aff} (modulo T_0). It yields a functor from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G,I_0))$ to the category of (φ^{18},Γ)-modules over $\mathcal{O}_{\mathcal{E}}$. Again this functor identifies the set of quasi supersingular $\mathcal{H}(G,I_0)_k$-modules bijectively with a set of certain E-symmetric étale (φ^{18},Γ)-modules over $k_{\mathcal{E}}$ of dimension 2. We leave the details to the reader.

Remark: The multiplicities of the s_i in $\phi^2\in W_{\mathrm{aff}}$ are the coefficients of the $\alpha_i^\vee$ in

$$\alpha_0^\vee+\alpha_7^\vee+2(\alpha_1^\vee+\alpha_2^\vee+\alpha_6^\vee)+3(\alpha_3^\vee+\alpha_5^\vee)+4\alpha_4^\vee=0.$$

5.3 Affine root systems $\tilde{G}_2$, $\tilde{F}_4$, $\tilde{E}_8$

If the underlying root system of G is G_2, F_4 or E_8 then co minuscule coweights do not exist, and there don't exist ϕ and τ satisfying the conclusions of Lemma 3.1.

One may nevertheless ask the following question: Is there a reduced expression (7) of a power multiplicative element $\phi\in N(T)$, some power of which maps to a dominant coweight in $N(T)/T_0=\widehat{W}$, such that the corresponding functor $M\mapsto\mathbf{D}(\Theta_*\mathcal{V}_M)$ from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G,I_0))$ to (φ^r,Γ_0)-modules has the following property: for *any* M, the (φ^r,Γ_0)-module structure on $\mathbf{D}(\Theta_*\mathcal{V}_M)$ extends (possibly in several ways) to an (φ^r,Γ)-module structure ?

Let us say that such expressions (7) have the *extension property*. We consider the question for the supersingular characters M of $\mathcal{H}(G,I_0)_k$. Suppose we are given a character $\lambda:\overline{T}\to k^\times$ and a subset $\mathcal{J}\subset S_\lambda$, with corresponding numbers $0\le k_i=k_i(\lambda,\mathcal{J})\le p-1$, not all of them equal to 0 and not all of them equal to $p-1$. As we are in case $\tilde{G}_2$, $\tilde{F}_4$ or $\tilde{E}_8$, we have $W_{\mathrm{aff}}=\widehat{W}$ and hence $\mathcal{H}(G,I_0)_{\mathrm{aff},k}=\mathcal{H}(G,I_0)_k$, as follows e.g. from [7] Corollary 3. Let us write $M=M[\lambda,\mathcal{J}]$ for the one-dimensional $\mathcal{H}(G,I_0)_k$-module on the basis element e given by $\chi_{\lambda,\mathcal{J}}$.

Notice that the group $N(T)/T_0=W_{\mathrm{aff}}=\widehat{W}$ is canonically independent on the chosen prime number p. Consider the unique equation

$$\sum_{i=0}^{d}m_i\alpha_i^\vee=0 \tag{35}$$

with minimally chosen positive coefficients $m_i\in\mathbb{N}$.

Lemma 5.3. *Given an expression (7), in order that the (φ^r,Γ_0)-module structure on $\mathbf{D}(\Theta_*\mathcal{V}_{M[\lambda,\mathcal{J}]})$ extends to a (φ^r,Γ)-module structure for any choice of $(\lambda,\mathcal{J})$ and for infinitely many primes p, a necessary condition is that r be a multiple of $\sum_{i=0}^d m_i$ and that s_j for each $0\le j\le d$ shows up in (7) exactly with multiplicity $m_jr/(\sum_{i=0}^d m_i)$.*

PROOF: (Sketch) Put $n = \sum_{i=0}^{r-1} k_{\beta(i+1)} p^i$. As in the proof of Lemma 4.4 we see that

$$t^n \varphi^r e = \varrho e \tag{36}$$

in $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$, for some $\varrho \in k^\times$. Notice that formula (36) completely characterizes the action of Γ_0 and of φ^r on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ since we know that the action of Γ_0 respects the subspace M and acts trivially on it, and M generates $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ as a $k_{\mathcal{E}}^+[\varphi^r]$-module.

We need to investigate if there is a homomorphism $\epsilon : \mathbb{F}_p^\times \longrightarrow \mathbb{F}_p^\times$ such that the action of $\lfloor \mathfrak{N}_0, \varphi^r, \Gamma_0 \rfloor$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ extends to an action by $\lfloor \mathfrak{N}_0, \varphi^r, \Gamma \rfloor$ in such a way that for all $x \in \mathbb{F}_p^\times$ the action of $\gamma(x) \in \Gamma$ satisfies

$$\gamma(x) \cdot e = \epsilon(x) e. \tag{37}$$

The formula (37) provides a well defined action of Γ on $k.e = M$, with trivial restriction to Γ_0. An extension from $k.e = M$ to all of $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$, if it exists, is necessarily uniquely determined, since e generates $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ as a $k_{\mathcal{E}}^+[\varphi^r] = k_{\mathcal{E}}^+[\phi]$-module. More precisely, in order to extend it to all of $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ one must use the relations, in $k_{\mathcal{E}}^+[\phi, \Gamma]$, between $\gamma(x)$ and t and $\varphi^r = \phi$. Namely, given $m \geq 0$ and $c \geq 0$, if $\gamma(x) t^m \phi^c = \sum_{m'} \beta_{m'} t^{m'} \phi^c \gamma_{m'}$ in $k_{\mathcal{E}}^+[\phi, \Gamma]$ with certain $\beta_{m'} \in k$ and $\gamma_{m'} \in \Gamma$, then one must put

$$\gamma(x) \cdot (t^m \phi^c e) = \sum_{m'} \beta_{m'} t^{m'} \phi^c (\gamma_{m'} \cdot e)$$

and extend by linearity. In order to check if this yields a well defined action of Γ one must in particular check if this definition is compatible with formula (36). Thus one must do the following computation:

$$\gamma(x) \cdot t^n \phi e = x^n t^n \gamma(x) \cdot \phi e = x^n t^n \phi \gamma(x) \cdot e = x^n t^n \phi \epsilon(x) e = x^n \gamma(x) \cdot \varrho e.$$

Here the first equality follows from the fact that t^{n+1} annihilates ϕe (which it does because of formula (36)) together with the following

SUBLEMMA: (see [3]) *For all $k \geq 0$ we have $\gamma(x) \cdot t^k \gamma(x)^{-1} - x^k t^k \in t^{k+1} k_{\mathcal{E}}^+$.*

Thus, in order that the desired Γ-action on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ exists we need to have $x^n = 1$ for all $x \in \mathbb{F}_p^\times$. Now

$$x^n = x^{\sum_{i=0}^{r-1} k_{\beta(i+1)} p^i} = x^{\sum_{i=0}^{r-1} k_{\beta(i+1)}} = \lambda\Big(\prod_{i=0}^{r-1} \alpha_{\beta(i+1)}^\vee(x)\Big).$$

Thus, since λ is arbitrary we need that $\prod_{i=0}^{r-1} \alpha_{\beta(i+1)}^\vee = \mathbf{1}$ in $\mathrm{Hom}(\mathbb{F}_p^\times, \overline{T})$. In order that this be true for infinitely many primes p we exactly get our stated condition. $\square$

Remark: This discussion also shows that we have no problems with extending the (φ^r, Γ_0)-action to a (φ^r, Γ)-action for trivial λ. In particular, this applies to the category of modules over the Iwahori Hecke algebra $\mathcal{H}(G, I)_k$ (which is a direct summand of $\mathcal{H}(G, I_0)_k$).

(a) **Affine root system** $\tilde{G}_2$

Claim: No expression (7) has the extension property for infinitely many p.

We have the Coxeter generators s_1, s_2 of W and s_0, s_1, s_2 of $W_{\mathrm{aff}} = \widehat{W}$ (thus $s_0^2 = s_1^2 = s_2^2 = 1$) such that

$$(s_1 s_2)^6 = (s_0 s_2)^3 = 1$$

and moreover $(s_0 s_1)^2 = 1$. The equation (35) reads

$$\alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee = 0. \tag{38}$$

In view of Lemma 5.3 we conclude from formula (38) that we would need to find a power multiplicative reduced expression of length $r \in 4\mathbb{N}$ in which the factors s_0, s_1, s_2 appear with multiplicities in exact proportions $1 : 1 : 2$. But it is easy to see that such expressions do not exist.

(b) **Affine root system** $\tilde{F}_4$

Claim: No expression (7) has the extension property for infinitely many p.

We have Coxeter generators $s_1, \ldots, s_4$ of W and $s_0, \ldots, s_4$ of $W_{\mathrm{aff}} = \widehat{W}$ (thus $s_i^2 = 1$ for all i) such that

$$(s_2 s_3)^4 = (s_0 s_1)^3 = (s_1 s_2)^3 = (s_3 s_4)^3 = 1$$

and moreover $(s_i s_j)^2 = 1$ for all other $i \neq j$. The equation (35) reads

$$\alpha_0^\vee + \alpha_4^\vee + 2(\alpha_1^\vee + \alpha_3^\vee) + 3\alpha_2^\vee = 0. \tag{39}$$

Thus, in view of Lemma 5.3 we would need to find a power multiplicative reduced expression of length $r \in 9\mathbb{Z}$ in which the factors s_0, s_1, s_2, s_3, s_4 appear with multiplicities in exact proportions $1 : 2 : 3 : 2 : 1$. Such an expression does not exist. (By power multiplicativity we may assume that the desired expression represents a *translation*. As such it can be written as a linear combinaton with $\mathbb{Z}_{\geq 0}$-coefficients of the fundamental weights. Inspecting the list of all reduced expressions for the translations by the fundamental weights — this list can easily be produced using *sage* — the claim can be verified without much pain.)

(c) **Affine root system** $\tilde{E}_8$

Extrapolating from the cases $\tilde{G}_2$ and $\tilde{F}_4$ one might expect that again the answer is negative. We have the Coxeter generator $s_1, \dots, s_8$ of W and $s_0, \dots, s_8$ of $W_{\rm aff} = \widehat{W}$ (thus $s_i^2 = 1$ for all i) such that

$$(s_1s_3)^3 = (s_3s_4)^3 = (s_4s_5)^3 = (s_5s_6)^3 = (s_6s_7)^3 = (s_7s_8)^3 = (s_8s_0)^3 = (s_4s_2)^3 = 1$$

and moreover $(s_is_j)^2 = 1$ for all other pairs $i \neq j$. The equation (35) reads

$$\alpha_0^\vee + 2(\alpha_1^\vee + \alpha_8^\vee) + 3(\alpha_2^\vee + \alpha_7^\vee) + 4(\alpha_3^\vee + \alpha_6^\vee) + 5\alpha_5^\vee + 6\alpha_4^\vee = 0. \tag{40}$$

¿From here one might try to proceed as in the case of $\tilde{F}_4$; but unfortunately, this time the combinatorics seem to become too involved to be tractable by hand.

6 Appendix

Verification of the statement $\phi^{12} = \omega_1^3$ in the proof of Proposition 5.1.
In the computer algebra system *sage*, the input

R=RootSystem(["E",6,1]).weight_lattice()
Lambda=R.fundamental_weights()
omega1=Lambda[1]-Lambda[0]
*R.reduced_word_of_translation(3*omega1)*

prompts the output

$$\begin{aligned}&[0,2,4,3,5,4,2,0,6,5,4,2,3,1,4,3,5,4,2,0,6,5,4,2,\\&3,1,4,3,5,4,2,0,6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1].\end{aligned} \tag{41}$$

By definition of the function *reduced_word_of_translation* this means $s_{i_1}^* \cdots s_{i_{48}}^* = \omega_1^3$, with the string $[i_1, \dots, i_{48}]$ as given by (41). Here $s_i^* = s_i$ for $1 \leq i \leq 6$, but s_0^* denotes the affine reflection in the outer face of Bourbaki's fundamental alcove A. Since we deviate from these conventions in that our s_0 is the affine reflection in the outer face of the *negative* $C = -A$ of A, we must modify the above string (41) as follows. First, writing $s_i^{**} = s_0^*s_i^*s_0^*$ for $0 \leq i \leq 6$, conjugating the factors in the previous word by s_0^* and commuting some of its factors where allowed, the above says $s_{j_1}^{**} \cdots s_{j_{48}}^{**} = \omega_1^3$ where the string $[j_1, \dots, j_{48}]$ is given by

$$\begin{aligned}&[2,4,5,6,3,4,2,0,5,4,3,1,2,4,5,6,3,4,2,0,5,4,3,1,\\&2,4,5,6,3,4,2,0,5,4,3,1,2,4,5,6,3,4,2,0,5,4,3,1].\end{aligned} \tag{42}$$

The s_i^{**} are precisely the reflections in the codimension 1 faces of s_0^*A. But s_0^*A is a translate of C, and under this translation, the reflection $s_0^{**} = s_0^*$ corresponds to s_0, whereas for $1 \leq i \leq 6$ the reflection s_i^{**} corresponds to $w_0 s_i w_0$, where w_0 is the longest element of W. We have $w_0 s_i w_0 = s_i$ for $i \in \{0, 2, 4\}$, but $w_0 s_3 w_0 = s_5$ and $w_0 s_1 w_0 = s_6$. Thus, we obtain $s_{k_1} \cdots s_{k_{48}} = \omega_1^3$ where the string $[k_1, \ldots, k_{48}]$ is obtained from the string (42) by keeping its entry values $0, 2$ and 4, while exchanging the entry values 3 with 5 and 1 with 6. Using formulae (33) one checks that $s_{k_1} \cdots s_{k_{48}} = \phi^{12}$. $\square$

Verification of the statement $\phi^{12} = \omega_1^3$ for ϕ given by (34).
The argument is the same as in Proposition 5.1. The string returned by *sage* to

*R=RootSystem(["E",6,1]).weight_lattice(), Lambda=R.fundamental_weights(), omega6=Lambda[6]-Lambda[0], R.reduced_word_of_translation(3*omega6)*

reads

$$[0, 2, 4, 3, 1, 5, 4, 2, 0, 3, 4, 2, 5, 4, 3, 1, 6, 5, 4, 2, 0, 3, 4, 2,$$
$$5, 4, 3, 1, 6, 5, 4, 2, 0, 3, 4, 2, 5, 4, 3, 1, 6, 5, 4, 2, 3, 4, 5, 6].$$

Verification of the statement $\phi^6 = \tau^2$ in the proof of Proposition 5.2.
The string returned by *sage* to

*R=RootSystem(["E",7,1]).weight_lattice(), Lambda=R.fundamental_weights(), omega7=Lambda[7]-Lambda[0], R.reduced_word_of_translation(2*omega7)*

reads

$$[0, 1, 3, 4, 2, 5, 4, 3, 1, 0, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 0,$$
$$7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 7, 6, 5, 4, 2, 3, 4, 5, 6, 7]. \tag{43}$$

Now proceed as in Proposition 5.1. $\square$

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