Contents

0. Preface

The moduli space $\widehat{F}$ of polarized abelian threefolds with (compatible) imaginary quadratic multiplication (of type $(2,1)$ with a fixed imaginary quadratic number field K) is called a *Picard modular surface.* The corresponding period domain is the two-dimensional complex unit ball $\mathbb{B}$. So $\widehat{F}$ is the Baily-Borel compactification of a quotient surface $F = \mathbb{B}/\Gamma$, Γ an arithmetic group, called a *Picard modular group.*

Let $\widehat{C}$ be an irreducible compact curve on $\widehat{F}$ and C its intersection with F. We call C (also $\widehat{C}$) an *arithmetic curve*, if each abelian threefold A_P corresponding to any point P of C is not simple. We show that all arithmetic curves fill a thin dense set on $\widehat{F}$. More precisely, they can be realized as quotients of all K-rational linear subdiscs $\mathbb{D}$ of the ball $\mathbb{B}$ using the arithmetic $\mathbb{D}$-lattices $N_\Gamma(\mathbb{D})$ consisting of all elements of Γ acting on $\mathbb{D}$.

Arithmetic curves have both, an important geometric and arithmetic meaning. Geometrically, they can be used for the classification of Picard modular surfaces in the sense of Kodaira. For this purpose one defines special numerical invariants called (local) *orbital hights* of curves on $\widehat{F}$, principally, the *Euler hight* and the *signature height.* A simple proportionality relation between Euler and signature height yields an explicit criterion for a curve $\widehat{C}$ on F to be arithmetical. The determination of the surface type can be managed by finding a classifying configuration of arithmetic curves (together with the knowledge of Chern numbers of a smooth model of $\widehat{F}$ coming from global hights).

0.1 Theorem. There are two types of arithmetic curves C. In the *modular case* the threefolds A_P have an isogeny decomposition $E(P) \times E(P) \times E$ into three elliptic curves for all points P of C, where E is constantly an elliptic curve with complex K-multiplication and the $E(P)$'s can be organized to non-isotrivial elliptic curve families over (suitable) finite covers of C. The other (*quaternionic scew field types*) are interpreted in a similar manner by non-isotrivial families of abelian surfaces with scew field multiplication (in general points P) coming from isogeny decompositions $B(P) \times E$ of A_P, $B(P)$ a simple abelian surface of type II in Albert's list.

For the proof we classify the types of all possible endomorphism algebras of abelian threefolds with imaginary quadratic multiplication. There are precisely six types corresponding to different types of isogeny decompositions in the Picard modular case.

With regard to Hilbert's 7-th (transcendence) and 12-th (class field) problems one uses a strong theorem of Wüstholz for the fist part and Shimura's work to prove the second part of

0.2 Theorem. The point P on F and (each of) its preimage(s) τ on $\mathbb{B}$ are algebraic if and only if the abelian threefold A_P is simple with complex multiplication (CM) or it has an isogeny decomposition into three elliptic curves with complex multiplication. In the simple CM-case the field $K(P)$ is a class field over the CM-field $L = K(\tau)$ of A_P which is a cubic extension of K.

We call a point $\tau = (u : v : w)$ of $\mathbb{B} \subset \mathbb{P}^2$ *simply transcendental*, if $K(\tau)$ is a transcendental field extension of K and u, v, w are linearly dependent over K. With the decomposition results one gets

0.3 Theorem. The abelian threefold A_P not of (decomposed) CM-type is not simple, if and only if (each of) its preimage(s) τ on $\mathbb{B}$ is simply transcendental.

Shimura's class field theory works for simple abelian CM-varieties. It is not difficult to see that all other candidates for class field theory, namely the points P for which A_P has a decomposition into elliptic CM-curves, ly on arithmetic curves C. The previous result means that important class field extensions of $K(\tau)$, τ as above, for these points P comes from the theory of elliptic curves in the modular case or from the class field theory of quaternion division algebras over $\mathbf{Q}$ in the skew field case. This is a corollary of Theorem 1. Moreover the modular case happens in distinction to the quaternionic scew field case if and only if the corresponding curve $\widehat{C}$ goes through at least one of the cusp singularities collected in $\widehat{F} \backslash F$.

The results are applied in analogy to elliptic curves to Picard curves defined by equations of type $Y^3 = X^4 + aX^3 + bX^2 + cX + d$. The Jacobians are classified by the Picard modular surface of the field of Eisenstein numbers generated by a primitive third unit root over $\mathbf{Q}$. Here the quotient morphism th: $\mathbb{B} \longrightarrow$ F is described by explicit Theta constants and the inverse map by three (typical Picard) integrals of first kind along pathes on Picard curves.

0.4 Corollary. The following conditions are equivalent:
(i) The Picard integrals at a point P are linearly dependent over K;
(ii) $th(\tau) = P$ lies on an arithmetic curve;
(iii) the corresponding Jacobian threefold A_P is not simple;

1. Introduction

An abelian variety A has *complex multiplication* (CM), if its endomorphism algebra **End** $(A) := \mathbf{Q} \otimes End\,(A)$ is a CM-field (totally imaginary quadratic extension of a totally real number field) of absolute degree $2g$, where $g := dim\ A$. In our context abelian CM-varieties are understood as simple abelian varieties. From the basic work of SHIMURA and

TANIYAMA [ST] one knows that they play an important role in number theory. Let $\mathbf{A}/T$ be a family of abelian varieties over a complex (irreducible) algebraic variety T. We call $t \in T$ a *special point* of this family, if the abelian variety A_t splits up to isogeny into simple abelian varieties with complex multiplication. By our definition such abelian varieties are called *abelian DCM-varieties (decomposed complex multiplication)*. More generally we look for (irreducible) subvarieties S of T such that all members A_s of the restricted family $\mathbf{A}_S/S$ have a greater endomorphism algebra than a general member of $\mathbf{A}/T$. The restricted family $\mathbf{A}_S$ is called *specific* in this case. If the family is fixed once for all, then we call simply S a *specific subvariety* of T. The subvariety S (or the family $\mathbf{A}_S$) is called *very specific* with respect to the family $\mathbf{A}/T$, if $dim\ S > 0$ and for each $s \in S$ the abelian fibre variety $\mathbf{A}_s$ is isogeneous to a product of elliptic curves. The definitions transfer in obvious manner to the moduli spaces of polarized abelian varieties using the biunivoque correspondence between moduli points and isomorphy classes of polarized abelian varieties.

By means of the general classification theory of endomorphism algebras of abelian varieties due to ALBERT we introduce the *decomposition type* $DT(A)$ of an abelian variety. We say that the decomposition type $DT(B)$ is a *specialisation* of $DT(A)$, if there is a family $\mathbf{A}/T$ of abelian varieties such that the general members of $\mathbf{A}$ have decomposition type $DT(A)$ and there exists a point $s \in T$ with A_s of decomposition type $DT(B)$. Such a concrete specialisation is called of *codimension* c, where $c := dim\ T - dim\ S$. On this way one gets a *hierarchy structure* among all decomposition types of abelian varieties of a fixed dimension g. It seems to be an interesting problem to study the hierarchy structures in more detail.

This should also be done for classes of abelian varieties satisfying additional conditions. For example, one considers all abelian varieties whose endomorphism algebras contain a given $\mathbf{Q}$-algebra R. Such abelian varieties are said to have *R-multiplication*. With some restrictions and refinements (polarisations of certain lattice types) there are classifying algebraic varieties called (complex) SHIMURA varieties. For basic definitions and facts of the complex theory we refer to the monograph [BL]. The notion of "(very) specific" subvarieties of these moduli varieties is defined as above simply by the restrictive moduli interpretation.

In this article we restrict our attention to abelian varieties A with *imaginary quadratic multiplication*. By definition, **End** (A) contains an imaginary quadratic number field K. Most important are the principally polarized abelian threefolds with K-multiplication. The corresponding SHIMURA varieties (for each K) are called PICARD *modular surfaces*. For basic facts and advanced arithmetic studies we refer to the proceedings [Lg]. (Very) specific points and (very) specific curves are the main objects of our investigation. We want to describe in simple words in section 1 the results of the article and in ?. some open problems, which are all connected with each other. Proofs are given in sections 2. - ?.

Leading Example

Each PICARD modular surface $\mathbf{M}$ is a non-compact quotient $\mathbb{B}/\Gamma$ of the complex two-ball $\mathbb{B}$ by an arithmetic group Γ acting on $\mathbb{B}$. Its BAILY-BOREL compactification is denoted by $\widehat{\mathbf{M}} = \mathbb{B}/\widehat{\Gamma}$. The difference $\widehat{\mathbf{M}} \backslash \mathbf{M}$ consists of finitely many *cusp points*, which are normal surface singularities. Via Jacobian varieties $\mathbf{M}$ is interpreted as modular surface of certain curves of genus 3. The smooth curve correspond to a ZARISKI open subvariety $\mathbf{M}^{sm}$ of $\mathbf{M}$. The difference $\widehat{\mathbf{M}}\backslash\mathbf{M}^{sm}$ is an algebraic cycle of codimension 1 called the *compactification cycle* (with respect to smooth curves). At the end of this article we apply the above results to the PICARD modular surface $\mathbf{M}$ of the imaginary quadratic field of EISENSTEIN numbers and a specific one-dimensional subfamily $\mathbf{C}_T/T$ of the corresponding PICARD curve family $\mathbf{C}/\mathbf{M}'$, $\mathbf{M}'$ a suitable finite covering of the moduli space $\mathbf{M}$. Together with some earlier work the following interesting properties are verified:

1.1

$$\widehat{\mathbf{M}'} = \mathbb{P}^2,\ \widehat{\mathbf{M}'}\backslash\mathbf{M}' = \mathbb{P}^2\backslash\{\text{four (cusp) points in general position}\},$$
$$\widehat{\mathbf{M}'}\backslash\mathbf{M}'^{sm} = \{\textit{the six projective lines through pairs of the four cusp points}\}$$

1.2 *T is a specific curve in the above sense, more precisely, with respect to the Jacobian fibration* $\mathbf{J}(\mathbf{C})/\mathbf{M}'$.

1.3 *T is a projective line on $\mathbb{P}^2 = \widehat{\mathbf{M}'}$ not containing any cusp point. So it does not belong to the compactification cycle* $\widehat{\mathbf{M}'}\backslash\mathbf{M}'^{sm}$.

1.4 *T is the subquotient $\mathbb{D}/\Delta$ of the ball quotient surface $\mathbf{M}' = \mathbb{B}/\Gamma'$, $\mathbb{D}$ a linear subdisc of the two-ball $\mathbb{B}$, Δ a cocompact arithmetic group commensurable with the unitary group $\mathbf{U}((1,1),\mathbf{O})$, where $\mathbf{O} = \mathbf{Z} + \mathbf{Z}_\rho$, ρ a primitive unit root, is the ring of* EISENSTEIN *integers.*

1.5 *Explicitly the curve family $\mathbf{C}_T/T$ has the affine model*

$$Y^3 = X^4 - 4(\lambda^2+1)X^2 + 16\lambda^2,\ \lambda \in \mathbf{C},$$

of smooth (PICARD) curves of genus 3 *(in general).*

1.6 *The Jacobian threefolds J_λ of the Picard curves of the above family split in general up to isogeny into a product of the elliptic curve E with $\mathbf{Q}(\rho)$-multiplication and a simple abelian surface S_λ with a $\mathbf{Q}$-central skew field $D \cong \mathbf{End}\ S_\lambda$ as endomorphism algebra.*

2. Decomposition types of abelian varieties

Let A be an abelian variety. All our abelian varieties are defined over a field of characteristic 0. By POINCARÉ's Complete Reducibility Theorem, see e.g. [BL], V.3.7, there is an isogeny decomposition (unique up to isogeny)

$$A \approx A_1^{m_1} \times \ldots \times A_r^{m_r} \tag{2.1}$$

with simple abelian varieties A_i in the decomposition. We call the factors A_i (more precisely their isogeny classes) the *isogeny components* of A and the exponent m_i the *multiplicity of* A_i in the decomposition. The endomorphism algebra
$\mathbf{End}\ (A) = \mathbf{Q} \otimes End\ (A)$ has the decomposition

$$\mathbf{End}\ (A) \cong Mat_{m_1}(D_1) \oplus \ldots \oplus Mat_{m_r}(D_r) \tag{2.2}$$

where $D_i = \mathbf{End}\ (A_i)$ is a division algebra (a skewfield because of associativity) since A_i is simple. We denote by K an imaginary quadratic number field. We say that A has *K-multiplication*, if there exists a $\mathbf{Q}$-algebra homomorphism ι: $K \longrightarrow \mathbf{End}\ (A)$. This is a field embedding because it is non-trivial.

Now let A be simple. We remember to the classification list for endomorphism algebras D of simple abelian varieties due to ALBERT (see [AV], [Al], [BL]). The division algebra D has an involution $'$ (ROSATI involution) associated to a polarisation of A. The involution is positive, this means that the quadratic form $x \mapsto Tr(xx')$, $Tr = Tr_{D/\mathbf{Q}}$ the reduced trace on D, is positive definit. In general the involution is not uniquely determined, but uniquely determined is the JORDAN algebras $Jd(A) = Jd(D,\ '\ ) = \{x \in D;\ x' = x\}$ of elements of D fixed by the involution, because it is the image of a canonical homomorphism from the NERON - SEVERI group $NS(A)$ into D. There are four rough types of division algebras $(D,\ '\ )$ with positive involutions coming from simple abelian varieties A (of any characteristic). For their description we denote the center of D by Z, the *index* of D by d, that means $d^2 = dim\ \mathbf{Q}(D/Z)$, and set

$$\begin{aligned} g &= dim\ A, \\ e &= [Z : \mathbf{Q}], \\ Z^+ &= \{z \in Z;\ z' = z\}, \\ e^+ &= \left[Z : Z^+\right] \in \{1, 2\}, \\ \eta &= dim_{\mathbf{Q}}\ Jd(D,\ '\ )/dim_{\mathbf{Q}}\ D. \end{aligned}$$

Then we have the following complete list:

2.3. Rough types *of endomorphism algebras of simple abelian varieties*

Types		Conditions
I.	$e = e^+,\quad d = 1,\ h = 1;$	$e \mid g$
II.	$e = e^+,\quad d = 2,\ h = \frac{3}{4};$	$2e \mid g$
III.	$e = e^+,\quad d = 2,\ h = \frac{1}{4};$	$2e \mid g$ in characteristic 0 ($e \mid g$ in characteristic $p > 0$)
IV.	$e = 2e^+,\qquad h = \frac{1}{2};$	$e^+d^2 \mid g$ in characteristic 0 ($e^+d \mid g$ in characteristic $p > 0$)

Moreover, one has the following additional informations I', II', III', IV' for the above cases, respectively (in all characteristics):
I'. $D = Z = Z^+$ is a totally real number field.

II'. Totally indefinite quaternion type. $Z = Z^+$ is a totally real number field and D a quaternion field over Z (associative non-commutative division algebra of index $d = 2$ with center Z), such that for all field embeddings σ: $Z \longrightarrow \mathbf{R}$ it holds that

$$\mathbf{R} \otimes_\sigma D \cong Mat_2(\mathbf{R}). \tag{2.4}$$

The ROSATI involution can be represented by $x \mapsto x' = ax^*a^{-1}$, $a \in D$ with $a^2 \in Z$ totally negative, where $x \mapsto x^* = Tr(x) - x$ is the standard involution on D. The involution $'$ is the restriction of the transposition map $(X_1, \ldots, X_e) \mapsto ({}^tX_1, \ldots, {}^tX_e)$ along a suitable isomorphism $\mathbf{R} \otimes D \xrightarrow{\cong} Mat_2(\mathbf{R}) \times \ldots \times Mat_2(\mathbf{R})$.

III'. Totally definite quaternion type. $Z = Z^+$ is a totally real number field and D a quaternion field over Z again, such that

$$\mathbf{R} \otimes_\sigma D \cong \mathbf{H} = \mathbf{R} + \mathbf{Ri} + \mathbf{Rj} + \mathbf{Rk},$$

for all σ as above, $\mathbf{H}$ the HAMILTON quaternion field over $\mathbf{R}$. The involution $'$ is the standard involution $^* = Tr - id$ on D. A suitable isomorphism $\mathbf{R} \otimes D \xrightarrow{\cong} \mathbf{H} \times \ldots \times \mathbf{H}$ transforms $'$ to the componentwise conjugation on the HAMILTON quaternion field $\mathbf{H}$.

IV'. Z^+ is a totally real number field, Z a totally imaginary quadratic extension (a CM-field) of Z^+ and each field embedding σ: $Z \longrightarrow \mathbf{C}$ induces an isomorphism

$$\mathbf{R} \otimes_\sigma D \cong Mat_d(\mathbf{C}).$$

The automorphism $^-$ of complex conjugation on Z coincides with the involution $'$. Moreover, there is a positive involution *: $D \xrightarrow{\cong} D$ defined by restricting the canonical involution $(X_1, \ldots, X_{e^+}) \mapsto ({}^t\overline{X}_1, \ldots, t\overline{X}_{e^+})$ on $Mat_d(\mathbf{C}) \times \ldots \times Mat_d(\mathbf{C})$ along an isomorphism

$$\mathbf{R} \otimes D \xrightarrow{\cong} Mat_d(\mathbf{C}) \times \ldots \times Mat_d(\mathbf{C}).$$

Each positive involution $'$ on D has the form $x \mapsto x' = ax^*a^{-1}$ for a suitable element $a \in D^*$ with image $(A_1, \ldots, A_{e^+})$, A_i hermitian positive definit, in $Mat_d(\mathbf{C}) \times \ldots \times Mat_d(\mathbf{C})$.

2.5 Remark. Let D be a division algebra of rough type I, II, III or IV. By the classification results of ALBERT it can be realized as endomorphism algebra of a suitable simple abelian variety in characteristic 0, except for some of the types III or IV, namely $g = 2e$ or $4e$ (type III) and $g = e^+d^2$ or ed^2 (type IV). In these cases additional conditions to III' respectively IV' must be given.

2.6 Definition. A simple abelian variety A is called of

fine type (I, e), (II, e), (III, e) *or* (IV, d, e), $d, e \in \mathbf{N}$,

if it is of rough type $I.$, $II.$, $III.$, $IV.$, respectively, and the skewfield $\mathbf{End}\,(A)$ has the invariant $e = [Z : \mathbf{Q}]$, Z the center of $\mathbf{End}\,(A)$, and index $d = 1,\ 2$, ($d = 1$ unique in the first three cases).

2.7 Definition. Let A, B be two abelian varieties with isogeny decompositions (2.1) or

$$B \approx B_1^{n1} \times \ldots \times B_k^{nk},$$

respectively. We say that A, B have the same *decomposition type*, if and only if

1) $k = r$;
2) for a suitable numeration the multiplicities m_i, n_i coincide for $i = 1, \ldots, r$;
3) there is a suitable numeration as in 2) such that additionally for $i = 1, \ldots, r$ the fine types of $\mathbf{End}\ (A_i)$ and $\mathbf{End}\ (B_i)$ coincide.

The decomposition type of A is denoted by $DT(A)$.

3. Abelian varieties with imaginary quadratic multiplication

Let M be a number field. If the abelian variety has *M-multiplication*, then one knows that the absolute degree of M is not greater than $2dim\ A$. If, especially, $K = M$ is an imaginary quadratic number field, then we say shortly that A has *imaginary quadratic multiplication.*

3.1. Lemma. *Let A be an abelian variety with K-multiplication, K a number field, with isogeny decomposition* (2.1). *Then each primary component $A_i^{m_i}$ has K-multiplication. Especielly, if A_j is a simple component of multiplicity $m_j = 1$, then A_j has K-multiplication.*

Proof. The composition of ι: $K \longrightarrow \mathbf{End}\ (A)$ of the i-th projection

$$p_i\colon\ \mathbf{End}\ (A) \longrightarrow \mathbf{End}\ (A_i^{m_i})$$

in the direct sum (2.2) is a homomorphism of $\mathbf{Q}$-algebras not being trivial because 1 corresponds to 1. Therefore the kernel of $p_i \circ \iota$ is trivial and $p_i \circ \iota$ an embedding of K into $\mathbf{End}\ (A_i^{m_i})$. Hence $A_i^{m_i}$ has K-multiplication.

□

3.2. Proposition. *Let A be a simple abelian variety with imaginary quadratic K-multiplication and with endomorphism algebra D not of type IV. Then D is uniquely determined up to Z-isomorphy by K, the center Z and the L/Z-norm class of a suitable $z = z_D \in Z^* \backslash N_{L/Z}(L^*)$ having a square root $\mathbf{u}$ in D but not in $L := KZ$. More explicitly one has*

$$D \cong (L/Z,\ \sigma\ , z) \cong L \cdot 1 + L \cdot \mathbf{u} \text{ with relations } \mathbf{u}^2 = z,\ \ \mathbf{u}c = \sigma(c)\mathbf{u} \text{ for all } c \in L, \quad (3.3)$$

where σ denotes the complex conjugation. D is of type II (an indefinit quaternion field) if and only if $z > 0$ and of type III (a definit quaternion field) iff $z < 0$.

Proof. Without loss of generality we can assume that $K = \mathbf{Q}\left(\sqrt{-a}\right)$, $a \in \mathbf{Q}^+$, is a subfield of the skewfield D. Then $Z \cdot K = Z\left(\sqrt{-a}\right)$ is a subfield of D because Z is central in D. Since Z is totally real we conclude that $L := Z \cdot K \subset D$ is a quadratic field extension of K. So it is a maximal (commutative) subfield of D because $[D : Z] = d^2 = 4$, see II, III. (Type I is excluded because the endomorphism algebra is not a totally real number field). Therefore L splits D by the following

3.4 Theorem ((28.5) in [R]). Let D be a skewfield with center Z, and let $d = \sqrt{[D : Z]}$ be the index of D. Let L be a finite extension of Z.

(i) If L splits D, then $d \mid [L : Z]$.

(ii) There exists a smallest positive integer r for which there is an embedding $L \subset Mat_r(D)$ as Z-algebras. With this choice of r, L splits D if and only if (the image of) L is a maximal subfield of $Mat_r(D)$. Furthermore, the centralizer L' in $Mat_r(D)$ of the image of L is a skewfield or a field. Identify L with its image. Then L is a maximal subfield of $Mat_r(D)$ if and only if $L = L'$.

□

Since our $L = KZ$ lies in $D = Mat_1(D)$ we have $r = 1$ in (ii). "L splits D" means that $L \otimes_Z D$ is isomorphic to $Mat_d(L)$ as L-algebra.

Any simple algebra R with center Z is isomorphic to a matrix algebra $Mat_r(D)$ for a suitable $r \in \mathbf{N}^+$ and a skewfield D. Both, the natural number r and the isomorphy class of D, are uniquely determined by R, see [R]. D is called the *skewfield part* of R. Two simple Z-central algebras R and B are called *similar*, iff there is an isomorphism of Z-algebras

$$R \otimes_Z Mat_r(Z) \cong B \otimes_Z Mat_s(Z)$$

for suitable positive integers r and s. The similaryty classes of Z-central simple algebras form via tensor product a group $Br(Z)$ called the *Brauer group of* Z. The unit element is represented by the elements $Mat_r(Z)$, $r \in \mathbf{N}^+$.

For each field extension L of Z there is an exact sequence of groups

$$\begin{array}{ccccccc} 1 \longrightarrow & Br(L/Z) & \longrightarrow & Br(Z) & \longrightarrow & Br(L) \\ & & & R & \mapsto & L \otimes_Z R \end{array}$$

Thus, the similarity classes of Z-central simple algebras splitted by L appear as subgroup of $Br(Z)$. Moreover, if L/Z is a finite Galois extension, there is an isomorphism

$$Br(L/Z) \xrightarrow{\cong} H^2\left(Gal\left(L/Z\right), L^*\right),$$

see [R], theorem (29.12). On the other hand the cohomology theory of groups yields isomorphisms

$$H^2\left(G, L^*\right) \cong L^{*G}/L^{*1+\sigma+\ldots+\sigma^n} = L^{*G}/N_{L/Z}(L^*) = Z^*/N_{L/Z}(L^*),$$

if $G = \langle\sigma\rangle = Gal(L/Z) \cong \mathbf{Z}/n\mathbf{Z}$ is cyclic of order n. We refer to [R] again, 29. exerc. 12, 13.

Now we turn back to the field $L = KZ$ in Prop. 3.2. It is a (cyclic Galois) extension of Z of degree 2 splitting D. In [R], section 30, one can find the explicit description of the our quaternion field D as described in (3.3). Indeed, one has the the exact group sequence

$$\begin{array}{ccccccc} 1 \longrightarrow N_{L/Z}(L^*) & \longrightarrow & Z^* & \longrightarrow & Br(L/Z) & \longrightarrow 1 \\ & & z & \mapsto & [L/Z,\ \sigma,\ z] & \end{array}$$

where $[\ldots]$ denotes the similarity class of the corresponding simple agebra described in (3.3). For this fact we refer to [R], section 30, ex. 1. It means that

3.5 the quaternion fields $(L/Z,\ \sigma,\ z)$ and $(L/Z,\ \sigma,\ z')$ are Z-isomorphic if and only if z/z' is in $N_{L/Z}(L^*)$.

Since A is assumed to be simple, hence $D = \mathbf{End}\ (A) \neq Mat_2(Z)$, it cannot happen in our case that $z \in N_{L/Z}L^*$. The main part of Prop. 3.2 is proved.

For the last statement we show that D is of type III, if $z < 0$. The tensor product of (3.3) with $\mathbf{R}$ yields explicitly

$$\mathbf{R} \otimes D = \mathbf{C} \cdot 1 + \mathbf{C} \cdot \mathbf{u} = (\mathbf{C}/\mathbf{R},\ \sigma,\ z)\,.$$

Comparing with the HAMILTON quaternion field

$$\mathbf{C} \cdot 1 + \mathbf{C} \cdot \mathbf{j} = \mathbf{H} = (\mathbf{C}/\mathbf{R},\ \sigma,\ -1)$$

we see that $\mathbf{R} \otimes D \cong \mathbf{H}$ because $-1/z \in \mathbf{R}_+ = N_{\mathbf{C}/\mathbf{R}}(\mathbf{C}^*)$. This means that D is of type III.

If $z > 0$ then $\mathbf{R} \otimes D \cong (\mathbf{C}/\mathbf{R},\ \sigma,\ +1)$ splits because $+1$ is a norm. This means $\mathbf{R} \otimes D \cong Mat_2(\mathbf{R})$, hence D is of type II. The Proposition 3.2 is proved.

$\square$

3.6 Definition. A simple abelian variety A of dimension g has *complex multiplication*, iff the endomorphism algebra $\mathbf{End}\ (A)$ contains a subfield F of (maximal possible) degree $[F : \mathbf{Q}] = 2g$.

In this case one knows that $\mathbf{End}\ (A) = F$, see [CM], I, Lemma 3.2, and F is a CM-field. Usually non-simple abelian varieties A are called of CM-type, if the condition of the definition is satisfied. In this case one knows that A has a primary isogeny decomposition

$$A \approx B \times \ldots \times B = B^m,\ B \text{ a simple abelian CM} - \text{variety},$$

see [CM], I, Theorems 3.1, 3.3. From (2.2) we get

$$\mathbf{End}\ (A) \cong Mat_m(F) \text{ where } F = \mathbf{End}\ (A) \text{ is a CM} - \text{field.} \tag{3.7}$$

For our purposes it is convenient to use the following generalizing

3.8 Definition. An abelian variety A has *decomposed complex multiplication* (or is *of DCM-type*), if all the simple components A_i of any isogeny decomposition (2.1) of A have complex multiplication.

Together with (2.1),(2.2) we see that the endomorphism algebra of an abelian DCM-variety A has the form

$$\mathbf{End}\ (A) \cong Mat_{m_1}(F_1) \oplus \ldots \oplus Mat_{m_r}(F_r)\,,\ F_i \cong \mathbf{End}\ (A_i) \text{ CM} - \text{fields.} \tag{3.9}$$

4. Endomorphism algebras of simple abelian surfaces with imaginary quadratic multiplication

4.1. Proposition. *If S is a simple abelian surface (over a field of characteristic 0), then* **End** (S) *has one of the following types:*
$(i,1)$ **End** $(S) \cong \mathbf{Q}$;
$(i,2)$ **End** $(S) \cong k$ *a real quadratic field;*
$(ii,1)$ **End** $(S) \cong D$ *an indefinite quaternion field over* $\mathbf{Q}$.
$(iv,2)$ **End** $(S) \cong F$ *a CM-field without imaginary quadratic subfields.*
Especially, S has imaginary quadratic multiplication if and only if it is of type $(ii,1)$. In this case it has K-multiplication for each imaginary quadratic number field K.

Proof. We have $g = dim\ S = 2$. Assume that S is of rough type I. Then **End** (S) is a totally real field of degree $e \mid 2$, therefore $e = 1$ or 2, hence $(i,1)$ and $(i,2)$ are the only possibilities. Both types can be realized. The first by a general abelian surface, the second by a general member of a family of abelian surfaces parametrized by a suitable finite covering of the Hilbert modular surface of the field k.

The quaternion types II, III live with the condition $2e \mid g$. Therefore $e = 1$ and **End** S has to be a quaternion field over $\mathbf{Q}$. But the case $(III,1)$ of a totally definit quaternion field over $\mathbf{Q}$ is not possible by a result of SHIMURA. We refer to [BL], IX, Ex. 1, or, more originally, to [Shm 1], Theorem 5 (a) and Prop. 15.

Now let S be of type IV. The condition $e^+d^2 \mid g$ in IV. yields $d = 1$ and $e^+ = 1$ or 2. By IV' **End** S has to be a CM-field F of absolute degree $e = 2e^+ = 2$ or 4. The case $(IV,1)$ of an imaginary quadratic field F cannot occur, see [BL], IX, Ex. 4, or, more originally, [Shm 1], Theorem 5 (c), (d), Propositions 14 and 18. The remaining possibility is the CM-type $(IV,2)$ because $[F:\mathbf{Q}] = 4 = 2g$.

We consider complex abelian surfaces S of this type with imaginary quadratic K-multiplication and show that they cannot be simple. For this purpose we remember to the definition of types of complex multiplications. Let (A,ι) be an abelian variety of CM-type

$$\iota\colon\ F \longrightarrow \mathbf{End}\ (A),\ \ [F:\mathbf{Q}] = 2dim\ A = 2g.$$

The number field F acts on the complex tangent space $T(A)$ of A at $0 \in A(\mathbf{C})$.

Diagonalizing this action we get g characters $F^* \longrightarrow \mathbf{C}^*$ or, equivalently, field embeddings $\varphi_i\colon\ F \longrightarrow \mathbf{C},\ i = 1,\ldots,g$. It can be shown that they are pairwise different and not conjugated, this means that $\varphi_i \neq \overline{\varphi_j}$ for all $i,j \leq g$. Such a g-tuple $\Phi = (\varphi_1,\ldots,\varphi_g)$ is called a *CM-type*. Usually it is denoted by (F,Φ) or Φ_F. Sometimes it is convenient to write

$$\Phi = \Phi_F = \sum_{i=1}^{g} \varphi_i$$

The abelian CM-varieaty (A,i) with complex multiplication we started with is called *of CM- type* (F,Φ). If M/F is a finite field extension, then

$$\Phi_M = \sum_{i=1}^{g} \sum \{\text{all extensions of } \varphi_i \text{ to } M\}$$

is a type on M. It is called the *lift* of Φ_F to M. A CM-type is called *simple*, if it is not lifted from a subfield. It can be proved that all CM-types lifted from suitable CM-subfields are endomorphism algebras of abelian varieties with decomposed complex multiplication (see [CM], I, Theorem 4.4). Moreover, we need the following

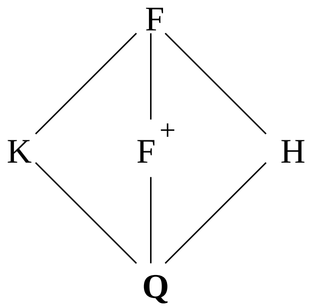
4.2. Proposition *(see [CM], I, Theorem 3.5). The abelian CM-variety A of type (F, Φ) is simple if and only if the type (F, Φ) is simple.*

$\square$

We assume that S is simple of CM type and has K-multiplication. Then F is a CM-field with totally real quadratic subfield F^+. Furthermore, there exists an imaginary quadratic field

$$K \subset F = \mathbf{End}\ S,\ K \neq F^+.$$

So we dispose on quadratic field extensions



Let (φ_1, φ_2) be the CM-type of F-multiplication on S. Without loss of generality we can assume that $\varphi_1 = id$ is the identical embedding. The restrictions of φ_2 to K and H cannot be both complex conjugated to id_K or id_H, respectively. Otherwise φ_2 would be complex conjugated to φ_1, which is not possible by a basic property of CM-types of abelian varieties. Without loss of generality we have $\varphi_1 = \varphi_2$ on H. Therefore the type (φ_1, φ_2) is the lift of the type (H, id_H). We get a contradiction to Proposition 4.2. The only possibility for endomorphism algebras of type IV of simple abelian surfaces is $(iv, 2)$.

The last statement of 4.1 will be proved and a littlebit extended in the following subsection, see Prop. 4.4.

$\square$

5. Abelian threefolds with imaginary quadratic multiplication

5.1 Theorem. *Let A be an abelian threefold over a field of characteristic 0 with imaginary quadratic K-multiplication, E an elliptic curve with K-multiplication. There are only the following types of isogeny decompositions of endomorphism algebras of A, respectively:*

End $(A) \cong K$, *A simple not of CM-type;* (A, K)
End $(A) \cong F$, *F a CM-field, $[F : K] = 3$, A simple abelian CM-threefold;* (A, F)
End $(A) \cong K \times Mat_2(\mathbf{Q}(\sigma))$, $\sigma \in \mathbb{H} = \{z \in \mathbf{C};\ \mathrm{Im}\ z > 0\}$, $A \approx E \times E_\sigma$, $E_\sigma = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\sigma$ *an elliptic curve with imaginary quadratic $\mathbf{Q}(\sigma)$-multiplication, $\mathbf{Q}(\sigma) \neq K$;* $(E \times E_\sigma^2)$
End $(A) \cong Mat_3(K)$, $A \approx E^3$; (E^3)
End $(A) \cong K \times D$, *D an indefinite $\mathbf{Q}$-central quaternion field, $A \approx E \times S$, S a simple abelian surface of type II;* $(E \times S)$
End $(A) \cong K \times Mat_2(\mathbf{Q})$, $A \approx E \times E_\tau^2$, $\tau \in \mathbb{H}$, $E_\tau = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$ *an elliptic curve without imaginary quadratic multiplication;* $(E \times E_\tau^2)$
End $(A) \cong Mat_3(\mathbf{Q})$, $A \approx E_\tau^3$, *E_τ as above an elliptic curve not of CM-type.* (E_τ^3)

Proof. First let A be simple. Then it is not of type I. The types II and III are impossible because of the condition $2e \mid g = 3$. Type IV comes with condition $e + d^2 \mid g = 3$, hence $d = 1$, this means that **End** (A) coincides with its center Z and is therefore a number field containing K. Since $e^+ \mid g = 3$, there are only the possibilities $e^+ = 1$ or 3, hence $Z = K$ or $Z = F$, F a CM-field of absolute degree 6.

Now assume that A has a decomposition $A \approx E \times S$ in two simple abelian varieties of dimension 1 or 2, respectively. Both factors have multiplicity 1. Therefore they have K-multiplication by Lemma 3.1. In characteristic 0 we know that a simple abelian surface S with K-multiplication has type $(ii, 1)$ by Lemma 4.1.

It remains to investigate the case when A has an isogeny decomposition $A \approx A_1 \times A_2 \times A_3$ into three elliptic curves. If these curves are not all isogeneous to each other, then at least one, say A_1, appears with multiplicity 1 in the isogeny decomposition of A. By Lemma 3.1 it has K-multiplication, hence $A_1 \approx E$ with the above notations. Furthermore, the other components cannot appear with multiplicity 1 because, by the same conclusion, they have to be isogeneous to E, hence to A_1, in this case. This contradicts to the 1-multiplicity of A_1. The only possibilities for an elliptic splitting with a component of multiplicity 1 are therefore the cases $(E \times E_\sigma^2)$ and $(E \times E_\tau^2)$.

It remains to check the isogeny case $A \approx A_1^3$. If A_1 has K-multiplication, then we have case (E^3). If A_1 is not a CM-curve, then we get (E_τ^3). Finally, it remains to exclude the case $A \approx Y^3$, Y an elliptic CM-curve with multiplication field $H \neq K$. We show that **End** $(A) \cong Mat_3(H)$ is not compatible with K-multiplication. Namely, in this case **End** (A) is a H-central algebra containing the subfield K. Then **End** (A) contains the compositum $L = H \cdot K$ of degree 4 over $\mathbf{Q}$, hence $4 = [L : \mathbf{Q}]$ divides $2dim\ A = 6$ by [CM], I,Thm.3.1. This is a contradiction.

□

Let us consider a family $\mathbf{A}/T$ of abelian varieties over an irreducible variety T of positive dimension. Let t be a general and s an arbitrary point of T (the general point of a subvariety S of T). Both fibres A_t and A_s are abelian varieties. The isogeny decomposition type of the fibre variety A_s is called a *specialisation* of the decomposition type of A_t. If A, B are arbitrary abelian varieties. Then we call $DT(B)$ a *specialisation* of $DT(A)$, if there is a family $\mathbf{A}/T$ as above with general fibre A and special fibre B. It is equivalent to say that the decomposition type of B is *more special* than that of A. In this case we write

$$DT(A) \longrightarrow DT(B).$$

Moreover, we call $DT(A)$ a *specialisation of* $DT(B)$, if there is a chain of specialisations

$$DT(A_1) \longrightarrow DT(A_2) \longrightarrow \ldots \longrightarrow DT(A_k)$$

with $A = A_1$ and $B = A_k$.

If we restrict our view to families of abelian varieties with K-multiplication, K a fixed number field, then it may happen, that the decomposition type $DT(B)$ of an abelian variety B with K-multiplication is the specialisation of $DT(A)$, A also an abelian variety with K-multiplication, but there is no family $\mathbf{A}/T$ of abelian varieties with K-multiplication joining A, B in the above sense. If such a K-family exists, then we say that $DT(B)$ is a *K-specialisation of* $DT(A)$ writing $DT(A) \xrightarrow{K} DT(B)$. The definition is extended by means of chains of K-specialisations in analogy to the absolute case above.

For example, the decomposition type of an elliptic CM-curve E is a specialisation of the decomposition type of the elliptic curve E_τ without complex multiplication. Therefore also $DT(E^3)$ is a specialisation of $DT(E_\tau^3)$. Let K be the CM-field of E. Since there are embeddings $K \longrightarrow Mat_2(\mathbf{Q}) \longrightarrow Mat_3(\mathbf{Q}) \cong \mathbf{End}\ (E_\tau^3)$ we see that E_τ^3 has also K-multiplication. But there is no irreducible K-family of abelian varieties of positive parameter dimension joining the types $DT(E^3)$ and $DT(E_\tau^3)$. Namely, it is easy to check that the type $DT(E_\tau^3)$ can only occur in a constant K-family. To see this we assume that their is a K-family $\mathbf{A}/T$, $dim\ T > 0$, with A_τ of isogeny decomposition type $DT(E_\tau^3)$ at a general point τ of T. Since the imaginary quadratic numbers $\sigma \in \mathbb{H}$ are dense in $\mathbb{H}$, in a small neighbourhood of τ there are members A_σ of the family of DCM-type $DT(E_\sigma^3)$, $\mathbf{Q}(\sigma) \neq K$. So A_σ has both K-multiplication and $\mathbf{Q}(\sigma)$-multiplication. But this has been already excluded at the end of the proof of Theorem 5.1. So we notice with the above notations that

$$DT(E_\tau^3) \longrightarrow DT(E_\sigma^3) \text{ but not } DT(E_\tau^3) \xrightarrow{K} DT(E_\sigma^3). \tag{5.2}$$

An isogeny decomposition type of abelian varieties is called *K-rigid* (K any number field), if there are only constant irreducible families $\mathbf{A}/T$ of abelian varieties with K-multiplication containing only members of the given type or of its specialisations. Besides of the above example one knows that all DCM-types are rigid, that means $\mathbf{Q}$-rigid, because it is well-known that abelian DCM-varieties have no moduli, see e.g. [BHP], IV.3.
Together with Theorem 5.1 we receive the following

5.3 Corollary. *The hierarchy of K-specialisations of decomposition types of abelian threefolds with imaginary quadratic K-multiplication is described in the following diagram:*

(5.4)

$$\begin{tikzcd}
 & (A, K) \arrow[dl] \arrow[dr] \arrow[ddl] \arrow[ddr] \arrow[ddd] & \\
(E \times B) \arrow[drr] & & (E \times E_\tau^2) \arrow[dll] \\
(E^3) & & (E \times E_\sigma^2) \\
 & (A, F) &
\end{tikzcd}$$

On the bottom row we placed the 0-modular cases (moduli dimension 0); the types (A, K), is 2-modular and the types $(E \times S)$ and $(E \times E_\tau^2)$ are 1-modular.

Remark. It is not clear to me whether the sporadic type (E_τ^3) can appear as K-specialisation of the type (A, K) in a suitable family. The K-specialisations $DT(E \times E_\tau^2) \longrightarrow DT(E \times E_\sigma^2)$ and $DT(E \times E_\tau^2) \longrightarrow DT(E^3)$ can be realized in obvious manner. For the other possible specialisations we did not prove the existence until now. This will be done by examples in ?.

5.5 Corollary-Definition. *Each specific (not isotrivial!) subfamily $\mathbf{A}_C/C$ of an arbitrary family $\mathbf{A}/S$ of abelian threefolds with imaginary quadratic K-multiplication is of type $(E \times S)$ or $(E \times E_\tau^2)$. Each very specific subcurve C of S with respect to $\mathbf{A}/S$ is of decomposition type $(E \times E_\tau^2)$ at general points.*

In the very specific case we call $\mathbf{A}_C/C$ a *modular (sub)family* and C a *modular curve* in S. In the other specific case we call $\mathbf{A}_C/C$ a *Kuga (sub)family* and C a *Kuga curve* in S.

5.6 Corollary. *With the notations of* 5.5 *the following conditions are equivalent:*

(i) *C is specific;*

(ii) *E is an isogeny component of all fibres A_c, $c \in C$;*

(iii) *E is an isogeny component of a general fibre A_γ of $\mathbf{A}_C$;*

(iv) *C is a modular or a Kuga curve.*

Now it is clear that we proved with Theorem 5.1 and Corollaries 5.3 - 5.6 the statements 1.1, 1.2, 1.3 and 1.4 of section 1. The criterion 1.4 (or 5.6 (ii), (iii)) is helpful for discovering explicitly curves of genus 3 with splitting Jacobian threefold with K-multiplication. This will be applied in the main example below. In order to be more precise and more flexible in connection with special and general fibres and for the sake of completeness, we prove

5.7 Lemma. *Let $\mathbf{A}/C$ be a family of abelian varieties over a smooth algebraic curve C, all defined over an algebraically closed field k of characteristic 0. If there is a point $\tau \in C$ outside of $C(k)$ such that A_τ is not simple, then the general fibre $A_\gamma/k(C)$, $\gamma = Spec\ k(C)$ the general point of C, is not simple. Moreover, all fibres A_c, $c \in C$, are not simple.*

Proof. The point τ corresponds to a morphism $Spec\ k(\tau) \longrightarrow C$, where $k(\tau) \neq k$ has to be a transcendental extension of K. Without loss of generality we can assume that C is an affine scheme over k. Then τ is interpreted as k-algebra homomorphism $k[C] \longrightarrow k(\tau)$ not factorizing through k. Therefore the kernel is a non-maximal prime ideal, hence equal to 0 because $k[C]$ is a DEDEKIND domain. The homorphism extends to the cofinte field embedding $k(C) \longrightarrow k(\tau)$.

The endomorphism algebra $End\ (A_\tau)$ has a faithful linear representation in the space of differentials of first kind. This is a vector space with finite basis defined over $k(C)$. Therefore the endomorphisms are stabilized by each element of the GALOIS group $Aut\ \left(k\left(\tau\right)^a / k\left(C\right)\right)$ acting on $A_\tau\left(k\left(\tau\right)^a\right)$, where a denotes the algebraic closure of a field. The elements of $End\ (A_\tau)$ are defined over $k(C)^a$, therefore $End\ (A_\tau)$ and $End\ (A_\gamma)$ coincide. Since an abelian variety A over a field is simple if and only if its endomorphism $\mathbf{Q}$-algebra is a skewfield, the general fibre A_γ inherits this (or the opposite) property from A_τ.

For the last statement we assume that the general fibre A_γ is not simple. Then there is an isogeny $A_\gamma \longrightarrow B \times G$ onto a product of two abelian varieties $B,\ G$ of positive dimension. Product and isogeny are defined over $k(C)^a$, hence already over a finite extension K' of $k(C)$. The field K' is the quotient field of the DEDEKIND domain R' obtained by normalisation of $R := k[C]$ in K'. Therefore we can work with the NERON models $(B \times G)_{R'}$ and $A_{R'}$ over $C' = Spec\ R'$. The latter model is nothing else but the lift $A_{C'}$ of A/C along the finite covering $C' \longrightarrow C$, see [Ar], Cor. 1.4. For existence, uniqueness and other basic properties of NERON models we refer to M. ARTIN's survey article [Ar]. The morphism $A_{K'} \longrightarrow B_{K'} \times G_{K'}$ extends uniquely to a morphism $A_{R'} \longrightarrow (B \times G)_{R'} = B_{R'} \times_{R'} G_{R'}$. By definition, the abelian variety A_K has good (non-degenerate) reduction at each point $c \in C$. This property pulls back to $A_{K'}$ and all points $c' \in C'$, and descents along isogenies, see [CM], II, Cor. 3.5. Therefore the fibres of $(B \times G)_{R'}$ at all $c' \in C'$ are abelian varieties, hence products of two abelian varieties $B_{c'},\ G_{c'}$ of positive dimension. Moreover, the fibre morphism $A_{c'} \longrightarrow B_{c'} \times G_{c'}$ cannot be trivial because projections to factors restrict to projections. Now we see that the fibres $A_{c'}$ are not simple. Let $c \in C$ and $c' \in C'$ a preimage point. Together with $\mathbf{Q} \otimes End\ (A_{c'})$ also $\mathbf{Q} \otimes End\ (A_c)$ is not a skewfield. Namely, both $\mathbf{Q}$-algebras coincide because $k(c')/k(c)$ is a finite field extension. $\square$

6. Semi-period matrices

Shimura defined in [Shm 1] Shimura varieties as quotients of certain bounded domains by arithmetic subgroups acting on them. They parametrize isomorphy classes of polarized abelian varieties with specified multiplication by a fixed finite-dimensional $\mathbf{Q}$-algebra with (anti)involution. For a more recent version we refer to [BL], ch. IX. Abelian threefolds with K-multiplication of type (2,1) explained below, $K = \mathbf{Q}(\sqrt{-d})$ an imaginary quadratic number field, correspond to Picard modular surfaces of K. We repeat Shimura's definition for this case and transfer it to a three-dimensional language, which is more convenient and handable for our purposes. In order to avoid some unessential complications with elementary linear algebra we restrict ourselves to the pricipal case using the $\mathbf{Z}$-lattise $\mathbf{O}^3$

of K^3 and and the skew-symmetric diagonal matrix $T = diag\ \left(\sqrt{-d},\ \sqrt{-d},\ -\sqrt{-d}\right)$, see [BL], IX, 6. The complex unit ball

$$\mathbb{B} = \left\{(\mathrm{u},\mathrm{v});\ |\mathrm{u}|^2 + |\mathrm{v}|^2 < 1\right\} (= \mathbb{H}_2,\ 1 \text{ in [BL]})$$

with the canonical embeddings $\mathbb{B} \subset \mathbf{C}^2 \subset \mathbb{P}^2 = \mathbb{P}^2(\mathbf{C})$ is the corresponding period domain. The (transitive) action of the unitary group

$$\mathbf{U}\left((2,1), \mathbf{C}\right) = \left\{ A \in \mathbf{Gl}_3(\mathbf{C});\ {}^tA \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}$$

on $\mathbb{B}$ is explained as restriction of the canonical action of $\mathbb{P}\mathbf{Gl}_3(\mathbf{C})$ on $\mathbb{P}^2$. The arithmetic subgroup

$$\Gamma := \mathbf{U}\left((2,1), \mathbf{O}\right) = \left\{ A \in \mathbf{Gl}_3(\mathbf{O});\ {}^tA \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\},\quad \mathbf{O} = \mathbf{O}_K,$$

is a *Picard modular group*. The quotient surface $\mathbb{B}/\Gamma$ (or its compactification) is called a *Picard modular surface*. It parametrizes isomorphism classes of polarized abelian threefolds with K-multiplication, which we want to describe now. Let $\tau = (u, v)$ be a point of $\mathbb{B}$ and Λ_τ the $\mathbf{Z}$-lattice in $\mathbf{C}^3$ generated by all vectors

$$\begin{pmatrix} u & v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & u \\ 0 & 0 & 0 & 0 & 1 & v \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \overline{\alpha} \\ \overline{\beta} \\ \overline{\gamma} \end{pmatrix}, \qquad \alpha,\ \beta,\ \gamma \in \mathbf{O}. \tag{6.1}$$

Then the torus $\mathbf{C}^3/\Lambda_\tau$ is an abelian threefold. For the polarization we refer to [Shm 1] or [BL] again. Now let us introduce for $\mu \in \mathbf{C}$ the notation

$$\widehat{\mu} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \overline{\mu} & 0 \\ 0 & 0 & \overline{\mu} \end{pmatrix}. \tag{6.2}$$

Then the above lattice vector can be written as

$$\widehat{\alpha} \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} + \widehat{\beta} \begin{pmatrix} v \\ 0 \\ 1 \end{pmatrix} + \widehat{\gamma} \begin{pmatrix} 1 \\ u \\ v \end{pmatrix}$$

By abuse of language we call

$$N := \begin{pmatrix} u & v & 1 \\ 1 & 0 & u \\ 0 & 1 & v \end{pmatrix}$$

a *normalized semi-period matrix.* It is regular because of $1 > |u^2| + |v^2| \geq |u^2 + v^2|$.

Observe that the last two row vectors $(1, 0, u)$, $(0, 1, v)$ generate the orthogonal complement of $(u, v, 1)$ in $\mathbf{C}^3$ with respect to the symmetric bilinear form represented by the diagonal matrix $diag(1, 1, -1)$. It is the same to say that their conjugates $(1, 0, \overline{u})$, $(0, 1, \overline{v})$ generate the orthogonal complement of $(u, v, 1)$ with respect to the hermitian form

$$\langle\ ,\ \rangle\colon \mathbf{C}^3 \times \mathbf{C}^3 \longrightarrow \mathbf{C},\ \mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},\ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \langle \mathbf{a}, \mathbf{x} \rangle := a\overline{x} + b\overline{y} - c\overline{z} \tag{6.3}$$

on $\mathbf{C}^3$ represented by the same diagonal matrix. Let Λ be a lattice in $\mathbf{C}^3$ and $G \in \mathbf{Gl}_3(\mathbf{C})$. Then the tori $\mathbf{C}^3/\Lambda$ and $\mathbf{C}^3/G\Lambda$ are isomorphic. Especially, for

$$G = d \times g = \begin{pmatrix} d & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \in \mathbf{C}^* \times \mathbf{Gl}_2(\mathbf{C}) \subset \mathbf{Gl}_3(\mathbf{C}) \text{ (canonically embedded)}$$

and $\Lambda = \Lambda_\tau$, the orthogonal relations of the rows in N are preserved. This means that

$$\Pi := GN = \begin{pmatrix} a_1 & a_2 & a_3 \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \\ \overline{c_1} & \overline{c_2} & \overline{c_3} \end{pmatrix} = \begin{pmatrix} {}^t\mathbf{a} \\ {}^t\overline{\mathbf{b}} \\ {}^t\overline{\mathbf{c}} \end{pmatrix} \tag{6.4}$$

$$\langle \mathbf{a}, \mathbf{a} \rangle < 0,\ \mathbf{a} \perp \mathbf{b},\ c \text{ (generating } a^\perp).$$

On this way we do not loose the ball point we started with, namely

$$\mathbb{P}\mathbf{a} = (\mathrm{a}_1 : \mathrm{a}_2 : \mathrm{a}_3) = (\mathrm{u} : \mathrm{v} : 1) \in \mathbb{B}. \tag{6.5}$$

A matrix Π with the conditions of (6.4) is called a *semi-period matrix.* Obviously, the matrices $d \times g$ and $\widehat{\mu}$, $\mu \in \mathbf{C}$, commute with each other. Therefore we can read off the lattice $\Lambda' = (d \times g)\Lambda_\tau$ directly from Π, namely

$$\begin{aligned}
\Lambda' &= d \times g \cdot \left\{ \widehat{\alpha} \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} + \widehat{\beta} \begin{pmatrix} v \\ 0 \\ 1 \end{pmatrix} + \widehat{\gamma} \begin{pmatrix} 1 \\ u \\ v \end{pmatrix};\ \alpha,\ \beta,\ \gamma \in \mathbf{O} \right\} \\
&= \left\{ \widehat{\alpha}(d \times g) \begin{pmatrix} u \\ 1 \\ 0 \end{pmatrix} + \widehat{\beta}(d \times g) \begin{pmatrix} v \\ 0 \\ 1 \end{pmatrix} + \widehat{\gamma}(d \times g) \begin{pmatrix} 1 \\ u \\ v \end{pmatrix};\ \alpha,\ \beta,\ \gamma \in \mathbf{O} \right\} \\
&= \left\{ \widehat{\alpha} \begin{pmatrix} a_1 \\ \overline{b_1} \\ \overline{c_1} \end{pmatrix} + \widehat{\beta} \begin{pmatrix} a_2 \\ \overline{b_2} \\ \overline{c_2} \end{pmatrix} + \widehat{\gamma} \begin{pmatrix} a_3 \\ \overline{b_3} \\ \overline{c_3} \end{pmatrix} \right\}.
\end{aligned}$$

Now we introduce the operation $\widehat{\ }$ for each column vector $\mathbf{x} = {}^t(x_1, \ldots, x_n) \in \mathbf{C}^n$ setting

$$\widehat{\mathbf{x}} := \begin{pmatrix} x_1 \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{pmatrix}, \quad n \geq 2. \tag{6.6}$$

Then it holds that

$$\widehat{\mu}\widehat{\mathbf{x}} = \widehat{(\mu\mathbf{x})} = \widehat{(\mathbf{x}\mu)} \text{ for } \mu \in \mathbf{C} \text{ and } \mathbf{x} \in \mathbf{C}^3. \tag{6.7}$$

The lattice Λ' can be written as

$$\Lambda' = \left\{ \left(\alpha \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}\right)^{\wedge} + \left(\beta \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}\right)^{\wedge} + \left(\gamma \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}\right)^{\wedge}; \ \alpha, \ \beta, \ \gamma \in \mathbf{O} \right\} = \left[\widehat{\Pi}\mathbf{O}^3\right]^{\wedge} \tag{6.8}$$

with $\widehat{\Pi}$ defined in obvious manner:

$$\widehat{\Pi} = \Pi\left(\mathbf{a}, \ \overline{\mathbf{b}}, \ \overline{\mathbf{c}}\right)^{l} and = \Pi\left(\mathbf{a}, \mathbf{b}, \mathbf{c}\right) := \ {}^{t}\left(\mathbf{a}, \mathbf{b}, \mathbf{c}\right). \tag{6.9}$$

Keep in mind that $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is (in general) not a semi-period matrix in our sense.

Altogether we get explicitly

$$\Lambda_{\tau} = \Lambda\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right) := \Lambda' = \left[\ {}^{t}\left(\mathbf{a}, \mathbf{b}, \mathbf{c}\right)\mathbf{O}^3\right]^{\wedge}. \tag{6.10}$$

Since the isomorphy class of the corresponding abelian variety depends only on the ball point $\tau = \mathbb{P}\mathbf{a}$ we introduce the notations

$$A_{\tau} = A_{\mathbf{a}} \cong A\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right) = \mathbf{C}^3/\Lambda\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right) = \mathbf{C}^3/\left[\widehat{\Pi}\mathbf{O}^3\right]^{\wedge}, \tag{6.11}$$

defined up to isomorphy by $\mathbb{P}\mathbf{a}$. Its $\mathbf{Q}$-span of the period lattice is explicitly described by

$$\mathbf{Q} \otimes \Lambda' = \left[\ {}^{t}\left(\mathbf{a}, \mathbf{b}, \mathbf{c}\right)K^3\right]^{\wedge} = \left[\widehat{\Pi}K^3\right]^{\wedge}. \tag{6.12}$$

This follows immediately from the considerations above substituting $\mathbf{O}$ by $K = \mathbf{Q} \otimes \mathbf{O}$.

Now we explain the *K-action of type* $(2,1)$ on A_{τ}. The element $\mu \in \mathbf{O}$ is applied as $\widehat{\mu}$ to Λ'. From (6.7) and (6.8) it follows that $\widehat{\mu}\Lambda' \subseteq \Lambda'$. This defines a natural morphism $\mathbf{O} \longrightarrow \mathbf{End}\ A_{\tau}$ which extends to the embedding

$$\iota\colon\ K \longrightarrow \mathbf{End}\ A_{\tau}, \ \iota(\mu) = \widehat{\mu}\colon\ \left[\ {}^{t}(\mathbf{a}, \mathbf{b}, \mathbf{c})\mathbf{x}\right]^{\wedge} \mapsto \left[\ {}^{t}(\mathbf{a}, \mathbf{b}, \mathbf{c})\mu\mathbf{x}\right]^{\wedge}, \mu \in K. \tag{6.13}$$

In terms of diagrams we notice for $\mu \in \mathbf{O}$

$$
\begin{tikzcd}
0 \arrow[r] & \Lambda(\mathbf{a}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) \arrow[r] \arrow[d] & \mathbf{C}^3 \arrow[r] \arrow[d, "\mu"] & A_{\mathbf{a}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda(\mathbf{a}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) \arrow[r] & \mathbf{C}^3 \arrow[r] & A_{\mathbf{a}} \arrow[r] & 0
\end{tikzcd} \tag{6.14}
$$

For finding interesting sublattices of $\Pi(\mathbf{a}, \mathbf{b}, \overline{\mathbf{c}})$, which possibly split the abelian threefolds $A_{\mathbf{a}}$ (up to isogeny), we introduce the notation

$$\Pi\left(\mathbf{a}, \overline{\mathbf{b}}\right) := {}^{t}\left(\mathbf{a}, \overline{\mathbf{b}}\right) = \begin{pmatrix} a_1 & a_2 & a_3 \\ \overline{b_1} & \overline{b_2} & \overline{b_3} \end{pmatrix} = \begin{pmatrix} {}^{t}\mathbf{a} \\ {}^{t}\overline{\mathbf{b}} \end{pmatrix} \tag{6.15}$$

for the submatrix of $\Pi\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right)$ consisting of its first two rows only. There is a $\mathbf{C}$-*action of signature* $(1,1)$ on $\mathbf{C}^2$ defined by multiplcation with

$$\begin{pmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{pmatrix}$$

which is also denoted by $\widehat{\mu}$ for $\mu \in \mathbf{C}$. With respect to this action restricted to $\mathbf{O}$ we denote the $\mathbf{O}$-module in $\mathbf{C}^2$ generated by the columns of $\Pi\left(\mathbf{a}, \overline{\mathbf{b}}\right)$ by $\Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right)$. As above we have the following relations:

$$\mathbf{Q} \otimes \Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right) = \left[\ {}^{t}\left(\mathbf{a}, \mathbf{b}\right) K^3\right]^{\wedge},\ \Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right) = \left[\ {}^{t}(\mathbf{a}, \mathbf{b}) O^3\right]^{\wedge} \tag{6.16}$$

and

$$d := dim_{\mathbf{Q}} \Pi\left(\mathbf{a}, \overline{\mathbf{b}}\right) \mathbf{Q}^6 = dim_{\mathbf{Q}} \mathbf{Q} \otimes \Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right) = 2 \cdot dim_K \left[\ {}^{t}(\mathbf{a}, \mathbf{b}) K^3\right]^{\wedge}.$$

We recognize that d is an even natural number ≤ 6. We exclude the cases $d \leq 2$. Assume the opposite, this means that the K-dimension is not greater than 1. Then the vectors $\mathbf{a}$, $\mathbf{b}$ must be K-linearly dependent. We obtain the contradiction $0 > \langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle = 0$, namely $\mathbf{a}$, $\mathbf{b}$ come from a semi-period matrix, see (6.4). We proved the first part of the following

6.17 Lemma. The $\mathbf{Z}$-rank (or $\mathbf{Q}$-dimension) of $\Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right)$ (of $\mathbf{Q} \otimes \Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right)$) is equal to 4 or 6. It is equal to 4 if and only if $\mathbf{a},\ \mathbf{b} \in \overline{\mathbf{c}}^{\perp}$ for a suitable $\mathbf{o} \neq \mathbf{c} \in K^3$.

Proof. Assume that $d = 4$, that means $dim_K \left[\ {}^{t}(\mathbf{a}, \mathbf{b}) K^3\right]^{\wedge} = 2$. This happens if and only if there is a non-trivial triple of numbers $\alpha,\ \beta,\ \gamma \in K$ such that

$$\begin{pmatrix} \overline{\alpha} a_1 \\ \alpha \overline{b_1} \end{pmatrix} + \begin{pmatrix} \overline{\beta} a_2 \\ \beta \overline{b_2} \end{pmatrix} - \begin{pmatrix} \overline{\gamma} a_3 \\ \gamma \overline{b_3} \end{pmatrix} = \mathbf{o}. \tag{6.18}$$

With

$$\mathbf{o} \neq \mathbf{c} := \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in K^3$$

this relation can be translated to $\mathbf{a}\perp\mathbf{c}$ and to $\mathbf{b}\perp\mathbf{c}$. Conversely, assume that $\mathbf{o} \neq \mathbf{c} \in K^3$ and $\mathbf{a},\ \mathbf{b}\perp\mathbf{c}$. Then one obtains a non-trivial relation (6.18) setting

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} := c.$$

$\square$

6.19 Definition-Remark. We call a semi-period matrix $\Pi = \Pi\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right)$ *orthogonally normalized* iff $\mathbf{a}\perp\mathbf{b}\perp\mathbf{c}\perp\mathbf{a}$. If Π is an arbitrary semi-period matrix, then we can choose in $\mathbf{a}^\perp$ an orthogonal basis $\mathbf{b}',\ \mathbf{c}'$. Also $\Pi' = \Pi\left(\mathbf{a}, \overline{\mathbf{b}'}, \overline{\mathbf{c}'}\right)$ is a semi-period matrix. There is an element $g \in \mathbf{Gl}_2(\mathbf{C})$ such that $(1 \times g)\Pi = \Pi'$. The corresponding lattices Λ and Λ' generated by the columns of Π or Π' as $\mathbf{O}$-modules are $\mathbf{C}^* \times \mathbf{Gl}_2(\mathbf{C})$-isomorphic. Therefore the corresponding abelian threefolds $\mathbf{C}^3/\Lambda$ and $\mathbf{C}^3/\Lambda'$ are isomorphic. So each isomorphy class of an abelian threefold A_τ with K-multiplication of type $(2,1)$ is represented by an orthogonally normalized semi-period matrix. An $\mathbf{O}$-submodule of type $\Lambda\left(\mathbf{a}, \overline{\mathbf{b}'}\right)$ of small $\mathbf{Z}$-rank 4 sits in Λ if and only if there exists a vector $\mathbf{c}' \in \mathbf{a}^\perp(K) := K^3 \cap a^\perp$. Namely, in this case $\mathbf{b}'$ is defined (uniquely up to a $\mathbf{C}^*$-factor) as basis vector of the line $\mathbf{a}^\perp \cap \mathbf{c}'^\perp \subset \mathbf{C}^3$, and $\Pi\left(\mathbf{a}, \overline{\mathbf{b}'}, \overline{\mathbf{c}'}\right)$ is an orthogonally normalized period matrix equivalent to Π with respect to $\mathbf{C}^* \times \mathbf{Gl}_2(\mathbf{C})$.

6.20 Proposition. With the above notations the abelian variety $A = A_{\mathbf{a}}$ is not simple if there exists $\mathbf{o} \neq \mathbf{c} \in K^3$ orthogonal to $\mathbf{a}$. An abelian subsurface can be realized up to isogeny as $S = \mathbf{C}^2/\Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right)$ coming from a orthogonally normalized semi-period matrix $\Pi\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right)$ corresponding to $\mathbf{a}$, $\mathbf{c}$. More precisely, S has K-multiplication, and there is an isogeny decomposition $A \approx E \times S$, where E is an elliptic curve with K-multiplication. If, moreover, also $\mathbf{a}$ (or $\mathbf{b}$) belong to K^3, then $A \approx E^3$.

Proof. The projection π: $\mathbf{C}^3 = \mathbf{C}^2 \times \mathbf{C} \longrightarrow \mathbf{C}^2$ onto the first factor yields a commutative diagram

(6.21)

$$\begin{tikzcd}
0 \arrow[r] & \Lambda(\mathbf{a}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) \arrow[r] \arrow[d] & \mathbf{C}^3 \arrow[r] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\bar{\pi}"] & 0 \\
0 \arrow[r] & \Lambda(\mathbf{a}, \bar{\mathbf{b}}) \arrow[r] & \mathbf{C}^2 \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

where S is a complex torus of dimension 2 by Lemma 6.17. Taking vertical kernels we recognize that A has an elliptic curve $E = Ker\ \overline{\pi}$ as subvariety. Therefore A is not simple.

The $\mathbf{O}$-multiplications (of type $(2,1)$ or $(1,1)$, respectively) on the varieties are compatible with the morphisms of the diagram. Therefore both, S and E have K-multiplication. Let S' be an abelian subsurface of A complementary to E. This means that there exists an isogeny $E' \times S' \longrightarrow A$ with isogeneous restriction $E \longrightarrow E'$. The composition with $A \longrightarrow S$ sends E' to 0, hence S' and S are isogeneous. Since S' is an algebraic subvariety of A, it has to be abelian. Therefore also its isogeneous image S is an abelian surface.

If $\mathbf{b}$ (or $\mathbf{a}$) $\in K^3$ then $\mathbf{a}$ (or $\mathbf{b}$) lies in the K-line $\mathbf{b}^\perp(K) \cap \mathbf{c}^\perp(K)$ (or $\mathbf{a}^\perp(K) \cap \mathbf{c}^\perp(K)$) in K^3. So we can choose also $\mathbf{a}$ (or $\mathbf{b}$) $\in K^3$, in any case an orthogonally normalized semi-period matrix $\Pi = \Pi\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right) \in Mat_3(K)$ not changing the isomorphy class of A. But then by the same argument as above, the projections of $\mathbf{C}^2$ onto the first and second coordinate axes in the lower row of diagram (6.21) split S in $E \times E$ up to isogeny.

$\square$

6.22 Definition-Remark. We consider the linear subdiscs $\mathbb{D}_\mathbf{c} = \mathbb{B} \cap \mathbb{P}\mathbf{c}^\perp$ of $\mathbb{B}$ for $\mathbf{c} \in \mathbf{C}^3$, $\langle \mathbf{c}, \mathbf{c} \rangle > 0$. $\mathbb{D}_\mathbf{c}$ is called a *K-disc* iff $\mathbf{c} \in K^3$. The *linear subdisc* $\mathbb{D}$ of $\mathbb{B}$ — defined as a non-void intersection of a projective line in $\mathbb{P}^2(\mathbf{C})$ with $\mathbb{B} \subset \mathbb{P}^2(\mathbf{C})$ — is a K-disc if and only if there are two different points $\mathbb{P}\mathbf{a}$, $\mathbb{P}\mathbf{a}' \in \mathbb{P}^2(\mathrm{K})$ on $\mathbb{D}$. Namely, for a K-disc $\mathbb{D}_\mathbf{c}$ the orthogonal complement of $\mathbf{c}$ in K^3 has K-dimension 2 and signature $(1,1)$. Therefore the set of points with coordinates in K is dense on $\mathbb{D}_\mathbf{c}$. Conversely, the orthogonal complement of $K\mathbf{a} + K\mathbf{a}'$ in K^3 has K-dimension 1, if $\mathbf{a}$, $\mathbf{a}' \in K^3$, $\mathbb{P}\mathbf{a} \neq \mathbb{P}\mathbf{b}$, both on $\mathbb{D}$. Now choose a vector $\mathbf{c} \in K^3$ generating $(K\mathbf{a} + K\mathbf{a}')^\perp$. Then the points $\mathbb{P}\mathbf{a}$, $\mathbb{P}\mathbf{a}'$ ly on $\mathbb{D}_\mathbf{c}$, hence $\mathbb{D} = \mathbb{D}_\mathbf{c}$ is a K-disc.

$\square$

6.23 Corollary. If $\tau = \mathbb{P}\mathbf{a} \in \mathbb{B}$ lies on a K-disc $\mathbb{D} \subset \mathbb{B}$, then the abelian variety $A_\tau = A_\mathbf{a}$ is of decomposition type $(E \times S)$, $\left(E \times E_\tau^2\right)$ $\left(E \times E_\sigma^2\right)$ or (E^3) in the sense of Theorem 5.1.

Proof. We know that

$$\begin{aligned} \mathbb{D} = \mathbb{D}_\mathbf{c},\ \mathbf{c} \in \mathrm{K}_+^3 &:= \left\{\mathbf{x} \in K^3;\ \langle \mathbf{x},\ \mathbf{x} \rangle > 0\right\}, \\ \mathbf{c} \perp \mathbf{a} \in K_-^3 &:= \left\{\mathbf{x} \in K^3;\ \langle \mathbf{x},\ \mathbf{x} \rangle < 0\right\}. \end{aligned} \tag{6.24}$$

Choose $\mathbf{o} \neq \mathbf{b} \in \mathbf{c}^\perp \cap \mathbf{a}^\perp$. The abelian variety $A_\mathbf{a} = A_\Pi$, $\Pi = \Pi\left(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}\right)$, splits up to isogeny by Proposition 6.20. The possible types of splitting have been listed in Theorem 5.1.

$\square$

7. Exceptional abelian threefolds with imaginary quadratic multiplication

7.1 Definition. An abelian threefold A with imaginary quadratic K-multiplication ι: $K \longrightarrow \mathbf{End}\, A$ is called *exceptional* if and only if the centralizer $Z_{\mathbf{End}\ A}\iota(K)$ of $\iota(K)$ in the endomorphism algebra of A is bigger than $\iota(K)$. We call also $\mathbf{End}\, A$ or the endomorphism ring $End\ A$ exceptional, if A is. In the opposite case these objects are called *general*.

Remark. Shimura proved in [Shm 1] that **End** $A \cong K$, if the isomorphism class of A represents a "general" point of a Picard modular surface of K. Therefore the set of general abelian threefolds with K-multiplication in the sense of 7.1 is not void.

7.2 Lemma. With the above notations it holds that A is general iff **End** $A \cong K$.

Proof. The direction ($\Leftarrow$) is trivial. For the other direction we first remark that a general abelian threefold A with K-multiplikation must be simple. Namely, if A is not simple, then the elliptic curve E with K-multiplication is an isogeny factor of A ($\approx E \times S$, S an abelian surface), see Theorem 5.1. But then $K \times \mathbf{O} \subset K \times \mathbf{End}\ S$ centralizes $\iota(K)$ additionally to the subfield $\iota(K)$ in contradiction to the assumption of the lemma. For simple A the endomorphism algebra **End** A is (isomorphic to) a number field by Theorem 5.1 again. Thus the centralizer of $\iota(K)$ coincides with **End** (A). The conclusion of the lemma follows immediately.

$\square$

With the notations of the previous section let

$$\tau = \mathbb{P}\mathbf{a} \in \mathbb{B},\ \mathbf{a} = {}^{t}(\mathrm{a}_1, \mathrm{a}_2, 1)\ \text{(w.l.o.g.)} \tag{7.3}$$

be the period point of

$$A = A_\tau = \mathbf{C}^3/\Lambda,\ \Lambda = \Lambda_\tau = \left[\ {}^{t}(\mathbf{a}, \mathbf{b}, \mathbf{c})\, \mathbf{O}^3\right]^\wedge,\ \mathbf{a}\perp\mathbf{b}, \mathbf{c} \in \mathbf{C}^3, \tag{7.4}$$

with respect to the corresponding hermitian form $\langle\ ,\ \rangle$ of signature $(2,1)$, representing a point of the Picard modular surface of K. Remember that

$$\mathbf{Q} \otimes \Lambda_\tau = \left[\ {}^{t}(\mathbf{a}, \mathbf{b}, \mathbf{c})\, K^3\right], \tag{7.5}$$

On this way we get a representation of **End** A in $Mat_3(K)$ in the following manner. We use the *complex representation* (or **C**-*representation*) of **End** A on $\mathbf{C}^3$ corresponding to the middle column of diagram (6.14) forgetting $\widehat{\mu}$ there. Let $C = (c_{ij}) \in Mat_3(\mathbf{C})$ represent an element of **End** A. It acts on $\mathbf{Q}\otimes\Lambda$ by left multiplication with column vectors. Looking at the generating columns and at (7.5) we see that there is a Matrix $M \in Mat_3(K)$ such that

$$C\ {}^{t}(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}) = \left[\ {}^{t}(\mathbf{a}, \mathbf{b}, \mathbf{c})\ {}^{t}M\right]^\wedge = {}^{t}\left[M\,(\mathbf{a}, \mathbf{b}, \mathbf{c})\right]^\wedge. \tag{7.6}$$

The faithful representation $J\colon C \mapsto M$ is called the *K-representation of* **End** A (on K^3 with respect to **a**,**b**,**c**).

7.7 Proposition. The restriction of the K-representation J yields a isomorphism of **Q**-algebras

$$J\colon Z_{\mathbf{End}\ A}\iota(K) \xrightarrow{\sim} End_K\left(\mathbf{a}; \mathbf{a}^\perp\right) := \left\{M \in End\ K^3;\ M\mathbf{a} \in C\mathbf{a},\ M\mathbf{a}^\perp \subset \mathbf{a}^\perp\right\}. \tag{7.8}$$

sending $\iota(K)$ to the center $K \cdot id$ of $End\ K^3$.

Proof. Let $C = (c_{ij}) \in Mat_3(\mathbf{C})$ be an element of **End** A centralizing $\iota(K) = \{diag\,(d, \overline{d}, \overline{d})\,;\ d \in K\}$. Comparing both sides of $C\ diag\,(d, \overline{d}, \overline{d}) = diag\,(d, \overline{d}, \overline{d})\ C$ it is clear that $c_{1i} = c_i = 0$ for $i = 2,\ 3$. Write $C = diag\,(a;\ \overline{B}) \in \mathbf{C}^* \times Mat_2(\mathbf{C})$. For $M = \rho_K(\mathbf{C})$ defined in (7.6) it holds that

$$diag\,(a,\ \overline{B})\ {}^t(\mathbf{a}, \overline{\mathbf{b}}, \overline{\mathbf{c}}) = \left[diag\,(a; \overline{B})\ {}^t(\mathbf{a}, \mathbf{b}, \mathbf{c})\right]^\wedge = -\ {}^t[M\,(\mathbf{a}, \mathbf{b}, \mathbf{c})]^\wedge\,,$$

hence

$$M(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b}, \mathbf{c})\,diag\,(a;\ {}^tB) = (\mathbf{a}, (\mathbf{b}, \mathbf{c})\ {}^tB)\,, \tag{7.9}$$

thus $M\mathbf{a} = a\mathbf{a}$, $M(\mathbf{b}, \mathbf{c}) = (\mathbf{b}, \mathbf{c})\ {}^tB$ and finally $M \in End_K(\mathbf{a}; \mathbf{a}^\perp)$.

Conversely, one gets easily back $C = diag\,(a,\ \overline{B})$ from $M \in End_K\,(\mathbf{a};\ \mathbf{a}^\perp)$.

$\square$

7.10 Corollary. The ball point $\tau = \mathbb{P}\mathbf{a}$ is exceptional if and only if $End_K\,(\mathbf{a};\ \mathbf{a}^\perp)$ is greater than $K \cdot id$. Especially, each exceptional ball point is a fixed point of a K-endomorphism acting effectively on $\mathbb{P}^2$.

$\square$

The following table relates the decomposition types of A listed in Theorem 5.1 with the types of centralizers of $\iota(K)$.

(7.11)

Dec. type of A	**End** A	max. subfields N (in big component)	$Z_{\mathbf{End}\ A}\iota(K) \cong End_K\,(\mathbf{a}; \mathbf{a}^\perp)$
general(A, K)	K	$N = K$	K
$CM(A, F)$	F	$N = F$	F
		CM – field, $[F : K] = 3$	
$(E \times S)$	$K \times \mathbf{Q}$	$[N : \mathbf{Q}] = 2$	$K \times K$
$(E \times E^2_\tau)$	$K \times Mat_2(\mathbf{Q})$	$[N : \mathbf{Q}] = 2$	$K \times K$
$(E \times E^2_\sigma)$ $\sigma \notin K$	$K \times Mat_2\mathbf{Q}(\sigma)$	$[N : \mathbf{Q}(s)] = 2$	$K \times K(\sigma)$
$(E \times E \times E)$	$Mat_3(K)$	$[N : K] = 3$	$K \times Mat_2(K)$

where S is a simple abelian surface with indefinite quaternionic endomorhism algebra $\mathbf{Q}$. If A is not simple, then the maximal subfields N in the greatest component of **End** A are not unique. The projection of $\iota(K)$ to the second component is a maximal subfield of $\mathbf{Q}$ or of $Mat_2(\mathbf{Q})$, respectively. It is its own centralizer there.

In the 5-th case N contains $\mathbf{Q}(\sigma)$ because $\mathbf{Q}(\sigma)$ is the center of the component. Each quadratic extension of $\mathbf{Q}(\sigma)$ can be embedded into $Mat_2(\mathbf{Q}(\sigma))$. A greater subfield is not possible because $dim_{\mathbf{Q}(\sigma)} Mat_2(\mathbf{Q}(\sigma)) = 4$. The centralizer Z' of $K' = p_2\iota(K) \cong K$ in $Mat_2(\mathbf{Q}(\sigma))$ is a $\mathbf{Q}$-algebra containing K' and $\mathbf{Q}(\sigma)$, but $Z' \neq Mat_2(\mathbf{Q}(\sigma))$ because the center of $Mat_2(\mathbf{Q}(\sigma))$ is $\mathbf{Q}(\sigma) \neq K'$. Comparing degrees we obtain $Z' = K'\mathbf{Q}(\sigma)$.

For the last case we use that all abelian varieties of isogeny type $(E \times E \times E)$ are isogeneous by definition and the corresponding endomorphism algebras are isomorphic. Therefore we can work with a special representant $A = \mathbf{C}^3/\Lambda$ as in (7.4) with $\mathbf{a}, \mathbf{b}, \mathbf{c} \in K^3$, see Prop. 6.20. The relation (7.9) and Proposition 7.7 show that the centralizer of $\iota(K)$ corresponds to all matrices $diag(a, B)$ with $a \in K,\ B \in Mat_2(K)$ because $diag(a, B)$ has to be $\mathbf{Gl}_3(K)$-conjugated to $M \in Mat_3(K)$.

□

7.12 Definition. We call $\tau = \mathbb{P}\mathbf{a} \in \mathbb{B}$ *exceptional of third, second* or *first degree*, if A_τ is exceptional and $K(\tau) := K(a_1, a_2)$ is a field extension of K of third, second or first degree. The exceptional point τ as above is called *isolated* if and only if $\mathbf{a}$ is eigenvector of a suitable $M \in End\ (\mathbf{a}; \mathbf{a}^\perp)$ belonging to simple eigenvalue of M.

7.13 Lemma. If τ is isolated exceptional of third degree, then $A = A_\tau$ is simple of CM-type. Moreover, τ is not a point of any K-disc on $\mathbb{B}$.

Proof. By 7.10 the ball point τ is a fixed point of $M \in End\ (\mathbf{a}, \mathbf{a}^\perp) \setminus K \cdot id$, hence $M\mathbf{a} = \alpha\mathbf{a},\ \mathbf{a} \in K(\alpha)^3$ and $K(\alpha) = K(\mathbf{a})$ has degree 3 over K. Then the characteristic polynomial of M is its minimal polynomial. The field $K[M]$ transfers along the isomorphism of 7.7 to a subfield of $Z_{\mathbf{End}\ A}\iota(K)$ of degree 3 over K. This is only possible in the CM-case by Table (7.11). It is clear that $K[M] \cong F = K(\alpha)$, where F denotes the CM-field in the table.

Assume that $\tau \in \mathbb{D},\ \mathbb{D} = \mathbb{P}\mathbf{c}'^\perp$ the K-disc of $\mathbf{c}' \in K$. Then the endomorphism algebra $\mathbf{End}\ A$ contains $Mat_2(\mathbf{Q})$ or a scewfield $\mathbf{Q}$ by Corollary 6.23. But this contradicts obviously to $\mathbf{End}\ A \cong F$.

□

7.14 Lemma. For the ball point τ the following conditions are equivalent:

(i) τ is a K-rational point on $\mathbb{B}$, that means $K(\tau) = K$;

(ii) A_τ is of decomposition type (E^3);

(iii) τ is exceptional of degree 1.

Proof. $(i) \Rightarrow (iii),\ (ii)$: If $\tau = \mathbb{P}\mathbf{a},\ \mathbf{a} \in \mathrm{K}^3$, then one finds easily a vector $\mathbf{c} \in K^3 \cap \mathbf{a}^\perp$. But then A_τ is isogeneos to E^3 by Prop. G.
$(ii) \Rightarrow (i)$: By Table (7.11) $\mathbf{a}$ is eigenvector of suitable $M,\ M' \in End\, K^3$ whose restrictions to $\mathbf{a}^\perp$ generate different quadratic extensions $F,\ F'$ of K in $End\ \mathbf{a}^\perp$. Neither M nor M' generate a cubic field extension of K. The eigenvalues of $M,\ M'$ generate F or F', respectively. If $\mathbf{a} \notin K^3$, then $K(\tau) = F = F'$. This is a contradiction.
$(iii) \Rightarrow (i)$ is trivial.

□

7.15 Lemma. If τ is isolated exceptional of second degree, then $A = A_\tau$ is of decomposition type $(E \times E_\sigma^2)$ and $K(\tau) = K(\sigma) \neq K$. Moreover, there is a K-disc $\mathbb{D}$ on $\mathbb{B}$ containing τ.

Proof. Choose again $M \in End\ \left(\mathbf{a}, \mathbf{a}^\perp\right) \backslash K \cdot id$ with $M\mathbf{a} = \alpha\mathbf{a}$, α a simple eigenvalue of M. The same argument as above yields a subring isomorphic to $K[M]$ in $Z = Z_{\mathbf{End}\ A^\iota}(K)$. The characteristic polynomial $\chi_M(T)$ splits into $(T-\alpha)(T-\alpha^\tau)(T-c)$, $c \in K$, $\{id, \tau\} = Gal\ K(\alpha)/K$ and $K(\tau) = K(\alpha)$. The presence of such subrings in Z is only possible in the last two cases of Table (7.11). The decomposition type (E^3) is excluded by by the previous lemma. So we obtain the decomposition type $(E \times E_\sigma)$, where the maximal subfield of Z is uniquely determined as $K(\sigma)$. Now it is easy to see that $K(\sigma) = K(\alpha)$.

For the second statement observe that the eigenvalues α^τ, c of M are the eigenvalues of $M|_{\mathbf{a}^\perp}$. The eigenvector c' corresponding to c is orthogonal to $\mathbf{a}$ and can be choosen in K^3. So we find $\tau = \mathbb{P}\mathbf{a}$ on the K-disc $\mathbb{D} = \mathbb{P}\mathbf{c}'^\perp$.

□

7.16 Proposition. If $\tau = \mathbb{P}\mathbf{a}$ is a non-isolated exceptional ball point, then it lies on a K-rational linear subdisc of $\mathbb{B}$. Except for the isolated exceptional (CM-)points of third degree all exceptional points are contained in suitable K-discs $\mathbb{D} \subset \mathbb{B}$.

Proof. We have $M\mathbf{a} = a\mathbf{a}$, M has precisely two different eigenvalues a, c, say, of order 2 or 1, respectively. The decomposition of the characteristic polynomial $(T-a)(T-a)(T-c) \in K[T]$ of M in prime polynomials in $K[T]$ shows that $a,\ c \in K$. The restriction $M|_{\mathbf{a}^\perp}$ has eigenvalues a, c. Since c is a simple eigenvalue we find a K-rational eigenvector c' of M in $\mathbf{a}^\perp$. This means that $\tau = \mathbb{P}\mathbf{a}$ belongs to the K-disc $\mathbb{D}\mathbf{c}'$.
Now the second statement comes from the Lemmas 7.13 - 7.15.

□

7.17 Theorem. Let C be a specific curve on the Picard modular surface $\mathbb{B}/\Gamma$ of the imaginary quadratic number field K. Then there exists a K-disc $\mathbb{D} \subset \mathbb{B}$ such that

$$C = \mathbb{D}/\Gamma := \{\mathrm{z} \bmod \Gamma;\ \mathrm{z} \in \mathbb{D}\}. \tag{7.18}$$

The K-disc $\mathbb{D}$ is uniqely determined up to Γ-equivalence. Moreover, the normalization $\widetilde{C}$ of C coincides with $\mathbb{D}/\Gamma_\mathbb{D}$, where $\Gamma_\mathbb{D}$ is the arithmetic group acting on $\mathbb{D}$ defined by

$$\begin{aligned} \Gamma_\mathbb{D} &= \mathbb{P}\mathrm{N}_\Gamma(\mathbb{D}) = \mathrm{N}_\Gamma(\mathbb{D})/\mathrm{Z}_\Gamma(\mathbb{D}), \\ N_\Gamma(\mathbb{D}) &= \{\gamma \in \Gamma;\ \gamma|_\mathbb{D} = \mathbb{D}\}, \\ Z_\Gamma(\mathbb{D}) &= \{\gamma \in \Gamma;\ \gamma|_\mathbb{D} = id_\mathbb{D}\}. \end{aligned} \tag{7.19}$$

Moreover, there is an algebraic group $N_\mathbb{D}$ defined over $\mathbf{Q}$ such that the arithmetic normalizer group $N_\Gamma(\mathbb{D})$ is commensurable with $N_\mathbb{D}(\mathbf{Z})$.

Proof. For a general point $P \in C$ the corresponding abelian threefold A_P has decomposition type $(E \times S)$ or $(E \times E_\tau^2)$. Let $z \in \mathbb{B}$ be a preimage of P. By Proposition 7.16 there is a K-disc $\mathbb{D}_\mathrm{z} \subset \mathbb{B}$ through z. Since $A_z = A_P$ is not of type (E^3) the ball point z does not belong to $\mathbb{B}(\mathrm{K})$ by Lemma 7.14. Especially, z is not an intersection point of two K-discs on $\mathbb{B}$. Therefore $\mathbb{D} = \mathbb{D}_\mathrm{z}$ is uniquely determined by z. The projective K-rational line through $\mathbb{D}$ is denoted by $L = L_z \subset \mathbb{P}$. Choose a small open neighbourhood U of z

such that the restriction of the quotient map p: $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ to U is a finite covering onto $p(U)$ (possibly branched along $C \cap p(U)$). Now take another point $P' \in C \cap p(U)$ nearby P such that $A_{P'}$ is also not of DCM-type (E^3). It has a finite number of preimages $z' \in U$. As for z there is a unique K-disc $\mathbb{D}_{z'}$ or K-line $L_{z'}$ through z', respectively. Assume that $\mathbb{D} \neq \mathbb{D}_z$, hence $L \neq L_{z'}$. The intersection point of L and $L_{z'}$ belongs to $L(K) \subset \mathbb{P}^2(\mathrm{K})$. There are only countable many of them. Therefore there are only countable many points $P' \in C \cap p(U)$ such that $\mathbb{D} \neq \mathbb{D}_{z'}$. So for almost all points $P' \in p(U)$ and their preimages $z' \in U$ we have $\mathbb{D} = \mathbb{D}_{z'}$. All these points ly on $\mathbb{D}/\Gamma$. By contineouity we conclude that C and $\mathbb{D}/\Gamma$ coincide in $p(U)$. Now it is clear that C and $\mathbb{D}/\Gamma$ coincide also globally.

For the last statement of the theorem we refer to [Ho 2].

□

The main result of this (and previous) section is summerized in the follwing

7.20 Theorem. *The specific points P of an open Picard modular surface $\mathbb{B}/\Gamma$ of an imaginary quadratic field K have been characterized now as images along the quotient map of exceptional points τ on $\mathbb{B}$. The corresponding abelian threefold $A_P \cong A_\tau$ is not simple if and only if t belongs to a K-disc on $\mathbb{B}$. It splits up to isogeny completely into $E \times E \times E$ iff $\tau \in \mathbb{B}(\mathrm{K}) = \mathbb{B} \cap \mathbb{P}^2(\mathrm{K})$ or, equivalently, τ is the intersection point of two different K-discs on $\mathbb{B}$.*

8. K-discs on IB

We fix the imaginary quadratic number field $K = \mathbf{Q}\left(\sqrt{-d}\right)$, d a squarefree positive integer. The K-line on $\mathbf{C}^2$ through $0 = (0,0)$ and $(1,c)$, $c \in \mathbf{C}$ arbitrary, is denoted by $L(c)$, and $\mathbb{D}(\mathrm{c})$ denotes its intersection with the unit ball $\mathbb{B} = \mathbb{P}\mathrm{V}(\mathbf{C})^-$, $\mathrm{V} = \mathrm{K}^3$ with the canonical hermitian $(2,1)$-metric corresponding to $diag(+1,+1,-1)$. In canonical projective coordinates we get

$$\mathbb{D}(\mathrm{c}) = \left\{(\mathrm{z} : \mathrm{cz} : 1);\ \mathrm{z} \in \mathbf{C},\ (1 + |\mathrm{c}|^2)|\mathrm{z}|^2 < 1\right\}. \tag{8.1}$$

Setting

$$\mathbf{c} = {}^t(\overline{c}, -1, 0)\,,\ V_{\mathbf{c}} = \mathbf{c}^\perp \subset V, \tag{8.2}$$

we see that

$$\mathbb{D}(\mathrm{c}) = \mathbb{D}_{\mathbf{c}} = \mathbb{P}\mathrm{V}_{\mathbf{c}}(\mathbf{C})^- = \mathbb{P}\mathbf{c}^\perp(\mathbf{C})^-. \tag{8.3}$$

The hermitian vector space $V(\mathbf{C}) = \mathbf{C}^3$ is spanned by the orthogonal basis

$$\mathbf{a} = \mathbf{a}(z) = \begin{pmatrix} z \\ cz \\ 1 \end{pmatrix},\ \mathbf{b} = \mathbf{b}(z) = \begin{pmatrix} 1 \\ c \\ q\overline{z} \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} \overline{c} \\ -1 \\ 0 \end{pmatrix} \text{ with } q := 1 + |c|^2. \tag{8.4}$$

Assume that $\mathbf{a} \in V(\mathbf{C})^-$, that means $\mathbb{P}\mathbf{a} \in \mathbb{D}(\mathrm{c})$. Then we dispose on semi-period matrices

$$\Pi\left(\mathbf{a},\overline{\mathbf{b}},\overline{\mathbf{c}}\right)=\begin{pmatrix} z & cz & 1\\ 1 & \overline{c} & qz\\ c & -1 & 0\end{pmatrix}$$

and lattices

$$\Lambda\left(\mathbf{a},\overline{\mathbf{b}},\overline{\mathbf{c}}\right)=\widehat{\mathbf{O}}\widehat{\mathbf{x}_1}+\widehat{\mathbf{O}}\widehat{\mathbf{x}_2}+\widehat{\mathbf{O}}\widehat{\mathbf{x}_3},$$

where $\mathbf{x}_j$ is the j-th column of

$$\Pi\left(\mathbf{a},\mathbf{b},\mathbf{c}\right)=\begin{pmatrix} z & cz & 1\\ 1 & c & q\overline{z}\\ \overline{c} & -1 & 0\end{pmatrix}.$$

The corresponding abelian threefold $A_{\mathbf{a}}=\mathbf{C}^3/\Lambda\left(\mathbf{a},\overline{\mathbf{b}},\overline{\mathbf{c}}\right)$ has the abelian surface

$$S(z)=S(c;z):=S\left(\mathbf{a},\overline{\mathbf{b}}\right):=\mathbf{C}^2/\Lambda\left(\mathbf{a},\overline{\mathbf{b}}\right) \tag{8.5}$$

as isogeny component, see Proposirtion 6.20 and the definitions around. As usual we write

$$\begin{aligned}\Lambda(z)=\Lambda(c;z):=\Lambda\left(\mathbf{a},\overline{\mathbf{b}}\right)&=\left[\begin{pmatrix} z\ cz\ 1\\ 1\ \ c\ \ q\overline{z}\end{pmatrix}Mat_{3\times 2}(\mathbf{O})\right]^{\wedge},\\ \mathbf{Q}\otimes\Lambda(z)=\left[\begin{pmatrix} z & 1\\ 1 & q\overline{z}\end{pmatrix}Mat_2(K)\right]^{\wedge}&=\widehat{K}\widehat{\mathbf{z}}+\widehat{K}\widehat{\mathbf{y}},\\ \mathbf{z}=\begin{pmatrix} z\\ 1\end{pmatrix},\quad \mathbf{y}=\begin{pmatrix} 1\\ q\overline{z}\end{pmatrix}.&\end{aligned} \tag{8.6}$$

At general points P of the arithmetic curve $\mathbb{D}/\Gamma$ on the Picard modular surface $\mathbb{B}/\Gamma$ the abelian threefold A_P is of modular decomposition type $(E\times E_\tau^2)$ or of quaternionic (scew field) type $(E\times S)$. This depends only on $\mathbb{D}$. The notation of both types is transfered to the K-discs. We would like to distinguish both types of K-discs by a suitable arithmetic or geometric condition. For this purpose we will apply a criterion of Ruppert about splitting of abelian surfaces knowing a period matrix.

Let Λ be a lattice in $\mathbf{C}^2$ such that $S=\mathbf{C}^2/\Lambda$. The determinant det: $\mathbf{C}^2\times\mathbf{C}^2\longrightarrow\mathbf{C}$ defines by restriction to the elements of the 4-dimensional vector space $\mathbf{Q}\otimes\Lambda$ over $\mathbf{Q}$ an alternating form

$$\delta\colon\ (\mathbf{Q}\otimes\Lambda)\times(\mathbf{Q}\otimes\Lambda)\longrightarrow\mathbf{C},\ \ \delta(\mathbf{u},\mathbf{v})=det(\mathbf{u},\mathbf{v})=\ {}^t\mathbf{u}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v} \tag{8.7}$$

This form is called hyperbolic, if there is a direct decomposition

$$\mathbf{Q}\otimes\Lambda=V\oplus W \tag{8.8}$$

into two δ-isotropic $\mathbf{Q}$-subspaces of dimension 2.

8.9 Proposition (Ruppert′s criterion). The torus $S=\mathbf{C}^2/\Lambda$ has an isogeny decomposition into elliptic curves if and only if δ is hyperbolic.

The simple proof is given in see [BL], X,(6.1), where it is not necessary to suppose that the torus S is an abelian surface.

$\square$

We apply the criterion to $S = S(z)\ S(c;z)$, $\Lambda = \Lambda(z) = \Lambda(c;z)$ as defined above, connected with arithmetic discs $\mathbb{D} = \mathbb{D}(c)$ through $0 \in \mathbb{B}$. For arbitrary pairs

$$\widehat{\mathbf{u}} = \widehat{a}\widehat{\mathbf{z}} + \widehat{b}\widehat{\mathbf{y}},\ \ \widehat{\mathbf{v}} = \widehat{c}\widehat{\mathbf{z}} + \widehat{d}\widehat{\mathbf{y}} \in \mathbf{Q} \otimes \Lambda,\ \ a,\ b,\ c,\ d \in K,$$

we can write

$$\delta\left(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}\right) = det \left[(\mathbf{z}, \mathbf{y}) \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right]^{\wedge}.$$

We work especially with

$$\mathbf{z} = \ {}^{t}(z, 1) \text{ and } \mathbf{y} = \ {}^{t}(1, q\overline{z}),\ \ q \in \mathbf{Q}^{*},\ \ \infty \geq [K(z) : K] > 2.$$

Then one gets

$$\begin{aligned}\delta\left(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}\right) &= det \left[\begin{pmatrix} z & 1 \\ 1 & q\overline{z} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right]^{\wedge} = det \begin{pmatrix} az + b & cz + d \\ \overline{b}qz + \overline{a} & \overline{d}qz + \overline{c} \end{pmatrix} \\ &= q\left(\overline{d}a - \overline{b}c\right) z^{2} + \left[q\left(\overline{d}b - \overline{b}d\right) + \left(\overline{c}a - \overline{a}c\right) \right] z - \left(\overline{d}a - \overline{b}c\right).\end{aligned}$$

Since z cannot satisfy a non-trivial quadratic equation over K we conclude that

$$\delta\left(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}\right) = 0 \text{ iff} \tag{8.10}$$

(i)
$$det \begin{pmatrix} a & c \\ \overline{b} & \overline{d} \end{pmatrix} = 0 \text{ and}$$

(ii)
$$q \cdot det \begin{pmatrix} b & d \\ \overline{b} & \overline{d} \end{pmatrix} = det \begin{pmatrix} a & c \\ \overline{a} & \overline{c} \end{pmatrix}.$$

The first condition is equivalent to $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & ta \\ b & \overline{t}b \end{smallmatrix}\right)$ or

$(i)'$
$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ta \\ \overline{t}b \end{pmatrix} \text{ for a suitable } t \in K$$

or, more symmetrically, to

$(i)''$
$$\widehat{K}\begin{pmatrix} c \\ d \end{pmatrix} = \widehat{K}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Substituting $c,\ d$ by ta or$\overline{t}b$, respectively, the second condition transforms to

$$q|b|^2 det \begin{pmatrix} 1 & \bar{t} \\ 1 & t \end{pmatrix} = q det \begin{pmatrix} b & t\bar{b} \\ \bar{b} & t\bar{b} \end{pmatrix} = \ldots = -det \begin{pmatrix} a & ta \\ \bar{a} & \bar{t}\bar{a} \end{pmatrix} = -|a|^2 det \begin{pmatrix} 1 & t \\ 1 & \bar{t} \end{pmatrix}.$$

This is equivalent to

$$(ii)' \qquad\qquad t \in \mathbf{Q} \text{ or } |a|^2 = q|b|^2.$$

Now we distinguish two cases:

1.) q is <u>not a norm</u> of an element of K: Then the conditions are equivalent to:

$$t = 0 \text{ or } \begin{pmatrix} c \\ d \end{pmatrix} = t \cdot \begin{pmatrix} a \\ b \end{pmatrix}, \; t \in \mathbf{Q}^*.$$

This means that $\delta(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}) = 0$ is only possible for $\widehat{\mathbf{v}} \in \mathbf{Q}\widehat{\mathbf{u}}$. In this case there cannot exist a two-dimensional δ-isotropic $\mathbf{Q}$-subspace of $\mathbf{Q} \otimes \Lambda$. By Ruppert's criterion the torus $S = \mathbf{C}^2/\Lambda$ is simple.

2.) q <u>is a norm</u> of an element of K: Then we denote by μ the $\mathbf{Q}$-linear isomorphism

$$\mu\colon\; K^2 \longrightarrow \left[(\mathbf{z}, \mathbf{y})\, K^2\right]^\wedge = \mathbf{Q} \otimes \Lambda, \; \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \widehat{\mathbf{u}} = \left[(\mathbf{z}, \mathbf{y}) \begin{pmatrix} a \\ b \end{pmatrix}\right]^\wedge.$$

Next we satisfy the second of the conditions (ii') setting $q = |a|^2/|b|^2$ for suitable $a,\; b \in K$. This is possible because q is a norm. Each pair $\widehat{\mathbf{u}},\; \widehat{\mathbf{v}} \in \mu\left(\widehat{K}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right) =: V$ satisfies obviously the condition $(i)''$. Together with (8.10) it follows that V is a δ-isotropic $\mathbf{Q}$-plane in $\mathbf{Q} \otimes \Lambda$.

It is easy to find

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \notin \mathbf{Q} \begin{pmatrix} a \\ b \end{pmatrix}$$

such that also $a',\; b' \in K$ and $q = |a'|^2/|b'|^2$; take for example

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \; \lambda \in K \backslash \mathbf{Q}.$$

Then also $W := \mu\left(\widehat{K}\left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right)\right)$ is a two-dimensional δ-isotropic $\mathbf{Q}$-subspace of $\mathbf{Q} \otimes \Lambda$ and $V \cap W = \{0\}$ because the preimages of $V,\; W$ along μ are different K-vector spaces $\widehat{K}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ or $\widehat{K}\left(\begin{smallmatrix} a' \\ b' \end{smallmatrix}\right)$, respectively, of dimension 1. Therefore they have trivial intersection. So we found a direct decomposition (8.8) into two δ-isotropic subspaces. Now Ruppert's criterion tells us that $S = \mathbf{C}^2/\Lambda$ has an isogeny decomposition into elliptic curves.

8.11 Theorem. Let $\mathbb{D} = \mathbb{D}(\mathrm{c}) = \mathbb{D}_{\mathbf{c}}$ be the K-subdisc of $\mathbb{B}$ and $S(z) = \mathbf{C}^2/\Lambda(z)$, $z \in \mathbb{D}(\mathrm{c})$, the corresponding two-dimensional abelian isogeny factor of A_z, all defined in (8.1),…,(8.6). Furthermore we denote by Γ the Picard modular group $\mathbf{U}((2,1), \mathbf{O})$,

$\mathbf{O} = \mathbf{O}_K$ and by $\Gamma_{\mathbb{D}} = N_\Gamma(\mathbb{D})/Z_\Gamma(\mathbb{D})$ the corresponding arithmetic group acting effectively on $\mathbb{D}$. Then the following conditions are equivalent:

- (i) $S(z)$ splits at general points $z \in \mathbb{D}$, hence everywhere on $\mathbb{D}$;
- (ii) A_z is of modular decomposition type $\left(K \times E_\tau^2\right)$ on $\mathbb{D}$ in general;
- (iii) $\langle \mathbf{c}, \mathbf{c}\rangle$ belongs to the norm group $N(K^*) \subset \mathbf{Q}^*$;
- (iv) the 4-dimensional $\mathbf{Q}$-algebra $\mathbf{Q} \otimes \Lambda(z)$ splits directly into two δ-isotropic $\mathbf{Q}$-subspaces of $\mathbf{Q}$-dimension 2;
- (v) the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ contains a Γ-cusp point $k \in \partial_\Gamma \mathbb{B} = \partial\mathbb{B}(\mathrm{K}) = \mathrm{K}^2 \cap \partial\mathbb{B}$;
- (v') the closure $(\mathbb{D}/\Gamma)^\wedge$ of the arithmetic curve $\mathbb{D}/\Gamma$ on the Baily-Borel compactified Picard modular surface $(\mathbb{B}/\Gamma)^\wedge$ goes through a cusp singularity $\widehat{\kappa} \in (\mathbb{B}/\Gamma)^\wedge\ \mathbb{B}/\Gamma$.
- (vi) $\Gamma_{\mathbb{D}}$ is a modular group, that means that a suitable $\mathbb{P}\mathbf{Gl}_2(\mathbf{C})$-conjugate of $\Gamma_{\mathbb{D}}$ is commensurable with $\mathbb{P}\mathbf{Sl}_2(\mathbf{Z})$.

The equality $\partial_\Gamma \mathbb{B} = \mathrm{K}^2 \cap \partial\mathbb{B}$ has been first proved in [Ho I].

The equivalence of the properties $(i) - (iv)$ has been proved above. The properties (v) and (v') are obviously equivalent. The first four and last three conditions are joined by some results of Shimura and an old classification result for hermitian vector spaces over number fields due to Landherr. We delegate this equivalence proof to the next section.

8.12 Remark. It is not necessary to restrict the proof of 8.11 to discs through 0. The equivalence of the Ruppert criterion (iv) with (i) and (iii) can be proved in a similar but not so convenient manner for all K-discs on $\mathbb{B}$, and all other equivalences are proved quite generally in this article. For a full proof of the first equivalences one parametrizes the K-discs in the following manner:

The projective line $L(b,c) \subset \mathbb{P}^2$ through $\binom{b}{0}$, $\binom{0}{-c} \in \mathbf{C}^2$, $b,\ c \in \mathbf{C}^*$, has the parametrization

$$L(b,c) = \{\mathbb{P}\mathbf{a}(\mathrm{z});\ \mathrm{z} \in \mathbf{C}\},\ \mathbf{a}(z) = {}^t\left(bz, c\left(b-z\right), b\right).$$

The vector $\mathbf{c} := {}^t\left(\overline{c}, \overline{b}, \overline{bc}\right)$ is orthogonal to all $\mathbf{a}(z)$, hence $L(b,c) = \mathbb{P}\mathbf{c}^\perp$, and $L(b,c)$ intersects $\mathbb{B}$ if and only if $\langle \mathbf{c}, \mathbf{c}\rangle/|c|^2 = 1 + |b/c|^2 - |b|^2 > 0$. Under this condition the disc $\mathbb{D}(\mathrm{b},\mathrm{c}) = \mathrm{L}(\mathrm{b},\mathrm{c}) \cap \mathbb{B} = \mathbb{D}_{\mathbf{c}}$ is defined. With $b,\ c \in K^*$ one parametrizes explicitly on this way all K-discs on $\mathbb{B}$ not containing 0. One obtains explicit expressions for $\mathbf{b}(z) \neq \mathbf{o}$ orthogonal to $\mathbf{a}(z)$ and $\mathbf{c} = \mathbf{c}(b,c)$, for $\Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right)$, $\mathbf{Q} \times \Lambda\left(\mathbf{a}, \overline{\mathbf{b}}\right)$ and so on. This leads finally to the isogeneous splitting condition $1 + |b/c|^2 - |b|^2 \in N(K^*)$ for all abelian surfaces $S(z) = \mathbf{C}^2/\Lambda\left(\mathbf{a}(z), \overline{\mathbf{b}}(z)\right)$, $z \in \mathbb{D}(\mathrm{b},\mathrm{c})$, sitting in the abelian threefolds A_z, respectively.

9. Q-central quaternion algebras and unitary groups

Let K be a CM-field with maximal total real subfield F; then $[K:F] = 2$. By (V,Φ) we denote a non-degenerate hermitian vector space over K of dimension n. The infinite places of F are numerated in the following manner:

$$F_l = \mathbf{R} \text{ for } 1 \leq r,\ K_l = \mathbf{C} \text{ for } 1 \leq l \leq t,\ K_l \text{ not a field for } t < l \leq r. \tag{9.1}$$

Then $V_l = K_l \times V$ together with the extension Φ_l of Φ to V_l is a hermitian vector space for $l \leq t$ of signature (p_l, s_l), $p_l + s_l = dim_K V$, say, where p_l denotes the positive elements of a diagonalizing matrix for Φ_l. For a second non-degenerate hermitian K-vector space (V', Φ') we use the obviously corresponding notations n', r', t', p'_l, s'_l. An *isometric embedding* μ: $V \longrightarrow V'$ is an injective K-linear map satisfying $\Phi'(\mu(\mathbf{a}), \mu(\mathbf{b})) = \Phi(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a}$, $\mathbf{b} \in V$. The isometric embedding μ is called an *isometry*, iff it is bijective. For all finite places $\mathbf{p}$ of F corresponding to prime ideals of $\mathbf{O}_F$ we dispose on norm maps $N_\mathbf{p}$: $K_\mathbf{p} \longrightarrow F_\mathbf{p}$, where $K_\mathbf{p} = K \times F$ is a field extension of degree 2 (iff $\mathbf{p}$ is inert in K) or of degree 1 (iff $\mathbf{p}$ is ramified in K) or $K_\mathbf{p}$ is $F_\mathbf{p}$-isomorphic to $F_\mathbf{p} \times F_\mathbf{p}$ (iff $\mathbf{p}$ is decomposed in K). The discriminant $d(\Phi)$ is the determinant of a Gram matrix $\left(\Phi(\mathbf{a}_i, \mathbf{a}_j)\right)_{i,j=1\ldots n}$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is a basis of V. It is uniquely determined up to multiplication with elements of the norm group $N(K^*) \subset F^*$, where N denotes the norm map $N_{K/F}$: $K \longrightarrow F$.

9.2 Theorem (Landherr, see [Ln 2]). With the above notations the following conditions (i), (ii) are equivalent:

(i) There exists an isometric embedding μ: $V' \longrightarrow V$.

(ii) $n > n'$ and for $1 \leq l \leq t$ it holds that $0 \leq s_l - s'_l \leq n - n'$ or $n' = n$, $s_l = s'_l$ for $1 \leq l \leq t$ and $d(\Phi) \in N_\mathbf{p}\left(K^*_\mathbf{p}\right) \cdot d(\Phi')$ for all primes $\mathbf{p}$ of F.

$\square$

In the case $n = n'$ each isometric embedding is an isometry because injective K-linear embeddings have to be surjective, hence isomorphic. For $n = n' = 1$ the theorem reduces to the well-known local characterizations of norms:

9.3 For two elements f, $f' \in F^*$ it holds that $f' \in N(K^*) \cdot f$ if and only if $f' \in N_\mathbf{p}(K^*_\mathbf{p}) \cdot f$ for all primes $\mathbf{p}$ of F. Especially, $f \in N(K^*)$ iff $f \in N_\mathbf{p}(K^*_\mathbf{p})$ for all primes $\mathbf{p}$ of F.

In the case of an imaginary quadratic number field K we have $F = \mathbf{Q}$, hence $1 = t = r$ in (9.1). So we can omit the only index $l = 1$ in our notations ($F_1 = \mathbf{R}$, $K_1 = \mathbf{C}$). For $n = n' = 2$ and $s = s' = 1$ we get

9.4 Corollary. Let K be an imaginary quadratic number field. The indefinite hermitian K-planes (V, Φ) and (V', Φ') are isometric if and only if $d(\Phi') \in N(K^*)d(F)$.

Our main objects are indefinite K-subplanes of the hermitian K-vector spaces $V = K^3$ with the metric $\langle\ ,\ \rangle$ corresponding to $diag(+1, +1, -1)$ of signature $(2, 1)$, K an imaginary quadratic number field. If $\mathbf{W}$ is any indefinite hermitian K-plane, then it is isometric to a hermitian K-subplane V' of V by Theorem 9.2. Moreover,

$$V = V' \cap K\mathbf{c} \text{ for any } \mathbf{o} \neq \mathbf{c} \in V'^{\perp} = K\mathbf{c} \subset V = K^3,\ V' = \mathbf{c}^{\perp}. \tag{9.5}$$

The multiplicativity of discriminants for orthogonal decompositions and isometry invariance up to $N(K^*)$-multiplication yield

$$\langle \mathbf{c}, \mathbf{c} \rangle d(W) \sim_N \langle \mathbf{c}, \mathbf{c} \rangle d(V') = \langle \mathbf{c}, \mathbf{c} \rangle d\left(\mathbf{c}^{\perp}\right) \sim_N d(V) = -1, \tag{9.6}$$

where $\sim_N$ denotes the $N(K^*)$-equivalence in $\mathbf{Q}^*$. Since V' is indefinite we know that $d(V') = d\left(\mathbf{c}^{\perp}\right) < 0$, hence $\langle \mathbf{c}, \mathbf{c} \rangle > 0$ that means $\mathbf{c} \in V^+$. Also from (9.6) follows that the $N(K^*)$-class of $\langle \mathbf{c}, \mathbf{c} \rangle$ depends only on the $N(K^*)$-class of W.

Take conversely a vector $\mathbf{b} \in V^+$ with $\langle \mathbf{b}, \mathbf{b} \rangle \sim_N \langle \mathbf{c}, \mathbf{c} \rangle$ $(\sim_N -d(W))$. Because of $K\mathbf{c} \cap \mathbf{c}^\perp = V = K\mathbf{b} \cap \mathbf{b}^\perp$ the orthogonal complements $\mathbf{b}^\perp$ and $\mathbf{c}^\perp$ are indefinite K-planes with norm-equivalent discriminants $-\langle \mathbf{b}, \mathbf{b} \rangle$ or $-\langle \mathbf{c}, \mathbf{c} \rangle$, respectively. Therefore $\mathbf{b}^\perp$ and $\mathbf{c}^\perp$ are isometric, but also $K\mathbf{b}$ and $K\mathbf{c}$ are by the above criteria. By linear extension we get an isometric endomorphism

$$\mu\colon\ V = K\mathbf{b} \cap \mathbf{b}^\perp \xrightarrow{\sim} K\mathbf{c} \cap \mathbf{c}^\perp = V$$

sending $c\mathbf{b}$ to $\mathbf{c}$ for a suitable $c \in K^*$. Vice versa each triple $\mathbf{b},\ \mathbf{c} \in V^+,\ c \in K^*$ with $\langle c\mathbf{b}, c\mathbf{b} \rangle = \langle \mathbf{c}, \mathbf{c} \rangle$ defines on this way an isometric endomorphism $\mu \in U(V) = U((2,1), K)$ sending $c\mathbf{b}$ to $\mathbf{c}$. Altogether we get the following

9.7 Corollary. With the above notations the following conditions are equivalent:

(i) $\quad d(V_\mathbf{b}) \in N(K^*)d(V_\mathbf{c})$;
(ii) $\quad$ the indefinite planes $V_\mathbf{b} = \mathbf{b}^\perp$ and $V_\mathbf{c} = \mathbf{c}^\perp$ are isometric;
(iii) $\quad \langle \mathbf{b}, \mathbf{b} \rangle \in \langle \mathbf{c}, \mathbf{c} \rangle \cdot N(K^*)$;
(iv) $\quad \mathbf{c} \in K^* \cdot U((2,1), K)\mathbf{b}$;
(v) $\quad V_\mathbf{b} = g(V_\mathbf{c})$ for a suitable $g \in U((2,1), K)$;

$\square$

Now we turn our attention to the connection of unitary groups as above with indefinite $\mathbf{Q}$-central quaternion skewfields. Proofs of facts listet below can be found in [Shm 3], ch. IX and [Shm 2]. For comparising first facts with the 1-dimensional modular case we remember to the following

9.8 Remark. Let Γ be a sublattice of

$$\mathbf{Gl}_{2+}(\mathbf{R}) = \{g \in Mat_2(\mathbf{R}),\ det\ g > 0\}$$

commensurable with $\mathbf{Sl}_2(\mathbf{Z})$. The group act on the Poincar upper half plane $\mathbb{H} = \{z \in \mathbf{C};\ \mathrm{Im}\ z > 0\}$ via linear fractions. The lattice Γ is a Fuchsian group of first kind, and the quotient $\mathbb{H}/\Gamma$ is a non-compact quasiprojective curve. The set of fixed points of elements of $\mathbf{Gl}_{2+}(\mathbf{R})$ on $\mathbb{H}$ coincides with the set of numbers $z \in \mathbb{H}$ generating an imaginary quadratic extension of $\mathbf{Q}$.

One has a similar situation for the $\mathbf{Q}$-central indefinit quaternion fields $D \subset Mat_2(\mathbf{R}) = \mathbf{R} \otimes D$. There is an algebraic group $\mathbf{D}$ defined over $\mathbf{Q}$ such that $\mathbf{D}(\mathbf{R}) = D^*$. The determinant defines the norm n on $\mathbf{D}$. The condition $n(g) = 1$ defines a $\mathbf{Q}$-subgroup $\mathbf{SD}$ of $\mathbf{D}$ and the condition $\mathrm{n}(g) > 0$ a subgroup D_+ of D^* acting on $\mathbb{H}$. Let T_f be an open compact subgroup of the finite valuation part of the group $\mathbf{SD}(\mathbb{A})$, $\mathbb{A} = \mathbb{A}\mathbf{Q}$ the adéles of $\mathbf{Q}$, $T = T_f\mathbf{SD}(\mathbf{R})$ and $\Gamma = \Gamma_T = T \cap \mathbf{SD}(\mathbf{Q})$. Then Γ is a Fuchsian group of first kind with compact quotient curve $\mathbb{H}/\Gamma$.

If $z \in \mathbb{H}$ is a fixed point of an element $g \in D_+(\mathbf{Q})\backslash\mathbf{Q}$, then $\mathbf{Q}(z)$ is an imaginary quadratic field. Conversely, each $\mathbf{Q}$-linear embedding $\iota\colon\ K \longrightarrow D$ defines a unique such fixed point with isotropy group $\iota(K^*) = \{g \in D_+(\mathbf{Q}); g(z) = z\}$.

9.9 Proposition. Let D be an indefinit quaternion field as described above. Then the quotient $\mathbf{SD}(\mathbf{Q})\backslash\mathbf{SD}(\mathbb{A})$ is compact. This is also true for the quotient curve $\Gamma\backslash\mathbb{H}$.

$\square$

For proofs we refer to [Shm 3], ch. IX, see also [GGP], ch.I, App. 1.4, for a more direct and explicit variant.

9.10 Remark. The only arithmetic Fuchsian groups (of first kind, acting proper discontineously on $\mathbb{H}$) are the groups commensurable with $\mathbf{Sl}_2(\mathbf{Z})$ or subgroups Γ of quaternion fields. This has been proved by A. Weil, see [GGP], ch.I, App. 1.1

Let K be an imaginary quadratic field and $V \cong K^3$ a three-dimensional K-vector space endowed with hermitian form $\langle\ ,\ \rangle$: $V \times V \longrightarrow K$ of signature $(2,1)$. Its extension to $V(\mathbf{R}) = \mathbf{R} \otimes V = \mathbf{C}^3$ is also denoted by $\langle\ ,\ \rangle$. As above we restrict our attention to the case where $\langle\ ,\ \rangle$ is given by the diagonal matrix $diag(1,1,-1) \in Mat_3(\mathbf{Q})$, that means

$$\langle \mathbf{v}, \mathbf{w}\rangle = {}^t\mathbf{v} \cdot diag(+1,+1,-1) \cdot \overline{\mathbf{w}} \text{ for } \mathbf{v},\ \mathbf{w} \in V(\mathbf{R}).$$

Cosider the complex 2-ball

$$\mathbb{B} = \left\{(\mathrm{u},\mathrm{v}) \in \mathbf{C}^2;\ |\mathrm{u}|^2 + |\mathrm{v}|^2 < 1\right\} = \mathbb{P}\mathrm{V}(\mathbf{R})^- = \mathrm{V}(\mathbf{R})^-/\mathbf{C}^*$$

with $V(\mathbf{R})^- = \{\mathbf{v} \in V(\mathbf{R});\ \langle v, v\rangle < 0\}$. There is an algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ such that $\mathbf{G}(\mathbf{Q}) = \mathbf{U}((2,1),K)$ and $\mathbf{G}(\mathbf{R}) = U((2,1),\mathbf{C})$. The Lie group $\mathbf{G}(\mathbf{R})$ acts on $\mathbb{B}$. Let $\mathbf{O} = \mathbf{O}_K$ denote the ring of integers in K. The quotient surface $\mathbb{B}/\Gamma$ of $\mathbb{B}$ by the *Picard modular group* $\Gamma := \mathbf{U}((2,1),O) = \mathbf{G}(\mathbf{Z})$ is the (complex, open) *Picard modular surface.*

For abbrevity we call a vector $\mathbf{c} \in V(\mathbf{R})$ *positive*, if it belongs to

$$V(\mathbf{R})^+ := \{\mathbf{v} \in V(\mathbf{R});\ \langle v, v\rangle > 0\}$$

The *negative* vectors $\mathbf{v}$ are those which belong to $V(\mathbf{R})^-$. The latter define points $\mathbb{P}\mathbf{v} \in \mathbb{B} = \mathbb{P}\mathrm{V}(\mathbf{R})^-$ by projection $\mathbf{v} \mapsto \mathbf{v}\ mod\ \mathbf{C}^*$. Remember to the parametrization of all linear subdiscs of $\mathbb{B}$ by positive vectors $\mathbf{c}$ and their notation $\mathbb{D}_\mathbf{c} := \mathbb{B} \cap \mathbf{c}^\perp$. It holds that

$$\mathbb{D}_\mathbf{c} = \mathbb{D}_\mathbf{b} \Longleftrightarrow \mathbb{P}\mathbf{c} = \mathbb{P}\mathbf{b} \Longleftrightarrow \mathbf{b} \in \mathbf{C}\mathbf{c}. \tag{9.11}$$

On $\mathbb{D}_\mathbf{c}$ acts the Lie group

$$G_\mathbf{c} := \mathbf{G}_\mathbf{c}(\mathbf{R}) = \{g \in \mathbf{G}(\mathbf{R});\ g(\mathbf{c}) \in \mathbf{C}\mathbf{c}\} = \{g \in \mathbf{G}(\mathbf{R});\ g\left(\mathbb{D}_\mathbf{c}\right) = \mathbb{D}_\mathbf{c}\}.$$

From the existence of orthogonal bases in hermitian vector spaces, especially in $\mathbf{c}^\perp$, it follows that

$$G_\mathbf{c} \cong \mathbf{U}((1,1),\mathbf{C}) \cong \mathbf{Gl}_{2+}(\mathbf{R}).$$

These two presentations of the Lie group $G_\mathbf{c}$ correspond to the action on $\mathbb{D}$ or $\mathbb{H}$, respectively, which are biholomorphic equivalent domains. The biholomorphic map

$g\colon \mathbb{H} \longrightarrow \mathbb{D}$ can be realized by an element $g \in \mathbf{Gl}_2\left(\mathbf{Q}\left(\sqrt{-1}\right)\right) \subset \mathbf{Gl}_2(\mathbf{C})$ acting by linear transformation on the complex projective line $\mathbb{P}^1(\mathbf{C})$.

The algebraic group $\mathbf{G_c}$ is defined over $\mathbf{Q}$, if $\mathbf{c} \in V^+ \subset K^3$. In this case $\mathbf{G_c}$ is a $\mathbf{Q}$-model of $\mathbf{Gl}_{2+}\mathbf{R}$. By classification of such $\mathbf{Q}$-models and corresponding Fuchsian groups, see Remark 9.10, there is a $\mathbf{Q}$-central indefinit quaternion algebra D such that $D^*_+ = \mathbf{G_c}(\mathbf{Q})$ and $\Gamma_\mathbf{c} := \Gamma \cap \mathbf{G_c}(\mathbf{Q})$ is a Fuchsian subgroup of D^* acting on $\mathbb{D}_\mathbf{c}$. The natural map $\mathbb{D}_\mathbf{c}/\Gamma_\mathbf{c} \longrightarrow \mathbb{D}/\Gamma \subset \mathbb{B}/\Gamma$ is a birational map of complex (i.g. open) algebraic curves, see [Ho 2].

It is clear that

$$\mathbb{D}_{\gamma\mathbf{c}} = \mathbb{P}(\gamma\mathbf{c})^\perp = \mathbb{P}\gamma^{-1}\left(\mathbf{c}^\perp\right) = \gamma^{-1}\mathbb{D}_\mathbf{c} \text{ for } \gamma \in \Gamma,$$

hence, by (9.11), we proved the first part of the following

9.12 Lemma.
(*i*) With the above notations it holds that

$$\gamma\mathbb{D}_\mathbf{b} = \mathbb{D}_\mathbf{c} \Longleftrightarrow \gamma(\mathbf{c}) \in \mathrm{K}^*\mathbf{b},\ \mathbb{D}_\mathbf{c}/\Gamma = \mathbb{D}_\mathbf{b}/\Gamma \Longleftrightarrow \mathbf{b} \in \Gamma\mathrm{K}^*\mathbf{c}$$

(*ii*) *There exist infinitely many subdiscs $\mathbb{D}_\mathbf{c}$ of $\mathbb{B}$ which are not Γ-equivalent.*

Proof. If $\mathbb{D}_\mathbf{c}$ and $\mathbb{D}_\mathbf{b}$ are Γ-equivalent, then $\gamma(\mathbf{c}) = c\mathbf{b}$ for a suitable $c \in K$, hence

$$|c|^2\langle \mathbf{b}, \mathbf{b}\rangle = \langle c\mathbf{b}, c\mathbf{b}\rangle = \langle \gamma(\mathbf{c}), \gamma(\mathbf{c})\rangle = \langle \mathbf{c}, \mathbf{c}\rangle,$$

thus $\langle \mathbf{c}, \mathbf{c}\rangle / \langle \mathbf{b}, \mathbf{b}\rangle \in N(K^*)$.

The group $\mathbf{Q}^*/N(K^*)$ is not finite. On the other hand $(K^3)^+ \longrightarrow \mathbf{Q}^*_+,\ \mathbf{c} \mapsto \langle \mathbf{c}, \mathbf{c}\rangle$, is a surjective map. Namely, the equation $\langle \mathbf{x}, \mathbf{x}\rangle = q$ for $q \in \mathbf{Q}$ can be understood as indefinite homogeneous quadratic diophantine equation over $\mathbf{Q}$ with six variables. By the Hasse-Minkowski theory (Theorem of Mayer) there exist proper $\mathbf{Q}$-solutions, see [Se], IV, 3.2, Cor. 2.

□

With the explicit background 9.7 we notice on this place the following important biunivoque correspondences in terms of $\mathbf{U}(V) = \mathbf{U}((2,1), K)$-equivalences:

$$\begin{aligned} \{K - \text{discs on } \mathbb{B}\}/\mathbf{U}(V) &= \left\{\mathbb{D}_\mathbf{c};\ \mathbf{c} \in \mathrm{V}^+\right\}/\mathbf{U}(V) \\ &\Longleftrightarrow \mathbf{Q}^*_+/N(K^*) \\ &\Longleftrightarrow V^+/K^* \cdot \mathbf{U}(V) \end{aligned} \tag{9.13}$$

This is much stronger than 9.12 (*ii*).

9.14 Definition. The positive vector $\mathbf{c} \in V^+$ is called of *modular type*, if $\mathbf{G_c} \cong \mathbf{Gl}_{2+}$ as algebraic group over $\mathbf{Q}$.

By Remark 9.10 we have the following characterizations:

9.15 *The vector* $\mathbf{c} \in V^+$ *is not of modular type if and only if one of the following conditions is satisfied:*

(i) $\mathbb{D}/\Gamma_{\mathbf{c}}$ *is a compact curve;*

(ii) *the above quaternion algebra* $D = D_{\mathbf{c}}$ *corresponding to* $\mathbf{c}$ *is a skewfield.*

□

Now we look more carefully at the value $\langle \mathbf{c}, \mathbf{c} \rangle$ or the discriminant of the two-dimensional hermitian K-vector space $V_{\mathbf{c}} := K^3 \cap \mathbf{c}^{\perp}(K)$ endowed with the restriction $\Phi_{\mathbf{c}}$ of $\langle\ ,\ \rangle$. We know for an imaginary quadratic subfield K of a $\mathbf{Q}$-central quaternion algebra D it holds that $K \otimes D \cong Mat_2(K)$, see Theorem 3.4 (ii). Shimura's article [Shm 2] rediscovers indefinit $\mathbf{Q}$-central quaternion algebras D sitting in $Mat_2(K)$ by means of two-dimensional hermitian K-vector spaces (W, Φ), Φ of signature $(1,1)$ on $W(\mathbf{R}) \cong \mathbf{C}^2$. We review the most useful results.

9.16 Definition. The hermitian K-vector space (W, Φ) is called *anisotropic* if and only if $\Phi(\mathbf{w}, \mathbf{w})$ is only satisfied for $\mathbf{w} = \mathbf{o}$. It is called *isotropic* iff it is not anisotropic.

9.17 Notations. Let H be a subalgebra of a matrix algebra $Mat_k(R)$, R a commutative ring with unit element 1. We assume that H contains the unit matrix. The multiplicative subgroup of H defined by $det = 1$ is denoted by $\mathbf{S}H$ and the group of units of H by H^*.

All hermitian similitudes of W form a subgroup $\mathbf{GU}(W) = \mathbf{GU}(W, \Phi)$ of $End_K(W)^*$. The factors $\iota(g)$ of the similitudes appear as values of the characters ι on $\mathbf{GU}(W)$. The preimage of 1 is the *unitary group* $\mathbf{U}(W, \Phi)$. Its intersection with $\mathbf{S}End_K(W)$ is the *special unitary group* $\mathbf{SU}(W, \Phi)$. Also important for our purpose is the subgroup

$$D\mathbf{U}(W, \Phi) := \{g \in \mathbf{GU}(W, \Phi);\ \iota(g) = det\ g\}.$$

From the definitions it is clear that $D\mathbf{U}(W, \Phi) \cap \mathbf{U}(W, \Phi) = \mathbf{SU}(W, \Phi)$. By [Shm 2], Prop. 2.5, it also holds that $D\mathbf{U}(W, \Phi) \cdot \mathbf{U}(W, \Phi) = \mathbf{GU}(W, \Phi)$ in our case $2 = dim_K W$.

The (simple) matrix algebra $Mat_2(K)$ has a canonical involution i, see [Shm 2], 1.4.

$$\begin{aligned} D &= D(W, \Phi, i) \\ &:= \{g \in End_K(W);\ \Phi(g \circ i(\mathbf{v}), \mathbf{w}) = \Phi(\mathbf{v}, g(\mathbf{w})) \text{ for all } \mathbf{v},\ \mathbf{w} \in W\} \end{aligned} \tag{9.18}$$

9.19 Proposition ([Shm 2], Prop. 2.6). If $dim_K W = 2$, then $D = D(W, \Phi, i)$ is a quaternion algebra over $\mathbf{Q}$ and $End_K(W) \cong K \otimes D$. Furthermore,

$$\mathbf{GU}(W, \Phi) = K^* \cdot D^*,\ D\mathbf{U}(W, \Phi) = D^* \text{ and } \mathbf{SU}(W, \Phi) = \{g \in D;\ ggi = 1\}.$$

□

Observe that the second, hence also the first and finally the third, do not depend on i. Especially, for $W = V_{\mathbf{c}}$, $\mathbf{c} \in (K^3)^+$ we set

$$D_{\mathbf{c}} := \{g \in End_K(V_{\mathbf{c}});\ \langle gi(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, g(\mathbf{w}) \rangle \text{ for all } \mathbf{v},\ \mathbf{w} \in V_{\mathbf{c}}\}; \tag{9.20}$$

then:

$$\mathbf{GU}\left(V_{\mathbf{c}},\Phi_{\mathbf{c}}\right)=K^{*}\cdot D_{\mathbf{c}}^{*},\; D\mathbf{U}\left(V_{\mathbf{c}},\Phi_{\mathbf{c}}\right)=D_{\mathbf{c}}^{*},\; \mathbf{SU}\left(V_{\mathbf{c}},\Phi_{\mathbf{c}}\right)=\{g\in D_{\mathbf{c}};\; ggi=1\}. \quad (9.21)$$

It follows that

9.22 Proposition. The arithmetic lattice $\mathbf{SG_c}$ of finite index in $\mathbf{G_c}$ is a subgroup of the unit group $D_{\mathbf{c}}^{*}$ of the quaternion algebra $D_{\mathbf{c}}$.

□

In comparision with earlier notations the arithmetic group $\Gamma_{\mathbf{c}}$ is nothing else but $N_{\gamma}\left(\mathbb{D}_{\mathbf{c}}\right)$ defined in (7.19). Dividing out its finite center we get the effectively on $\mathbb{D}=\mathbb{D}\mathbf{c}$ acting groups $\Gamma_{\mathbb{D}}=\mathbb{P}\Gamma_{\mathbf{c}}$. The latter group is isomorphic to $\mathbf{S}\Gamma_{\mathbf{c}}$ except for the field $K=\mathbf{Q}\left(\sqrt{-3}\right)$ of Eisenstein numbers, where it has to be substituted by $\mathbf{S}\Gamma_{\mathbf{c}}/Z_3$, where Z_3 is the cyclic center of $\mathbf{S}\Gamma_{\mathbf{c}}$ of order 3 generated by $diag(\rho,\rho,\rho)$, ρ a primitive third unit root.

9.23 Proposition ([Shm 2], Prop. 2.8). The quaternion algebra $D=D(W)$ as in Prop. 9.19 (especially $D_{\mathbf{c}}$) is isomorphic to $Mat_2(\mathbf{Q})$ if and only if W is isotropic (iff in $\mathbf{c}^{\perp}(K)$ exists an isotropy vector).

The proof is given in the appendix, see Cor. 12.21

□

9.24 Proposition ([Shm 2], Prop. 4.1). The indefinite hermitian K-plane W as above is isotropic if and only if the negative discrimanant $-d(W)$ belongs to $N(K^{*})$. Especially $\mathbf{c}^{\perp}$, $\mathbf{c}\in V^{+}\subset K^{3}$, is K-isotropic iff $\langle\mathbf{c},\mathbf{c}\rangle\in N(K^{*})$.

The second statement uses $d\left(\mathbf{c}^{\perp}\right)\cdot\langle\mathbf{c},\mathbf{c}\rangle\sim_{N}-1$, see (9.6).

□

We are now able to finnish the

Proof of Theorem 8.11 (continued). As already remarked it remains to prove only some equivalences, namely: the conditions (iii), (v), (vi) are each equivalent to

(iv') $\qquad V_{\mathbf{c}}=\mathbf{c}^{\perp}(K)$ is isotropic.

$(iii)\Longleftrightarrow(iv')$ follows from Prop. 9.24. $(v)\Longleftrightarrow(vi)$ comes from Prop. 9.15 (i).
$(v)\Longleftrightarrow(vi')$ is rather obvious because the set of rational boundary points of $\mathbb{D}_{\mathbf{c}}=\mathbb{P}\mathbf{c}^{\perp}(\mathbf{C})$ is nothing else but the projective set $\mathbb{P}\left\{\mathbf{a}\in\mathbf{c}^{\perp}(\mathrm{K});\;\langle\mathbf{a},\mathbf{a}\rangle=0\right\}$ of isotropy vectors in $\mathbf{c}^{\perp}(K)$.

□

At the end of the last section 12 we prove that the chain (9.13) of biunivoque correspondences can be extended to

$$
\begin{aligned}
&\{K-\text{discs on } \mathbb{B}\}/\mathbf{U}(V) = \left\{\mathbb{D}_{\mathbf{c}};\ \mathbf{c}\in \mathrm{V}^+\right\}/\mathbf{U}(V)\\
&\Longleftrightarrow V^+/K^*\cdot \mathbf{U}(V)\\
&\Longleftrightarrow Q_+^*/N(K^*)\\
&\Longleftrightarrow \left\{D(K,q);\ q\in Q_+^*\right\}/\text{iso}\\
&\Longleftrightarrow \{\text{indefinite } \mathbf{Q}-\text{central quaternion algebras}\}/\text{iso}\\
&\Longleftrightarrow \{\text{indefinte K}-\text{hermitian vector planes } (W,\Phi)\}/\text{isometries}
\end{aligned}
\tag{9.25}
$$

Notice that these are infinitely many chains, one for each imaginary quadratic number field K in spite of independence of the last isomorphism class. Along these chains one finds the simple explicit description $D(K,q)\cong(K/\mathbf{Q},\sigma,q)$ of the quaternion algebra belonging to any K-disc $\mathbb{D}=\mathbb{D}_{\mathbf{c}}$ and the corresponding arithmetic curve $\mathbb{D}/\Gamma\subset\mathbb{B}/\Gamma$, see (12.4) and (3.3).

It is a much harder problem to find or describe the arithmetic curves on a Picard modular surfaces $\mathbb{B}/\Gamma$ in algebraic geometric terms. An example ($K=\mathbf{Q}\left(\sqrt{-3}\right)$) will be given in section 11, where the surface is well-known and sufficiently simple to describe. We expect an intersting connection with a modular form on $\mathbb{H}$ with Fourier coefficients using intersection numbers and hights of arithmetic curves on our highly singular modular surfaces. For analogeous work without a special intersection theory we refer to some work of Hirzebruch-Zaiger [H-Z] and Kudla [Ku]. In the mean time the necessary intersection (hight) theory seems to be well-prepared in [Ho 6]. The application just mentioned will be appear in a forthcoming paper.

10. Elliptic curve subfamilies

We want to show that specific curves $C=\mathbb{D}/\Gamma_{\mathbb{D}}$ of modular type $\left(E\times E_\tau^2\right)$ on the Picard modular surfaces $\mathbb{B}/\Gamma$ of the imaginary quadratic field K give rise to elliptic curve families over finite coverings of C in a natural manner. In order to be precise we rember to some basic notions, see e.g. [Sha], VII.

A (complex) *elliptic bundle* is a triple (V,B,π), π: $V\longrightarrow B$ compact complex algebraic curve, such what the general fibre of π is an elliptic curve. The elliptic bundle is a *minimal model*, iff there are no exceptional curves of first kind in the fibres of π. In each birational equivalence class of elliptic bundles over B there exists a uniquely determined minimal model up to isomorphy over B. Birational automorphisms of a minimal model are biregular. Up to finitely many exceptions — called *exceptional fibres* — the invers images $\pi^*(b)$, $b\in B$ are the reduced fibres V_b of the above elliptic bundle. If $\pi^*(b)=mV_b$, $m>1$, then $\pi^*(b)$ is called *multiple fibre.* All types of exceptional fibres of minimal models have been classified by KODAIRA, see also [Sha], VII. Let $F_\beta/k(B)$, $\beta=Spec\ k(B)$, $k(B)$ function field of B be the general fibre of π. The elliptic bundle V/B has a section iff F_β has $k(B)$-rational point (t.m. $F_\beta(k(B))\neq\emptyset$). If not then there exists a finite covering $C\longrightarrow B$ such what $V\times_B C/C$ has sections. The birational classification theory of elliptic bundles due to KODAIRA is managed on this way.

Omitting intersection points of components in exceptional fibres of a minimal model with sections one can extend the group structure in the non-exceptional elliptic fibres to all fibres. On this way each elliptic bundle V/B with (0-)section becomes an object of the moduli theory of abelian schemes due essentialy to MUMFORD [GIT]:

Let S be a noetherian base scheme; By [GIT], Def. 6.1 (ch. VI) the relative group scheme X/S is called *abelian scheme*, if π: $X \longrightarrow S$ is simple, proper, with connected geometric fibres. Keep in mind that abelian schemes have as group schemes a 0-section over S (see [GIT], ch. 0, § 1). The classifying objects for the moduli space A_g of principally polarized abelian varities of dimension g are collected in the following manner ([GIT], VII, § 2): For all S as above set

$$A_g(S) := \{\text{principally polarized abelian schemes } /S \text{of dimension } g\} /\text{iso}.$$

As *rough moduli space* $A_g = M$ is uniquely determined up to isomorphy by the following properties:

(i) For each principally polarized abelian scheme A/T there exists a map of contravariant functors Φ: $A_g \longrightarrow Hom(\cdot, M)$, especially $A_g(T) \longrightarrow Hom(T, M)$, with bijective restrictions to algebraically closed points of T, see diagram (10.1);

(ii) Φ is universal with respect to all mappings $A_g \longrightarrow Hom(\cdot, N)$ with the same property as in (i).

(10.1)

<u>Rough moduli diagram</u>

$$\begin{tikzcd}
A_k \arrow[rrr] \arrow[dr] & & & k \arrow[dl] \arrow[ddd, equal] \\
& A \arrow[r] & T & \\
& M & T \arrow[l] & \\
m \arrow[ur] & & & k \arrow[lll] \arrow[ul]
\end{tikzcd}$$

(geometric fibre A_k of the family $A/T \mapsto$ unique moduli point m)

10.2 Proposition. *Up to completion, desingularisation and finite covering each specific curve C of modular type on the* PICARD *modular surface corresponding to K can be reinterpreted as base space of a non-isotrivial elliptic curve family* $\mathbf{E}$ *with the following property: Up to isogeny and some special points P the elliptic fibre E_P over $P \in C$ is uniquely determined as isogeny component without K-multiplication of the abelian threefold A_P cortresponding to the* PICARD *moduli point P of the surface.*

Proof. First we apply the above general moduli interpretation to $M = A_1$, the elliptic curve case. Let $\mathbf{E}/T$ be a family of elliptic curves over a curve T, not isotrivial. The pull back $\mathbf{E}'/T'$ along suitable finite covering f: $T' \longrightarrow T$ has a section. So we can restrict ourselves to minimal models of elliptic bundles over curves with section, or to abelian schemes over curves of relative dimension 1. The moduli diagram exists for $\mathbf{E}'$, T' instead of A, T. The elliptic fibres at $t \in T$ and $t' \in f^{-1}(t)$ coincide almost everywhere.

Now let C be a specific curve of modular type on the PICARD modular surface parametrizing principally polarized abelian threefolds with (admissible) imaginary quadratic K-multiplication. It is represented by a fibrewise splitting abelian scheme $\mathbf{A}/C'$, C' a finite covering of C. It is equivalent to say that the general fibre A_γ splits up to isogeny into the product of elliptic curves, see Lemma 5.7. Since C gives moduli dimension 1 we find an elliptic isogeny factor E_γ without K-multiplication, see Corollary 5.3.

This elliptic factor in the general fibre determines an elliptic curve family over an open part U' of C', which can be uniquely extended to a minimal model $\mathbf{E}/C'$ of elliptic bundles. The non-trivial homorphism $E_\gamma \longrightarrow A_\gamma$ extends to an open ZARISKI set on C'. Therefore almost everywhere the ellpitic curves E_c, $c \in C'$ appear as isogeny components of A_c. Thus $\mathbf{E}/C'$ is a hidden elliptic curve family over C' we look for sitting in $\mathbf{A}$ up to isogeny.

$\square$

11. The leading example

By (9.25) all $\mathbf{Q}$-central indefinite quaternion algebras appear in the theory of Picard modular surface of an imaginary quadratic number field K. Thereby the field can be chosen arbitrarily. In the case $K = \mathbf{Q}\left(\sqrt{-3}\right)$ the abelian threefolds corresponding to points of the Picard modular surface are Jacobians of explicitly known plane projective curves of degree 4, the so-called Picard curves. Moreover the correspondence of these curves with their moduli points is explicitly known and easy to describe, see (11.1) below. As levelled Picard modular surface we can choose the projective plane $\mathbb{P}^2$. For detailed proofs we refer to [Ho 3] and/or [Ho 5].

By the results of the previous sections the arithmetic curves C on $\mathbb{P}^2$ collect precisely all moduli points of Picard curves with (isogeneously) splitting Jacobians. These points are the image points of K-discs $\mathbb{D}$ on the uniformizing ball $\mathbb{B}$. The projection $\mathbb{B} \longrightarrow \mathbb{P}^2$ can be analytically expressed by the restriction along $\mathbb{B} \subset \mathbb{H}^3$ (the Siegel upper half space for abelian threefolds) of 4 explicitly known (linearly dependent) theta constants $th_1, \ldots th_4$ described precisely in [Ho 5]. Analytic-geometrically we established there a Schottky-Torelli diagram connecting $\mathbb{B}$, $\mathbb{H}^3$, $\mathbb{P}^2$ and the moduli space of abelian threefolds with these thetas.

The aim of this section is to give a first explicit example of an arithmetic curve corresponding to an explicit subfamily of curves with (isogeneously) splitting Jacobian threefolds of non-trivial quaternion type. It is described in (11.3). On the other hand we present also a subfamily of Picard curves along an arithmetic curve of modular type. The previous section teaches us that it discovers a hidden non-isotrivial family of elliptic curves sitting in the Jacobians.

Let us start with the explicit description the PICARD modular surface $\mathbf{M} := \mathbb{B}/\Gamma'$ of (the field $K = \mathbf{Q}\left(\sqrt{-3}\right)$) of EISENSTEIN numbers of level $1-\rho$, ρ a primitive third unit root. By definition, Γ' is the principal congruence subgroup of $\Gamma := \mathbf{U}((2,1), \mathbf{O})$ of the ideal $(1-\rho) = \left(\sqrt{-3}\right)$ of the ring of EISENSTEIN integers $\mathbf{O} + \mathbf{Z} + \mathbf{Z}\rho$ acting on the complex two-ball $\mathbb{B}$.

In [Ho 3] we proved that

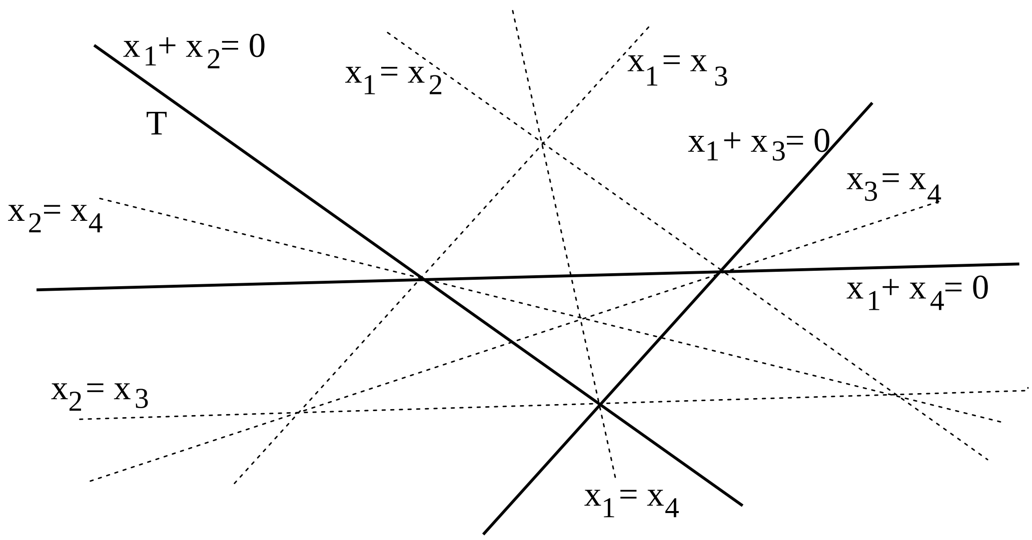
$$\mathbb{B}/\Gamma' \cong \mathbb{P}^2 \ \{4 \text{ points in general position}\}\,.$$

Furthermore $\mathbb{P}^2/\mathrm{S}_4$ is the compactified moduli space of PICARD curves, which can be defined as smooth curves of genus 3 with an automorphism group of order 3. Each of them has a plane model with normalized equation

$$C_x\colon\ Y^3 = (X-x_1)(X-x_2)(X-x_3)(X-x_4),\ \ x_1+x_2+x_3+x_4 = 0. \tag{11.1}$$

The situation is described in the following picture (11.2)

(11.2)



$\mathbb{P}^2 = \left\{(\mathrm{x}_1 : \mathrm{x}_2 : \mathrm{x}_3 : \mathrm{x}_4) \in \mathbb{P}^3 = \mathbb{P}^3(\mathbf{C});\ \mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3 + \mathrm{x}_4 = 0\right\}$

The symmetric group S_4 acts by permutation of coordinates. The six (thin) lines correspond precisely to non-smooth PICARD curves. The three (thick) lines through the double points of the six thin lines are determined as fixed points of $(12)(34),\ (13)(24),\ (14)(23) \in S_4$, respectively. Let T be the first of them. Along $p\colon\ \mathbb{B} \longrightarrow \mathbb{P}^2$ it is covered by a linear subdisc $\mathbb{D}$ of $\mathbb{B}$ fixed by a reflection $\sigma \in \Gamma$ with image $(12)(34)$ in $\Gamma/\Gamma' \cong S_4$, thus $T \cong \mathbb{D}/\Delta$, Δ an arithmetic group commensurable with $\mathbf{U}((1,1),\mathbf{O})$. This is a cocompact lattice of the disc $\mathbb{D}$ because T does not go through one of the four compactification points. For more details and proofs we refer to chapter I of the monograph [Ho 3].

Changing coordinates one gets a 2-parameter family over the affine (u,v)-plane

$$\mathbf{P}/\mathbf{A}^2\colon\ Y^3 = X(X-1)(X-u)(X-v),$$

the original PICARD curve family. It contains representants of all isomorphy classes of smooth PICARD curves. The correspondence $(x, y) \mapsto (x, \rho y)$ defines an automorphism of order 3 on each PICARD curve. Therefore the Jacobian threefolds of PICARD curves have K-multiplication, K the field of EISENSTEIN numbers. They form a two-dimensional family $\mathbf{J} = \mathbf{J}(\mathbf{P})$, wich cannot be lifted from a family of smaller dimension because the moduli space of PICARD curves has dimension 2 and by TORELLI's theorem. Looking at the hierarchy diagram (5.4) we see that a general member A of this family must have isogeny decomposition type (A, K). This means that A is simple and $\mathbf{End}\ (A) \cong K$.

Now we consider the 1-modular subfamily of (in general) smooth PICARD curves corresponding to $\mathbb{P}^2$-points of the cubic $X_1X_2X_3 + X_4X_2X_3 + X_1X_4X_3 + X_1X_2X_4 = 0$ in terms of (11.1). They are of equation type

$$Y^3 = X^4 + aX^2 + b. \tag{11.3}$$

The cubic is degenerate and consists of the three thick lines of picture (11.2). Especially, T: $X_1 + X_2 = X_3 + X_4 = 0$ is a component of the cubic not going through the four cusp points. Setting $x_1 = -x_2 = 2\lambda$, $x_3 = -x_4 = 2$ we get the 1-modular subfamily

$$\mathbf{C}/T\colon\ Y^3 = X^4 - 4(\lambda^2+1)X^2 + 16\lambda^2,\ \lambda \in \mathbf{C}, \tag*{(11.3)$'$}$$

of $\mathbf{P}$ (up to coordinate shift), see 1.5. For symmetric reasons all PICARD curves over the cubic are represented up to isomorphy by this family, hence all curves of the biquadratic equation type (11.3). The corresponding Jacobian family is denoted by $\mathbf{T} = \mathbf{J}(\mathbf{C})/T$.

One can immediately recognize that the general fibre of $\mathbf{T}/T$ is not of type (A, K). For this purpose we apply the criterion 5.6 to our family in order to show that T is an arithmetic curve. We work with the equations of type (11.3). The correspondence $(x, y) \mapsto (-x, y)$ defines an automorphism τ of order 2 on $\mathbf{C}$. The corresponding quotient curve $\mathbf{C}/\langle\tau\rangle$ is defined by the equation $Y^3 = V^2 + aV + b$, or, after change of coordinates, by $W^2 = Y^3 + c$. This is an elliptic curve E with K-multiplication. By Cor. 5.5 the parameter curve T has to be arithmetic. Since T is smooth it follows from Theorem 7.17 that

11.4 The arithmetic curve T is a quotient curve $T = \mathbb{D}/\Delta$ for a suitable K-disc $\mathbb{D} \in \mathbb{B}$ by the $\mathbf{Q}$-arithmetic $\mathbb{D}$-lattice $\Delta = \Gamma'_{\mathbb{D}}$.

11.5 Very special example. *The Jacobian threefold J of the smooth PICARD curve $Y^3 = X^4 + 1$ has decomposition type (E^3). This curve is represented by the point $(i : -i : 1 : -1)$ on $\mathbb{P}^2(\mathbf{C})$* (which is not visible in the real picture (11.2)).

Namely, the automorphism $(x, y) \mapsto (ix, y)$ of order 4 on the curve extends to $\mathbf{Q}(i)$-multiplication on J. So J has both $\mathbf{Q}\left(\sqrt{-3}\right)$- and $\mathbf{Q}(i)$-multiplication. By Lemma 3.1 each primary isogeny component of J has $\mathbf{Q}(i)$-multplication. Therefore E cannot be a primary isogeny component. The only possibility is the decomposition type (E^3) for J by the hierarchy diagram (5.4).

Setting $\lambda = i = \sqrt{-1}$ in (11.3)$'$ one obtains the representing point $(i : -i : 1 : -1)$.

□

In order to find a suitable K-disc $\mathbb{D} \in \mathbb{B}$ covering T some elementary things should be added to the picture (11.2). From [Ho 3] we know that the fixed points of $G := S_4 = \Gamma/\Gamma'$

on $\mathbb{P}^2 \setminus \{4 \text{ points}\}$ correspond biuniquely to the Γ'-orbits of the Γ-elliptic points on $\mathbb{B}$. Especially. the branch points of the subquotient morphism

$$\mathbb{D} \longrightarrow \mathbb{D}/\Gamma_{\mathbb{D}} = (\mathbb{D}/\Delta)/\mathrm{G_T} = \mathrm{T}/\mathrm{G_T}$$

are the same as those of the finite covering $T \longrightarrow T/G_T$. We use similar notations for T and $G \subset \mathbf{PGl}_3(\mathbf{C})$ acting on $\mathbb{P}^2$. Explicitly we have

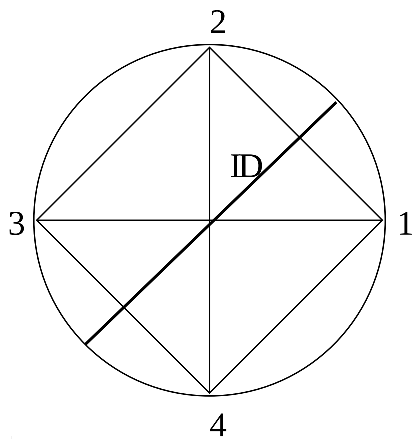
$$\begin{aligned} T := T_{12} = T_{34}\colon\ X_1 + X_2 = X_3 + X_4 = 0, \\ N_G(T) = \langle (1,2), (3,4), (1,4,2,3) \rangle \cong D_4 \text{ (dieder group)}, \\ Z_G(T) = \langle (1,2)(3,4) \rangle \cong Z_2,\ G_T \cong D_4/Z_2 \cong K_4. \end{aligned}$$

The generating reflection of $Z_G(T)$ lifts to a reflection in Γ', for example to

$$(12)(34)_{\mathbb{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma' \subset \Gamma = \mathbf{U}((2,1), O)$$

The corresponding K-disc $\mathbb{D}$ over T fixed pointwise by $(12)(34)_{\mathbb{B}}$ is the diagonal disc described by the equation $u = v$ on the ball. In the (real) picture (11.6) we draw six representing discs covering the six (thin) branch lines and $\mathbb{D}$ covering T along the quotient map $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma'$.

(11.6)



For more details we refer to [Ho 3].

Finally, one finds easily the branch and ramification points of the curve coverings starting from $\mathbb{D}$. It is clear that

$$\mathbf{P}^1 \cong T \longrightarrow T/K_4 \cong \mathbf{P}^1$$

has 3 branch points by the HURWITZ genus formula:

$$(2g' - 2) = d(2g - 2) + \sum_{P \in T} (e_P - 1),\ g' = \text{ genus of } T = 0 = g = \text{ genus of } \mathbf{P}_1,$$

$e_P = |G_{T,P}| = 2, = \quad d = |K_4| = 4$. Therefore there are 6 ramification points P on T or Γ'-inequivalent ramificatient points on $\mathbb{D}$, hence 3 branch points on $T/K_4 = \mathbb{D}/\Gamma'_{\mathrm{T}}$. Four of the six ramification points are obviously the intersection points of T with the six thin lines T_{ij}: $X_i = X_j$, $1 \leq i < j \leq 4$, see picture (11.2).

$$T \cap Fix(1,2) = (0:0:1:-1),\ Fix(1,3) \cap Fix(2,4) = (1:-1:1:-1),$$
$$T \cap Fix(3,4) = (1:-1:0:0),\ Fix(1,4) \cap Fix(2,3) = (1:-1:-1:1),$$

The other two sit in $Fix(1,4,2,3) = \{(1:-1:i:-i),\ (1:-1:-i:i)\}$.

11.7 Corollary. *The general members of the family of Jacobian threefolds of* PICARD *curves* $Y^3 = X^4 + aX^2 + b$ *have non-trivial quaternionic isogeny decomposition type* $(E \times S)$. *Special members are of type* $\left(E \times E_\sigma^2\right)$ *or* $\left(E^3\right)$.

Proof. This follows now directly from Theorem 8.11 because the parameter curve T: $X_1 + X_2 = 0$ doesn't cross the set of 4 cusp points.

□

Finally, let L one of the six thin lines in Picture (11.2), say L: $X_3 = X_4$. As already mentioned it is the compactification of a quotient curve of $\mathbb{B}/\Gamma'_{\mathbb{D}}$ of one of the K-discs $\mathbb{D}$ joining two of the Γ'-cusps $1, 2, 3, 4$ drawn in (11.6). Again from Theorem 8.11 it follows that

11.8 Corollary. *The general members of the family of Jacobian threefolds of* PICARD *curves* C_λ: $Y^3 = X(X-1)(X-\lambda)(X-\lambda)$, $\lambda \in \mathbf{C}$, *have modular decomposition type* $\left(E \times E_\tau^2\right)$.

□

It is easy to recognize the elliptic curve family sitting in the Jacobian $\mathbf{J}(\mathbf{C})$ of this Picard curve family $\mathbf{C}/L = \mathbb{P}^1$ according to Prop.10.2 of the previous section. Consider $\mathbf{C}/L$ as algebraic surface with function field $G = \mathbf{C}(x,y)(u)$, x, y transcendentally independent over $\mathbf{C}$, with only relation $u^2 = y^3/x(x-1)$. We substituted in the equation for C_λ the term $(X-\lambda)^2$ by U and use small lattices in the function field. It contains the function field $F = \mathbf{C}(x,y)(t)$, if we set $t = u^2$. Since $t = y^3/x(x-1)$ the field F is the function field of the most classical elliptic curve family $\mathbf{E}/L$: $Y^3 = X(X-1)(X-\lambda)$. There are surface models $\mathbf{C}$, $\mathbf{E}$ of our curve families allowing a two-sheeted covering $\mathbf{C} \longrightarrow \mathbf{E}$ corresponding to the quadratic field extension G/F. Using $\lambda = x - t$ as a parameter again we see that this covering yields curve coverings $C_\lambda \longrightarrow E_\lambda$ of degree 2. They extend to surjective morphisms of the Jacobians $J(C_\lambda) \longrightarrow E_\lambda$, hence E_λ is an isogeny component of $J(C_\lambda)$. By 11.8 we get

11.9 Corollary. For $\lambda \neq 0,\ 1$ the (generalized) Jacobians of the singular Picard curve C_λ of 11.8 has isogeny decomposition

$$J(C_\lambda) \approx E \times E_\lambda \times E_\lambda \text{ with elliptic curve } E_\lambda\text{: } Y^3 = X(X-1)(X-\lambda).$$

□

11.10 Remark-Problem. This elliptic curve family $\{E_\lambda\}$ appears as most classical example in the theory of (ordinary) Fuchsian differential equations and their solutions described by hypergeometric functions. The example fits into a nice correspondence between isomorphy classes of complex (minimal smooth) elliptic surfaces over $\mathbb{P}^1$ with precisely 3 exceptional fibres and the most simple Fuchsian equations, see [S-Z] for a nice modern algebraic-geometric treatment.

With regard to our splitting results it is quite natural to ask for a similar connection between abelian surface families of non-trivial quaternion type (over arithmetic curves of Picard modular surfaces). One should start with the above example of 11.7, wich is the most simple sitting in the Jacobians of Picard curves.

12. Appendix
K-embeddings into $\mathbf{Q}$-central quaternion algebras

We will prove that for each indefinit $\mathbf{Q}$-central quaternion algebra D and independently given imaginary quadratic number field K there exists a K-embedding. Moreover we would like to parametrize all of them.

Let D be a Z-central simple algebra, Z a number field. Consider it as element of the BRAUER group $Br(Z)$ and denote the localisations $D_v = \mathbf{Q}_v \otimes D$ at arbitrary places v of Z. The property that an algebra is central and simple is preserved under extension of ground field. Therefore the localitions define group homomorphisms $Br(Z) \longrightarrow Br(Z_v)$. The local Brauer groups are well-understood by certain invariants, wich are unit roots. In additive style of writing one disposes on local BRAUER group isomorphisms
μ_v: $Br(Z_v) \xrightarrow{\cong} \mathbf{Q}/\mathbf{Z}$. For more details and proofs we refer to [R] or [We].

With regard to our purposes we restrict to the case $Z = \mathbf{Q}$ and $n^2 = dim_{\mathbf{Q}} D = 4$, where we describe the local-global (Hasse) principle more explicitly. We have $\mathbf{Q}_v = \mathbf{Q}_p$ or $\mathbf{R}$ for $v = p$ a natural prime number or $v = \infty$, respectively. The local invariants $\mu_v(D)$ are described as follows

$$\mu_v(D) = +1 \text{ iff } D_v \cong Mat_2(\mathbf{Q}_v);$$
$$\mu_v(D) = -1 \text{ iff } D_v \text{ is a skew field.}$$

12.1 Theorem (Hasse-Brauer-Noether). The map $D \mapsto (\ldots, \mu_v(D), \ldots)_v$ corresponding each $\mathbf{Q}$-central quaternion algebra D its set of local invariants induces a bijection

$$\{\mathbf{Q}-\text{central quaternion algebras}\}/\text{iso}$$
$$\Longleftrightarrow \left\{ (\varepsilon_v) \in \prod_v \{\pm 1\};\ \varepsilon_v = +1 \text{ for almost all } v,\ \prod_v \varepsilon_v = +1 \right\}$$

$$\{\mathbf{Q}-\text{central indefinite quaternion algebras}\}/\text{iso}$$
$$\Longleftrightarrow \left\{ (\varepsilon_v) \in \prod_v \{\pm 1\};\ \varepsilon_\infty = +1,\ \varepsilon_v = +1 \text{ for almost all } v,\ \prod_v \varepsilon_v = +1 \right\}$$

□

12.2 Corollary.

(i) A $\mathbf{Q}$-central quaternion algebra is isomorphic to $Mat_2(\mathbf{Q})$ if and only if all local invariants $\mu_v(D)$ are equal to $+1$.

(ii) There exist infinitely many isomorphy classes of indefinit $\mathbf{Q}$-central quaternion algebras.

□

For $\mathbf{Q}$-central quaternion algebras D and imaginary quadratic number fields K we consider pairs (D,ι) or $(D,\iota(K))$, where $\iota\colon\ K \longrightarrow D$ is an embedding of algebras. In order to classify such pairs we define for any $q \in Q_+^*$ the $\mathbf{Q}$-central Matrixalgebra $D(K,q)$ by

$$D(K,q) = \left\{ \begin{pmatrix} a & qb \\ \overline{b} & \overline{a} \end{pmatrix};\ a,\ b \in K \right\}. \tag{12.3}$$

Set $u := \left(\begin{smallmatrix} 0 & q \\ 1 & 0 \end{smallmatrix}\right)$ and identify $diag\,(c,\overline{c})$ with $c \in K$. Then

$$D(K,q) = K + K\mathbf{u} \tag{12.4}$$

with relations

$$\mathbf{u}^2 = q,\ \overline{c}\mathbf{u} = \mathbf{u}c \text{ for all } c \in K.$$

12.5 Definition. If the quaternion algebra D is isomorphic to $K + K\mathbf{u}$ with relations (12.3) defining the $\mathbf{Q}$-algebra structure, then we call $K + K\mathbf{u}$ a K-presentation of D. The matrix algebra $D(K,q)$ is called a K-representation of D.

So we dispose on models for each of the isomorphy classes $(K/\mathbf{Q},\sigma,q)$ described in (3.3). From the Brauer group theory we also know that they represent all isomorphy classes of $\mathbf{Q}$-central indefinit quaternion algebras containing K, and two of these algebras $D(K,q)$, $D(K,q')$ are isomorphic if and only if $q'/q \in N(K^*)$, see 3.5.

In [K-S], § 11, for $K = \mathbf{Q}\left(\sqrt{-d}\right)$ the isomorphy class of $D = D(K,q)$ is denoted by $(-d,q)$. Using the notation of (6.2), $n = 2$, we can write our K-respresentation as

$$D = D(K,q) = \left\{ \widehat{a} + \widehat{b}\mathbf{u};\ a,\ b \in K \right\} = \widehat{K} + \widehat{K}\mathbf{u}. \tag{12.6}$$

of $(-d,q)$. The canonical anti-involution * on D is defined by

$$a + b\mathbf{u} \mapsto (a + b\mathbf{u})^* := (\overline{a} - b\mathbf{u})\,. \tag{12.7}$$

Indeed, it is easy to check that

$$[(a + b\mathbf{u})\,(a' + b'\mathbf{u})]^* = (a' + b'\mathbf{u})^*\,(a + b\mathbf{u})^*\,.$$

Furthermore,

$$\mathbf{N}(a + b\mathbf{u}) := (a + b\mathbf{u})(a + b\mathbf{u})^* = |a|^2 - q|b|^2 = det \begin{pmatrix} a & qb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathbf{Q}$$

and

$$\mathbf{T}(a+b\mathbf{u}) = (a+b\mathbf{u}) + (a+b\mathbf{u})^* = a+\overline{a} = \text{ trace of } \begin{pmatrix} a & qb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathbf{Q}$$

define *norm* and *trace* on $K+K\mathbf{u}$ extending the $K/\mathbf{Q}$-norm or -trace on K, respectively. The element $A \in D$ satisfies the characteristic equation $T^2 - \mathbf{T}(A)T + \mathbf{N}(A)$, which is independent of any imaginary quadratic representation of D. Therefore $\mathbf{T}$ and $\mathbf{N}$ are correctly defined on D. This is also true for the anti-involution * because $A^* = \mathbf{T}(A) - A$.

All elements $A \in D$ with $\mathbf{N}(A) \neq 0$ have an inverse in D, namely $A^{-1} = \mathbf{N}(A)^{-1}A^*$. On this way we proved directly the first part of

12.8 Proposition. $D = D(K,q)$ is a skewfield if and only if the equation $|a|^2 - q|b|^2 = 0$ has only the trivial solution $a = b = 0$ in K. This happens if and only if $q \notin N(K^*)$. With a fixed imaginary quadratic subfield K of a $\mathbf{Q}$-central quaternion algebra $D \cong K + K\mathbf{u}$ all K-presentations are given by $K + K\mathbf{v}$ with $\mathbf{v} \in N(K^*)\mathbf{u}$ and all K-representations by $D(K,q'),\ q' \in N(K^*)q$.

Proof. It suffices to prove the third statement. Let $K + K\mathbf{v}$ be a second K-presentation of D with relations $\mathbf{v}^2 = q'$ and $\overline{c}\mathbf{v} = c\mathbf{v}$ for $c \in K$. These relations

$$\overline{c}a + \overline{c}b\mathbf{u} = \overline{c}(a+b\mathbf{u}) = \overline{c}\mathbf{v} = \mathbf{v}c = (a+b\mathbf{u})c = ac + \overline{c}b\mathbf{u}$$

are only possible, if $a = 0$. Therefore $\mathbf{v} = b\mathbf{u}$ and

$$q' = \mathbf{v}^2 = b\mathbf{u}b\mathbf{u} = |b|^2\mathbf{u}^2 = |b|^2 q \in N(K^*)q.$$

$\square$

12.9 Lemma. *Let $K = \mathbf{Q} + \mathbf{Q}\sqrt{-d}$ be an imaginary quadratic number field and $D = D(K,q)$ a K-representation* (12.4) *of indefinit $\mathbf{Q}$-central quaternion algebras belonging to the norm class $qN(K^*) \in Q_+^*/N(K^*)$. The imaginary quadratic field $L = \mathbf{Q}\left(\sqrt{-k}\right)$, d, k squarefree natural numbers, can be embedded into D if and only if the diophantine equation*

$$qX^2 + dqY^2 + kV^2 = dT^2$$

has a rational solution (x,y,v,t) with $v \neq 0$.

Proof. Let $\alpha = \left(\begin{smallmatrix} a & qb \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$ be an arbitrary element of D^*. Its square is

$$\alpha^2 = \begin{pmatrix} a^2 + q|b|^2 & q\,(ab + \overline{a}b) \\ a\overline{b} + \overline{a}\overline{b} & \overline{a}^2 + q|b|^2 \end{pmatrix}$$

It is an element of $\mathbf{Q}^* \subset D^*$ iff $a \in \mathbf{Q}^*\sqrt{-d}$. With $a = t\sqrt{-d}$, $t \in \mathbf{Q}$, and $b \in K$ we parametrize on this way all imaginary quadratic subfields

$$L = L(t,b) = \mathbf{Q} + \mathbf{Q}\alpha \cong Q\left(\sqrt{-dt^2 + q|b|^2}\right),\ 0 > -dt^2 + q|b|^2.$$

sitting in D. We set $b = x + y\sqrt{-d}$, x, $y \in \mathbf{Q}$. Then $L \cong \mathbf{Q}\left(\sqrt{-k}\right)$ iff $-dt^2 + q(x^2 + dy^2) = -v^2k$ for a suitable $v \in \mathbf{Q}^*$.

$\square$

12.10 Proposition. *Let D be an indefinit $\mathbf{Q}$-central quaternion algebra. Each imaginary quadratic number field L can be embedded into D.*

Proof. By the lemma, it remains to check that the homogeneous diophantine equation

$$Q\colon\ qX^2 + dqY^2 + kV^2 - dT^2 = 0$$

has a non-trivial $\mathbf{Q}$-solution $\mathbf{a}_0 = (x_0, y_0, v_0, t_0)$. If this is done, then one finds also a rational solution (x, y, v, t) with $t \neq 0$ because in this case the projective rational solutions fill a dense subset on the corresponding projective quadric $\mathbb{P}Q(\mathbf{R}) \subset \mathbb{P}^3(\mathbf{R})$. In order to see this one chooses an arbitrary $\mathbf{Q}$-rational plane E in $\mathbb{P}^3$ not containing $P_0 := \mathbb{P}\mathbf{a}$. The central projection with center P_0 restricted to E yields defines correspondences

$$E(\mathbf{Q}) \longrightarrow \mathbb{P}\mathrm{Q}(\mathbf{Q}) \setminus \{\mathrm{P}_0\} \text{ and } \mathrm{E}(\mathbf{R}) \longrightarrow \mathbb{P}\mathrm{Q}(\mathbf{R}) \setminus \{\mathrm{P}_0\}.$$

The discriminant of the quadratic form is the product of the coefficients $discr(Q) = -d^2q^2k$. This is not a square in $\mathbf{Q}$. The existence of a non-trivial $\mathbf{Q}$-solution follows now from the following special result 12.11 of the Hasse-Minkowski theory for quadratic forms over $\mathbf{Q}$:

12.11 Proposition (see e.g. [Se], IV, § 2, Theorem 6, (iii)). *The diophantine equation*

$$aX^2 + bY^2 + cU^2 + dV^2 = 0,\ a,\ b,\ c,\ d \in \mathbf{Q},$$

has a non-trivial rational solution, if its discriminant abcd is not a rational square.

□

12.12 Remark. Looking back to Theorem 3.4 (ii) we proved that for each imaginary quadratic field L the smallest number r for which there exists an embedding $L \longrightarrow Mat_r(D)$ is equal to 1.

Moreover, for the set of all embeddings of a given field into an algebra the following general result is known, which is much more intrinsical than Lemma 12.2:

12.13 Theorem. *Let R be an algebra over a field F, $dim_F R = n^2$, R^* its group of units, L a field extension of F of degree n over F and $f\colon L \longrightarrow R$ a F-linear embedding. Each other such embedding $f'\colon L \longrightarrow R$ is R^*-conjugated to f.*

This means that there is an element $a \in R^*$ such that $f'(l) = a^{-1}f(l)a$ for all $l \in L$. The commutator of $f(L)$ in R is $f(L)$ itself. Thus, there is a bijective correspondence between all F-linear embeddings of L into R and the coset $f(L)^* \backslash R^*$. For a proof of the theorem we refer to [We], Appendix 3 (of the russian edition).

□

12.14 Corollary. Let D be a $\mathbf{Q}$-central indefinite quaternion algebra and K an arbitrary imaginary quadratic number field. Starting from one $\mathbf{Q}$-algebraic embedding $K \subset D$ all embeddings of K into D are precisely parametrized by the coset $K^* \backslash D^*$.

Proof. First we need the existence of a K-embedding into D. This comes from Prop. 12.10. Now the statement follows immediately from 12.13 setting $R = D,\ F = \mathbf{Q}$, $n = 2,\ L = K$.

□

12.15 Theorem. Let K be an arbitrary fixed imaginary quadratic number field. Then one has the followig bijective correspondences:

$$\{\text{indefinite } \mathbf{Q} - \text{central quaternion algebras}\}/\text{iso} = \left\{D(K,q);\ q \in \mathbf{Q}^+\right\}/\text{iso}$$
$$\Longleftrightarrow$$
$$Q_+^*/N(K^*) = Q^*/\pm N(K^*),$$

where $D(K,q) \cong (K/Q,\sigma,q)$ are defined in (12.4) and (3.3), N is the norm map of K/Q; and $q \mapsto qN(K^*)$ is the explicit description of the biunivoque map.

Proof. Indeed, let $K + K\mathbf{u}$ and $K' + K'\mathbf{u}'$ be two K-presentations with possibly different but isomorphic subfields $K,\ K'$ of D, $\mathbf{u}^2 = q$, $\mathbf{u}'^2 = q'$. The existence has been proved above. By 12.13/14 there exists $g \in D^*$ such that $K = g^{-1}K'g$. Set $\mathbf{v} = g^{-1}\mathbf{u}'g$. Then

$$\overline{c}\mathbf{v} = g^{-1}\overline{c'}\mathbf{u}'g = g^{-1}\mathbf{u}'c'g = \mathbf{v}c \text{ for all } c' \in K'.$$

Therefore we can apply Prop. 12.8, which says that $q' \in qN(K^*)$.

$\square$

Now we are able to establish more directly for any fixed imaginary quadratic number field K the bijective correspondence

$$\{\text{indefinite } \mathbf{Q} - \text{central quaternion algebras}\}/\text{iso}$$
$$\Longleftrightarrow$$
$$\{\text{indefinte K} - \text{hermitian vector planes } (W,\Phi)\}/\text{isometries}$$

which is part of the chain (9.25). For given $(W,\langle\ ,\ \rangle)$ the quaternion algebra sitting inside has been defined by Shimura, see 9.19.

$$D^* = \{g \in \mathbf{Gl}_K(W);\ \langle g(\mathbf{x}), g(\mathbf{y})\rangle = (det\ g)\langle \mathbf{x},\mathbf{y}\rangle \text{ for all } \mathbf{x},\ \mathbf{y} \in W\}. \tag{12.16}$$

It determines $D = D(W,\Phi)$ as smallest $\mathbf{Q}$-subalgebra of $End_K(W)$ containing D^*. According to the considerations after (12.7) the minimal polynomial of $g \in D\backslash\mathbf{Q}$ is $T^2 - \mathbf{T}(g)T + \mathbf{N}(g) \in \mathbf{Q}[T]$. On the other hand g satisfies as element of $End_K(W)$ the characteristic equation $T^2 - tr(g)T + det\ g = 0$. It follows that

$$tr(g) = \mathbf{T}(g) \in \mathbf{Q},\ det\ g = \mathbf{N}(g) \in \mathbf{Q},\ g^* = tr(g) - g. \tag{12.17}$$

We want to join the norm $\mathbf{N}$ on D with the indefinite hermitian form $\langle\ ,\ \rangle$ on W. For this purpose we consider W as a 4-dimensional $\mathbf{Q}$-vector space with D-multiplication. For any fixed $\mathbf{p} \in W$ the map π: $g \mapsto g\mathbf{p}$ is a $\mathbf{Q}$-linear homomorphism from D into W, both with $\mathbf{Q}$-dimension 4. For each $g \in D$ it holds that

$$n(g\mathbf{p}) := \langle g\mathbf{p}, g\mathbf{p}\rangle = (det\ g)\langle \mathbf{p},\mathbf{p}\rangle = \langle \mathbf{p},\mathbf{p}\rangle\mathbf{N}(g) = \mathbf{N}(g)n(\mathbf{p}). \tag{12.18}$$

If $\langle \mathbf{p}, \mathbf{p} \rangle = 0$, then π cannot be surjective because the hermitian form $\langle\ ,\ \rangle$ is not trivial. If $\langle \mathbf{p}, \mathbf{p} \rangle \neq 0$, then from $\pi(g) = g\mathbf{p} = \mathbf{o}$ follows $\mathbf{N}(g) = det\ g = 0$, hence $\mathbf{R} \times Ker\ \pi$ is an $\mathbf{R}$-linear subspace of $\mathbf{R} \times D = Mat_2(\mathbf{R})$ outside of $\mathbf{Gl}_2(\mathbf{R})$. This is only possible for $Ker\ \pi = \mathbf{O}$. Therefore

12.19 Lemma. The $\mathbf{Q}$-linear map $\pi = \pi_{\mathbf{p}}$ defined above is an isomorphism iff $\langle \mathbf{p}, \mathbf{p} \rangle \neq 0$. Equivalent are both of the conditions $W = D\mathbf{p}$, respectively

$$\langle \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{p}, \mathbf{p} \rangle \mathbf{N}\left(\pi^{-1}(\mathbf{w})\right) \text{ for all } \mathbf{w} \in W. \tag{12.20}$$

For the last statement set $\mathbf{w} = g\mathbf{p}$ and apply (12.18).

□

12.21 Corollary. For the quaternion algebra $D = D(W, \Phi)$ the following conditions are equivalent:

(i) D is a skewfield;

(ii) $det\ g \neq 0$ for all $g \in D\backslash \{0\}$;

(iii) W doesn't contain any isotropy vector.

Proof. We know that D is a skewfield if and only if $\mathbf{N}(g) \neq 0$ for all $g \in D\backslash \{0\}$, see Prop. 12.8. The equivalence of (i) and (ii) follows now from (12.17). The equivalence with (iii) comes from (12.20).

□

In the next step we give a geometric explanation for constructing all subfields of $D = D(W)$ isomorphic to K. Let $L^- \subset W$ be a K-line generated by a negative vector $\mathbf{a}$, that means $\langle \mathbf{a}, \mathbf{a} \rangle < 0$. Its orthogonal complementary line is denoted by $L^+ = K\mathbf{b}$, say. The nontrivial vectors of L^+ are positive and $W = L^- \textcircled{\perp} L^+$ (orthogonal sum). We consider the subalgebra of all $g \in D$ having L^- as a eigenline and prove that

$$\widehat{K}\left(L^-\right) := \left\{g \in D;\ L^- \text{ is an eigenline of } g\right\} \cong \widehat{K} = \left\{\begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix};\ c \in K\right\}. \tag{12.22}$$

Indeed, the linear extension of $\mathbf{a} \mapsto c\mathbf{a},\ \mathbf{b} \mapsto \overline{c}\mathbf{b}$ is an element $\gamma = \gamma_c$ of D because of

$$\begin{aligned} \langle \gamma(\mathbf{l} + \mathbf{m}), \gamma(\mathbf{l}' + \mathbf{m}') \rangle &= \langle c\mathbf{l} + \overline{c}\mathbf{m}, c\mathbf{l}' + \overline{c}\mathbf{m}' \rangle \\ &= |c|^2 \left[\langle \mathbf{l}, \mathbf{l} \rangle + \langle \mathbf{m}, \mathbf{m} \rangle\right] \\ &= (det\ \gamma)\langle \mathbf{l} + \mathbf{m}, \mathbf{l}' + \mathbf{m}' \rangle \end{aligned}$$

and the definition (12.6). On the other hand, any $g \in D^*$ preserves orthogonality by (12.6) again. If, additionally, L^- is an eigenline of g, then also L^+ is. Therefore $\widehat{K}\left(L^-\right)^*$ is a commutative group generating the commutative $\mathbf{Q}$-subalgebra $\widehat{K}\left(L^-\right)$ of D. The maximal subfields of D are quadratic, therefore $\widehat{K}(L^-) = \{\gamma_c;\ c \in K\} \cong K \cong \widehat{K}$.

It is clear that different negative K-lines L^- in W define different subfields $\widehat{K}\left(L^-\right)$ of D isomorphic to K. Conversely, the line L^- is uniquely determined as negative eigenline of by its associated field $\widehat{K}(L^-)$. By (12.6) the subgroup

$$D_+^* = \{g \in D;\ \mathbf{N}(g) > 0\}$$

of index two in D^* acts on the set $\mathbb{P}W^-$ of negative K-lines $L^- \subset W$. The whole group D^* acts on the set of pairs $\{L^-,\ L^+\}$, hence, via conjugation, on the corresponding set of subfields $\widehat{K}(L^-)$. On the other hand, any two subfields of D isomorphic to K are D^*-conjugated by Corollary 12.14. Therefore the above actions of D^* or D_+^* are transitive. Alltogether we notice the bijective correspondences

12.23 Internal conjugations.

$$\begin{aligned} &\mathbb{P}\mathrm{W}^- = \left\{\text{negative K}-\text{lines } \mathrm{L}^- \subset \mathrm{W}\right\} \\ \Longleftrightarrow\ & K^* \backslash D_+^* \\ \Longleftrightarrow\ & \left\{\text{subfields } \widehat{K}(L^-) \text{ of } D\right\} = \{\text{subfields of } D \text{ isomorphic to } K\} \end{aligned}$$

We constructed from each indefinit hermitian K-vector plane W an indefinit quaternion algebra D with K-subfield. Let us start now conversely from $D = K + K\mathbf{u}$ with the relations described in (12.3). On this K-vector plane we define an indefinite hermitian form in a natural manner. For this purpose let p: $D \longrightarrow K$ denote the projection onto the first summand. For $X = a + b\mathbf{u} \in D$, its conjugate $X^* = \overline{a} - b\mathbf{u}$ and $c \in K$ it holds that

$$p(X^*) = \overline{p(X)},\ p(cX) = cp(X).$$

12.24 Lemma-Definition. The *canonical hermitian form* on $D = K + K\mathbf{u}$ is defined by

$$\langle X, Y\rangle = p(XY^*) \in K,\ X,\ Y \in D.$$

It is negative definit; the discriminant (of the canonical generators 1, $\mathbf{u}$) is equal to $-q = -\mathbf{u}^2 = \mathbf{u} * \mathbf{u} \in -Q_+^*$. Furthermore $K + K\mathbf{u}$ is an orthogonal decomposition of D with respect to $\langle\ ,\ \rangle$.

Proof. The definition is correct, namely

$$\begin{aligned} \langle cX, Y\rangle &= p(cXY^*) = cp(X, Y^*) = c\langle X, Y\rangle,\ c \in K; \\ \langle X + Z, Y\rangle &= p((X+Z)Y^*) = p(XY^*) + p(ZY^*) = \langle X, Y\rangle + \langle Z, Y\rangle; \\ \langle Y, X\rangle &= p(Y^*X) = p\left((X^*Y)^*\right) = \overline{p(X^*Y)} = \overline{\langle X, Y\rangle}. \end{aligned}$$

Therefore $\langle\ ,\ \rangle$ is a K-hermitian form on D. Furthermore for the canonical generators we get

$$\langle 1, \mathbf{u}\rangle = p(\mathbf{u}^*) = 0,\ \langle 1, 1\rangle = p(1 \cdot 1^*) = p(1) = 1,\ \langle \mathbf{u}, \mathbf{u}\rangle = p(\mathbf{u}\mathbf{u}^*) = p(-\mathbf{u}^2) = -q.$$

Therefore $K \perp K\mathbf{u}$ and the discriminant is $-q$.

$\square$

12.25 Proposition. The isometrie class of $W = W(D) = (K + K\mathbf{u}, \langle\ ,\ \rangle)$ defined in 12.24 does not depend on the K-presentation of D.

Proof. If $D = K + K\mathbf{u}'$ with the relations analogeous to (12.3), then $\mathbf{u}' = c\mathbf{u}$ for a suitable $c\ K^*$ and $q' = \widehat{\mathbf{u}'} = N(c)q$, see 12.8. Therefore the discriminant of $\langle\ ,\ \rangle$ constructed with $\mathbf{u}'$ instead of $\mathbf{u}$ does not change up to $N(K^*)$-multiplication. The isometry class is the same by Landherr's Theorem, see 9.4.

Now use another subfield $K' \cong K$ of $D = K' + K'\mathbf{u}'$ for the construction of $\langle\ ,\ \rangle$, say. By 12.14 there is an element $A \in D^*$ such that $K' = A^{-1}KA$. The old K-presentation comes with $\mathbf{u} = A\mathbf{u}'A^{-1}$. Using the notation $Z' = A^{-1}ZA$ for $Z \in D$ and p' for the canonical projection of D onto K' along $K\mathbf{u}'$ it is clear that $p'(Z') = p(Z)'$. Since

$$\mathbf{u}'\mathbf{u}'^* = A^{-1}\mathbf{u}\mathbf{u}^*A = -qA^{-1}A = -q$$

the discriminants of $\langle\ ,\ \rangle'$ and $\langle\ ,\ \rangle$ coincide. Therefore one gets the same isometry class of hermitian forms. It is so easy to see that $'$ defines an isometry.

$\square$

Without change of notation we extend the hermitian form $\langle\ ,\ \rangle$ to $W(\mathbf{R}) \cong \mathbf{C}^2$. Also the action of D on W is extended to $W(\mathbf{R})$. The group

$$\mathbf{U}(W(\mathbf{R})) := \mathbf{U}(W(\mathbf{R}), \langle\ ,\ \rangle) \cong \mathbf{Gl}_{2+}(\mathbf{R}) \cong \mathbf{U}((1,1), \mathbf{C})$$

acts transitively on the set $\mathrm{I\!PW}(\mathbf{R})^-$ of negative $\mathbf{C}$-lines $L^- \subset W(\mathbf{R})$. We get the bijective correspondences

12.24 External conjugations.

$$\begin{aligned} &\mathrm{I\!PW}(\mathbf{R})^- = \{\text{negative } \mathbf{C} - \text{lines } \mathrm{L}^- \subset \mathrm{W}(\mathbf{R})\} \\ \Longleftrightarrow\ &\mathbf{C}^* \backslash \mathbf{U}((1,1), \mathbf{C}); \end{aligned}$$

$$\begin{aligned} &\{\ {}^gD := gDg^{-1};\ g \in \mathbf{U}(W(\mathbf{R}))\} \\ \Longleftrightarrow\ &\mathbf{Gl}_{2+}(\mathbf{R})/D^* \\ \Longleftrightarrow\ &(\mathbf{R} \times D)^*/D^*. \end{aligned}$$

Let $(\mathbf{a}', \mathbf{b}') \in W^- \times W^+$ be a fixed orthogonal pair; L^- a negative $\mathbf{C}$-line in $W(\mathbf{R})$ and $g \in \mathbf{U}(W(\mathbf{R}))$ such that $\mathbf{a} := g(\mathbf{a}') \in L^-$. Set $\mathbf{b} = g(\mathbf{b}')$. The quaternion algebra $D \cong {}^gD$ acts on the hermitian K-vector space $U := K\mathbf{a} \bigoplus K\mathbf{b}$ via g-conjugation. Since $\langle \mathbf{a}, \mathbf{a} \rangle \neq 0$ we can write $U = D\mathbf{a}$, see Lemma 12.19. Using a coordinate map κ: $W(\mathbf{R}) \xrightarrow{\sim} \mathbf{C}^2$ we can assume that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in D\mathbf{a}$, and D acts K-linearly on $K\mathbf{a} + K\mathbf{b}$.

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