Zeta Dimension Formula for Picard Modular Cusp Forms of Neat Natural Congruence Subgroups

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Abstract

Let $\Gamma_K = \mathbb{U}((2,1), \mathfrak{O}_K)$ be the full Picard modular group of the imaginary quadratic number field K. For all natural congruence subgroups $\Gamma_K(m), m \geqslant 3$, acting freely on the two-dimensional complex unit ball, we prove an explicit polynomial formula for the dimensions of spaces of cusp forms of weight $n \geqslant 2$. The coefficients of these polynomials in the natural variables m, n are expressed by higher third and first Bernoulli numbers of the Dirichlet character χ_K of K and by values of Euler factors of the Riemann Zeta function and such factors of the L-series of χ_K at 2 or 3, respectively. The proof is based on detailed knowledges about classification of Picard modular surfaces. It combines algebraic geometric methods (Riemann-Roch, Vanishing- and Proportionality Theorem, curvature, structure of algebraic groups) with modern and classical number theoretic ones (representation densities, Tamagawa measure, strong approximation, functional equation for L-series).

1 Basic notions, definitions and the main result

We denote by

$$\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 1\}$$

the two-dimensional complex unit ball. Up to biholomorphic equivalence it is the only irreducible symmetric domain of complex dimension 2. Its group of biholomorphic automorphisms is the projective group $\mathbb{PU}((2,1),\mathbb{C}) = \mathbb{PSU}((2,1),\mathbb{C})$ of fractional linear transformations. With obvious notations the corresponding *(special) unitary group* is defined by

$$\mathbb{U}((2,1),\mathbb{C}) := \{A \in \mathbb{G}l_3(\mathbb{C});\ {}^t\bar{A} \cdot diag(1,1,-1) \cdot A = diag(1,1,-1)\},$$

$$\mathbb{SU}((2,1),\mathbb{C}) := \mathbb{U}((2,1),\mathbb{C}) \cap \mathbb{S}l_3(\mathbb{C}).$$

All these real Lie groups

$$\mathbb{U}((2,1),\mathbb{C}),\ \mathbb{SU}((2,1),\mathbb{C}),\ \mathbb{PSU}((2,1),\mathbb{C}) = \mathbb{PU}((2,1),\mathbb{C})$$

act transitively on $\mathbb{B}$. Mainly we prefer to work with $\mathbb{G} = \mathbb{G}_{\mathbb{R}} = \mathbb{SU}(2,1)$. Then $\mathbb{G}(\mathbb{R}) = \mathbb{SU}((2,1),\mathbb{C})$ is a simple simply-connected Lie group, see [Hel], Basic definitions and results can be also transfered to other Lie groups. The ball $\mathbb{B}$ can be identified with the coset space $\mathbb{G}(\mathbb{R})/\mathbf{K}$, where $\mathbf{K} = \mathbb{S}(\mathbb{U}(2) \times \mathbb{U}(1))$ is the maximal subgroup of $\mathbb{G}(\mathbb{R})$ stabilizing the zero point $O = (0,0) \in \mathbb{B}$. In order to describe the action of the Lie groups above in a convenient algebraical manner we remark that the notation $\mathbb{U}((2,1),\mathbb{C})$ can be used more generally for the unitary group $\mathbb{U}(V)$ of a hermitian vector space $(V, <,>)$ with $\dim_{\mathbb{C}}(V) = 3$ and a hermitian form $<,>$ of signature $(2,1)$. The ball $\mathbb{B}$ appears as subspace

$$\mathbb{B} = \mathbb{P}\{v \in V; < v, v > < 0\} \subset \mathbb{P}V \cong \mathbb{P}^2(\mathbb{C})$$

of all complex lines in V generated by a "negative" vector v. The group $\mathbb{U}((2,1),\mathbb{C})$ acts on $\mathbb{B}$ via the natural composition

$$\mathbb{U}((2,1),\mathbb{C}) \subset \mathbb{G}l(V) \longrightarrow \mathbb{PG}(V) = Aut_{hol}(\mathbb{P}V) \cong \mathbb{PG}l_3(\mathbb{C}) \cong Aut_{hol}\mathbb{P}^2(\mathbb{C}). \tag{1}$$

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic number field, d a square-free positive integer. Then there exists a $\mathbb{Q}$-defined algebraic group $\mathbb{G} = \mathbb{G}^{(d)}$ such that $\mathbb{G}(\mathbb{Q}) = \mathbb{SU}((2,1),K)$. Using the hermitian metric on $\mathbb{C}^3$ defined by the diagonal matrix $diag(1,1,-1)$ we can choose in a natural and explicit manner for each d a model of $\mathbb{G}^{(d)}$ defined over $\mathbb{Z}$. The arithmetic groups

$$\Gamma^{(d)} = \mathbb{G}^{(d)}(\mathbb{Z}) = \mathbb{SU}((2,1),\mathfrak{O}_K), \tag{2}$$

$\mathfrak{O}_K$ the ring of integers in K, are called *special Picard modular groups.* Moreover we call also $\mathbb{U}((2,1),\mathfrak{O}_K)$, all congruence subgroups Γ of them and their images $\mathbb{P}\Gamma$ in $\mathbb{PU}((2,1),\mathbb{C})$ *Picard modular groups*, sometimes more precisely *full, special or projective Picard modular groups*, respectively, with obvious meanings. For $\Gamma_K = \mathbb{U}((2,1),\mathfrak{O}_K)$ and an $\mathfrak{O}_K$-ideal $\mathfrak{a}$ with $\bar{\mathfrak{a}} = \mathfrak{a}$ the *principal congruence subgroup* $\Gamma_K(\mathfrak{a})$ is defined by the exact sequence

$$1 \longrightarrow \Gamma_K(\mathfrak{a}) \longrightarrow \Gamma_K \longrightarrow \mathbb{U}((2,1),\mathfrak{O}_K/\mathfrak{a}).$$

Especially, for each positive integer m the *natural principal congruence subgroups* $\Gamma_K(m) = \Gamma_K(m\mathfrak{O}_K)$ are defined. As discrete subgroups of $\mathbb{U}((2,1),\mathbb{C})$ (or $\mathbb{PU}((2,1),\mathbb{C})$) the Picard modular groups Γ act on $\mathbb{B}$. The action of $\mathbb{P}\Gamma$ is effective. All Picard modular groups are ball lattices. This means that they act proper discontineously on $\mathbb{B}$ and the volume of a Γ-fundamental domain with respect to the $\mathbb{G}(\mathbb{R})$-invariant hermitian (Bergmann) metric on $\mathbb{B}$, which is uniqely determined up to a nontrivial constant factor, is finite. The quotient surface $\Gamma\diagdown\mathbb{B}$ can be compactified by means of finitely many cusp singularities to a (normal complex projective) algebraic surface $\widehat{\Gamma\diagdown\mathbb{B}}$, the Baily-Borel compactification. By Baily-Borel's theorem [B-B] one has

$$\widehat{\Gamma\diagdown\mathbb{B}} = Proj\, R(\Gamma),$$

where

$$R(\Gamma) = \bigoplus_{n=0}^{\infty} [\Gamma, n]$$

is the ring of Γ-automorphic forms with the finitely dimensional $\mathbb{C}$-vector spaces $[\Gamma, n]$ of all Γ-automorphic forms of weight m as summands. These forms are defined as follows. The lattice Γ acts via $Aut_{hol}\mathbb{B} = \mathbb{PU}((2,1),\mathbb{C})$ on the $\mathbb{C}$-vector space $H^0(\mathbb{B}, \mathcal{O}_{\mathbb{B}})$ of holomorphic functions on $\mathbb{B}$ corresponding to each $f(z_1, z_2)$ the function $\gamma^*(f)(z_1, z_2) = f(\gamma(z_1, z_2))$. For each n one gets a representation

$$\rho_n : \Gamma \longrightarrow Aut\, H^0(\mathbb{B}, \mathcal{O}_{\mathbb{B}}), \quad \Gamma \ni \gamma : f \mapsto j_\gamma^{-n} \cdot \gamma^*(f) \tag{3}$$

with the Jacobi determinants

$$j_\gamma(z_1, z_2) = det(\frac{\partial \gamma(z_1, z_2)}{\partial(z_1, z_2)})$$

Then $[\Gamma, n] \subset H^0(\mathbb{B}, \mathcal{O}_{\mathbb{B}})$ is defined to be the eigensubspace of $\rho_n(\Gamma)$ of the eigenvalue 1, that means

$$[\Gamma, n] = \{f \in H^0(\mathbb{B}, \mathcal{O}_{\mathbb{B}}); \gamma^*(f) = j_\gamma^n \cdot f \; for\; all\; \gamma \in \Gamma\} \tag{4}$$

For instance, $[\Gamma, 0] = \mathbb{C}$ because Γ-invariant holomorphic functions on $\mathbb{B}$ factorize through $\Gamma\diagdown\mathbb{B}$ and extend to holomorphic functions on $\widehat{\Gamma\diagdown\mathbb{B}}$ because the finitely many added cusp singularities are normal and we dispose on Hartog's extension theorem, hence

$$\mathbb{C} \subseteq [\Gamma, 0] \subseteq H^0(\widehat{\Gamma\diagdown\mathbb{B}}, \mathcal{O}_{\widehat{\Gamma\diagdown\mathbb{B}}}) = \mathbb{C}.$$

In order to find the ring structure of $R(\Gamma)$ it is important to know $dim[\Gamma, n]$ for each $n > 0$. For reasons of proof technics we concentrate our attention to the subspace $[\Gamma, n]_0 \subset [\Gamma, n]$ of cusp forms. Roughly speaking cusp forms are automorphic forms which vanish at infinity (at the cusps). To be more precise, let us first interprete automorphic forms as holomorphic sections of sheaves of higher differential form bundles $\mathfrak{K} = \mathfrak{K}_{\mathbb{B}}^n := \mathfrak{K}_{\mathbb{B}}^{\otimes n}$ with the sheaf $\mathfrak{K}_{\mathbb{B}}$ of holomorphic differential forms on $\mathbb{B}$. The canonical action of Γ on $\mathbb{B}$ is defined by

$$\gamma: \; \omega = f dz_1 \wedge dz_2 \; \mapsto \; \gamma^*(\omega) = \gamma^*(f)\gamma^*(dz_1 \wedge dz_2) = \gamma^*(f) \cdot j_\gamma^{-n} \cdot dz_1 \wedge dz_2.$$

The embeddings

$$H^0(\mathbb{B}, \mathcal{O}_{\mathbb{B}}) \longrightarrow H^0(\mathbb{B}, \mathfrak{K}^n), \quad f \mapsto f \cdot (dz_1 \wedge dz_2)^{\otimes n} \tag{5}$$

are compatible with the corresponding Γ-actions (ρ_n on the preimage space) and

$$[\Gamma, n] \cong H^0(\mathbb{B}, \mathfrak{K}^n)^\Gamma. \tag{6}$$

The latter space has the advantage to go down to the quotient space $\Gamma\diagdown\mathbb{B}$:

$$H^0(\mathbb{B}, \mathfrak{K}_{\mathbb{B}}^n)^\Gamma \subseteq H^0(\Gamma\diagdown\mathbb{B}, \mathfrak{K}_{\Gamma\diagdown\mathbb{B}}^n), \tag{7}$$

if we assume that Γ acts freely on $\mathbb{B}$, that means $\mathbb{B} \longrightarrow \Gamma\diagdown\mathbb{B}$ is a universal covering. The *space of cusp forms* $[\Gamma, n]_0 \subseteq [\Gamma, n]$ is defined by corresponding to forms $\omega \in H^0(\Gamma\diagdown\mathbb{B}, \mathfrak{K}_{\Gamma\diagdown\mathbb{B}}^n)$ which can be extended to zero at all boundary (cusp) points $P \in \widehat{\Gamma\diagdown\mathbb{B}} \setminus \Gamma\diagdown\mathbb{B}$.

The aim of this paper is to present a universal dimension formula for cusp forms essentially for all natural congruence subgroups $\Gamma(m)$ of all Picard modular groups $\Gamma = \Gamma_K$ and for all weights n with the restrictions $m > 2$ to ensure that $\mathbb{B}$ acts freely on $\mathbb{B}$, and $n > 1$. For its final formulation we need generalized Bernoulli numbers, (semilocal) Zeta functions and/or L-series. Let $D = D_{K/\mathbb{Q}} < 0$ be the discriminant $K/\mathbb{Q}$ and

$$\chi = \chi_K = \chi_D : \mathbb{Z} \longrightarrow \{0, \pm 1\}, \quad m \mapsto (\frac{D}{m}) \text{ (\textit{Jacobi symbol})}$$

be the corresponding multiplicative function (Dirichlet character, see [I-R], XVI, §4) factorizing (precisely) through $\mathbb{Z}/D\mathbb{Z}$ with quadratic rest values $(\frac{D}{p})$ at primes p (0 iff $p|D$). The *generalized Bernoulli numbers* $B_{n,\chi}$ are defined as coefficients of a power series $F_\chi(t) \in \mathbb{Q}[[t]]$, namely

$$\sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!} = F_\chi(t) := \sum_{a=1}^{|D|} \frac{\chi(a) t e^{at}}{e^{|D|t} - 1}. \tag{8}$$

Remark 1.1 *For $|D| = 1$ in the sums (which doesn't occur for quadratic number fields) and $\chi = id$ one gets the usual Bernoulli numbers $B_n = B_{n,id}$ corresponding to the trivial field extension $\mathbb{Q}/\mathbb{Q}$.*

The *Dedekind Zeta function* of any number field K is defined by

$$\zeta_K(s) = \prod\nolimits_{\mathfrak{p} \in Spec\, \mathfrak{O}} (1 - N(\mathfrak{p})^{-s})^{-1} = \sum\nolimits'_{\mathfrak{a} \in \mathfrak{I}(\mathfrak{O})} N(\mathfrak{a})^{-s}, \tag{9}$$

where $\mathfrak{O} = \mathfrak{O}_K, \mathfrak{I}(\mathfrak{O})$ is the semigroup of ideals of $\mathfrak{O}, N(\mathfrak{a}) \in \mathbb{N}$ denotes the absolute norm of $\mathfrak{a}$ and $\sum'$ means that the zero ideal is excluded from the sum. For $K = \mathbb{Q}$ one gets the *Riemann Zeta function*

$$\zeta(s) = \zeta_{\mathbb{Q}}(s) = \prod\nolimits_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}. \tag{10}$$

For simplicity we will restrict ourselves to imaginary quadratic number fields K. The Zeta function $\zeta_K(s)$ converges absolutely for $Re\, s > 1$. It has a meromorphic extension to the whole complex plane $\mathbb{C}$ with precisely one pole, namely at s = 1, and the pole order there is equal to 1. We refer to [I-R], XVI, §6 including the literature given there, to [B-S] V, and to [Lan], XIV. The *Dirichlet L-series* of the field K or of the Dirichlet character χ is defined by

$$L(s,\chi) = \prod\nolimits_p (1-\chi(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \chi(n)n^{-s}. \tag{11}$$

It has an analytic extension (without poles) on $\mathbb{C}$. Taking into account that $\chi(p) = 0$ iff $p|D$ one gets the relation

$$\zeta_K(s) = \zeta(s)L(s,\chi). \tag{12}$$

We set

$$\zeta_K^{(m)}(s) := \prod\nolimits_{\mathfrak{p}|m} (1-N(\mathfrak{p})^{-s})^{-1} \tag{13}$$

and

$$\zeta^{(m)}(s) := \prod\nolimits_{p|m} (1-p^{-s})^{-1} \quad for\ all\ integers\ m \neq 0, \pm 1, \tag{14}$$

see [B-S], V, §2 (12). We need also the m-th Euler factor of the L-series

$$L^{(m)}(s,\chi) = \prod\nolimits_{p|m} (1-\chi(p)p^{-s})^{-1} \tag{15}$$

Main Theorem 1.2 *With the above notations for all imaginary quadratic-number fields $K = \mathbb{Q}(\sqrt{-d})$, d a natural squarefree number, and for all natural numbers $n > 1, m > 2$ (except for the cases $2|m, D$ but $4 \nmid m$), the following dimension formulas for spaces of Picard modular cusp forms hold:*

$$dim[\Gamma_K(m), n]_0 =$$

$$\tfrac{1}{288\delta_m}\zeta^{(m)}(2)^{-1}L^{(m)}(3,\chi_D)^{-1}\left[B_{3,\chi}(9n^2-9n+2)m^8 + \tfrac{6\delta_D}{\varepsilon_D}B_{1,\chi}m^6\right]$$

Thereby and later we use the following

elementary field constants

$\delta = \delta_D = \delta_K := \frac{|D|}{d} \in \{1,4\}$

$$\delta_m = \delta_{m,D} := \begin{cases} 4, & if \quad 2|m, D \\ 1, & \quad else \end{cases}$$

$$\varepsilon = \varepsilon_D = \varepsilon_K := \begin{cases} 3, & if\ D = -3\ (K\ is\ the\ field\ of\ Eisenstein\ numbers) \\ 1, & \quad else \end{cases}$$

The proof of the Main Theorem consists of a long sequence of conclusions, using strong methods of algebraic geometry and number theory. The structure of the formula is reflected by the main steps of proof:

I. Polynomial structure: - Riemann-Roch theory on algebraic surfaces; - Hirzebruch-Mumford proportionality theory; - Kodaira-Mumford vanishing theorem.

II. First higher Bernoulli number: - cusp geometry; - selfintersections of compactifying curves via local euclidean volumina; - cusp numbers via class numbers.

III. Values of Euler factors of Zeta- and L-series: - finite unitary groups; - representation densities and their splitting; - index formula for local congruence subgroups; - strong approximation; - index formula for global congruence subgroups.

IV. Third higher Bernoulli numbers: - p-adic volumes; - adele groups; - Tamagawa measure and -number; - curvature calculations; - non-euclidean (Euler-Bergmann) volume of fundamental domains; - functional equation for L-series.

The proof has been well-prepared by the classification theory of Picard modular surfaces. For instance, a basic reference for the cusp part is [Ho80], which is not everywhere available. A new and broader basic reference will be the monograph [Ho97], where also more detailled calculations can be found. In this article the reader may regard thouse omitted here as exercises. We started with the definition of cusp form spaces, wash the dimensions stepwise by the procedures announced in I ... IV, take care on each appearing constant and come to the beautyful explicit and purely $\mathbb{Q}$-rational end formula of the Main Theorem, not disturbed by longer but straightforward calculations.

2 Cusp Geometry

Let Γ be a neat Picard modular group, for instance $\Gamma = \Gamma^{(d)}(m)$, $m > 2$, see [Ho80], Lemma 4.3. In order to determine cusp contributions to our formulas we look for uniformizations of small open analytic punctured neighbourhoods $\hat{U} \setminus \{\hat{\kappa}\}$ around cusp singularities $\hat{\kappa}$ on $\widehat{\Gamma \diagdown \mathbb{B}}$. The set of boundary points of the ball $\mathbb{B}$ is denoted by $\partial\mathbb{B}$. There are biunivoque correspondences with the set $\mathfrak{P} = \mathfrak{P}_{\mathbb{R}}(\mathbb{G})$ of minimal parabolic $\mathbb{R}$-defined subgroups of $\mathbb{G}$ (Borel subgroups) and with the set $\mathfrak{U} = \mathfrak{U}_{\mathbb{R}}(\mathbb{G})$ of maximal unipotent $\mathbb{R}$-defined subgroups of $\mathbb{G}$. Each boundary point κ corresponds to the parabolic stabilizer subgroup $\mathbb{P}_\kappa \subset \mathbb{G}$, or to the unipotent radical $\mathbb{U}_\kappa$ of $\mathbb{P}_\kappa$, respectively. The Lie group $\mathbb{G}(\mathbb{R})$ acts transitively on $\mathfrak{P}$ and $\mathfrak{U}$ via conjugation by their elements. For details we refer to [Ho97], IV. The ball $\mathbb{B}$ can be moved in $\mathbb{P}^2(\mathbb{C})$ by a projective transformation g to the unbounded *Siegel domain*

$$\mathbb{V} := \{(z,u) \in \mathbb{C}^2; 2.Imz - |u|^2 > 0\} \tag{16}$$

with special boundary point $\infty := {}^t(1,0,0)$. The group $g\mathbb{G}g^{-1}(\mathbb{R})$ acting on $\mathbb{V}$ is denoted by G, the corresponding transform of Γ is denoted by Γ again. The isotropy group Γ_∞ of Γ at ∞ is a lattice in $U_\infty \subset P_\infty$, see 2.1 below. Explicitly the unipotent Lie group is described by

$$U_\infty = \left\{ \begin{pmatrix} 1 & i\bar{a} & \frac{i|a|^2}{2} + r \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} =: [a, r]; a \in \mathbb{C}, r \in \mathbb{R} \right\}. \tag{17}$$

For $U := U_\infty$ we have an exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta(U) := [U,U] \cong \mathbb{R} & \longrightarrow & U & \longrightarrow & \mathbb{C} \cong U/[U,U] & \longrightarrow & 1 \\ & & & & [a,r] & \mapsto & a & & \end{array} \tag{18}$$

and the rule

$$[a, r].[b, s] = [a + b, r + s + i\frac{\bar{a}b - \bar{b}a}{2}] = [a + b, r + s - Im\,\bar{a}b] \tag{19}$$

from which follows that $\Delta(U)$ is the center of U.

Now we change over to ball lattices Γ and their maximal unipotent subgroups.

Definitions 2.1 *Let $U = U_\kappa$ be the (maximal) unipotent subgroup group of G associated to $\kappa \in \partial\mathbb{B}$. The discrete subgroup Γ'_κ of U is called a* neat ball cusp lattice*, if Γ'_κ is a sublattice of U or, equivalently, if $\Gamma'_\kappa \diagdown U$ is compact. A discrete subgroup Γ_κ of $P = P_\kappa$ is a* ball cusp lattice*, if $\Gamma'_\kappa := U \cap \Gamma_\kappa$ is a neat ball cusp lattice. For a discrete sublattice in $\mathbb{G}(\mathbb{R})$ we call $\kappa \in \partial\mathbb{B}$ a* $\Gamma-$cusp*, if $\Gamma_\kappa := P_\kappa \cap \Gamma$ is a ball cusp lattice. The set of Γ-cusps is denoted by $\partial_\Gamma\mathbb{B}$.*
The Γ-conjugation class $\Gamma\diagdown\partial_\Gamma\mathbb{B}$ is a finite set called the set of cusp points *of Γ. It coincides with $\widehat{\Gamma\diagdown\mathbb{B}} \setminus (\Gamma\diagdown\mathbb{B})$. Its cardinality is denoted by $h(\Gamma)$. If Γ is a sublattice of Γ_K, then $\partial_\Gamma\mathbb{B}$ coincides with $\partial_K\mathbb{B} := \partial\mathbb{B} \cap (K \times K)$.*

Theorem 2.2 *([Fe79],[Zin]). The number $h(\Gamma_K)$ of cusp points of the full Picard modular group Γ_K of the imaginary quadratic number field K coincides with the class number h(K).*

□

Now let Γ_κ be a neat ball cusp lattice. Then the sequence (18) extends to the commutative diagram (20) of group homomorphisms.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R} & \longrightarrow & U_\kappa & \longrightarrow & \mathbb{C} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta_\kappa & \longrightarrow & \Gamma_\kappa & \longrightarrow & \Lambda_\kappa & \longrightarrow & 1 \end{array} \tag{20}$$

where $\Delta_\kappa \cong \mathbb{Z}$ and $\Lambda_\kappa \cong \mathbb{Z}^2$ are lattices in the additive groups of $\mathbb{R}$ or $\mathbb{C}$, respectively. We restrict our attention now to $\kappa = \infty$. Coming from a ball lattice, Γ_κ acts on the Siegel domain $\mathbb{V} \subset \mathbb{C}^2$ defined in (16). By (17) the action extends linearly to $\mathbb{C}^2$. In two steps we factorize $\mathbb{C}^2$ first by Δ_κ and then the quotient by Λ_κ. The group Δ_κ acts on the first factor of $\mathbb{C}\times\mathbb{C}$ by translations. Therefore $\Delta_\kappa\diagdown\mathbb{C}^2 \cong \mathbb{C}^* \times \mathbb{C} \subset \mathbb{C}\times\mathbb{C}$. We will see that the action of Λ_κ on $\mathbb{C}^*\times\mathbb{C}$ extends to $\mathbb{C}\times\mathbb{C}$. On the second factor it acts by translation (see (20)). Therefore $F_\kappa = F_\kappa(\Gamma) = F(\Gamma_\kappa) := \Lambda_\kappa\diagdown\mathbb{C}^2$ is a line bundle F_κ/T_κ over an elliptic curve $T_\kappa = T_\kappa(\Gamma) = T(\Gamma_\kappa) := \Lambda_\kappa\diagdown\mathbb{C}$. The situation is described in diagram (21).

$$
\begin{array}{ccccc}
 & & \mathbb{C}\times\mathbb{C} & \longrightarrow & F_\kappa = \Lambda_\kappa\diagdown\mathbb{C}\times\mathbb{C} \\
 & & \uparrow & & \uparrow \\
\mathbb{C}\times\mathbb{C} & \longrightarrow & \mathbb{C}^*\times\mathbb{C} & & \overline{\Gamma_\kappa\diagdown\mathbb{V}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{V} & \longrightarrow & \Delta_\kappa\diagdown\mathbb{V} & \longrightarrow & \Gamma_\kappa\diagdown\mathbb{V}
\end{array}
\tag{21}
$$

On this way we can find explicitly the (local) toroidal compactifacation of $\Gamma_\kappa\diagdown\mathbb{V}$ corresponding to the cusp κ filling in the elliptic curve T_κ as image of $(0\times\mathbb{C})$ along the Λ_κ-quotient morphism. The quotient $\Gamma_\kappa\diagdown\mathbb{V}$ appears as bundle of punctured discs in F_κ around (over) the zero section T_κ. The (Baily-Borel) point compactification $\widehat{\Gamma_\kappa\diagdown\mathbb{V}} = \Gamma_\kappa\diagdown\mathbb{V}\cup\{\hat{\kappa}\}$ is received by contracting T_κ to the cusp point $\hat{\kappa}$. In order to be more precise we define Δ_κ-invariant neighbourhoods

$$\mathbb{V}_C = \{(z,u)\in\mathbb{C}^2; 2\,Im\,z - |u|^2 > C\}.$$

For $C >> 0$ the subgroup of a ball lattice Γ acting on $\mathbb{V}_C$ is nothing else but Γ_κ; for other elements $\gamma\in\Gamma$ it holds that $\gamma\mathbb{V}_C\cap\mathbb{V}_C = \emptyset$. Therefore $\Gamma_\kappa\diagdown\mathbb{V}_C$ is an open analytic neighbourhood of the cusp singularity $\hat{\kappa}\in\widehat{\Gamma\diagdown\mathbb{V}}$. The diagram (21) can be extended to (22).

$$
\begin{array}{ccccc}
F_\kappa & & & & \\
\uparrow & & & & \\
\overline{\Gamma_\kappa\diagdown\mathbb{V}} & \longleftarrow & \overline{\Gamma_\kappa\diagdown\mathbb{V}_C} & \longrightarrow & \overline{\Gamma\diagdown\mathbb{V}} \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma_\kappa\diagdown\mathbb{V} & \longleftarrow & \Gamma_\kappa\diagdown\mathbb{V}_C & \longrightarrow & \Gamma\diagdown\mathbb{V}
\end{array}
\tag{22}
$$

We want to calculate the selfintersection (T_κ^2) of T_κ in $\overline{\Gamma\diagdown\mathbb{V}}$ by means of euclidean volumes of fundamental domains of Δ_κ in $\mathbb{R}$ and Λ_κ in $\mathbb{C}$, see (20). First we characterize ball cusp groups $\Gamma = \Gamma_\infty \subset U = U_\infty$ as abstract groups. Γ is a non-commutative torsion free nilpotent group of rank 3 (3 generators). By (20) it is a central extension of $\mathbb{Z}^2(\cong\Lambda)$ by $\mathbb{Z}$ (with $\Delta\cong\mathbb{Z}$ as center).

Proposition 2.3 *([Ho97], IV.2).*

(i) The abstract group structure of a neat ball cusp lattice Γ is uniquely determined by the negative integer t satisfying

$$\gamma^{-1}\beta^{-1}\gamma\beta = \delta^t \tag{23}$$

with generators β,γ,δ of Γ, $\delta\in\Delta$.

(ii) If $vol(\Lambda)$ and $vol(\Delta)$ denote the positive euclidean volume of a fundamental domain of $\Lambda \subset \mathbb{C}$ or $\Delta \subset \mathbb{R}$, respectively, then

$$t = -2\frac{vol(\Lambda)}{vol(\Delta)}. \tag{24}$$

(iii) Let $F = F(\Gamma)$ be the cusp bundle over the elliptic curve T constructed above, see (21), (22). Identifying T with the zero section of F it holds that

$$t = (T^2) < 0. \tag{25}$$

where $(T^2) = (T^2)_F$ is the selfintersection number of T in F.

□

Now let $K = \mathbb{Q}(\sqrt{-d})$ be a fixed imaginary quadratic number field as above, $\kappa \in \partial_K \mathbb{B}$,

$$\Gamma := \Gamma_\kappa, \quad \Gamma_{\kappa,u} := \Gamma \cap U_\kappa = \Gamma_\kappa \cap U_\kappa$$

the corresponding unipotent ball cusp lattice. Feustel proved in [Fe80] that the selfintersection of the elliptic zero section T_κ in the cusp bundle $F(\Gamma_{\kappa,u})$ does not depend on κ but only on $d \bmod 4$. At the special cusp (1:0:1) one can calculate

Proposition 2.4 *([Ho80], 4.8). With the above notations it holds that*

$$(T_\kappa^2) = -\delta_K = \frac{D}{d} \in \{-1, -4\},$$

where $D = D_{K/\mathbb{Q}}$ is the discriminant of K.

□

Let Γ' be a neat normal sublattice of Γ. Then Γ'_κ is unipotent. With obvious notations we get from (20) a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_\kappa & \longrightarrow & \Gamma_{\kappa,u} & \longrightarrow & \Lambda_\kappa & \longrightarrow & 1 \\
 & & \cup & & \cup & & \cup & & \\
1 & \longrightarrow & \Delta'_\kappa & \longrightarrow & \Gamma'_\kappa & \longrightarrow & \Lambda'_\kappa & \longrightarrow & 1
\end{array}$$

It is not difficult to find the relation

$$\frac{(T_\kappa^{'2})}{(T_\kappa^2)} = \frac{[\Lambda_\kappa : \Lambda'_\kappa]}{[\Delta_\kappa : \Delta'_\kappa]}, \tag{26}$$

see [Ho80], proof of 4.7. We calculated also there (Lemma 4.5) for $\Gamma' = \Gamma(m), \Delta(m) =: \Delta'$ that in any case

$$[\Delta_\kappa : \Delta'_\kappa] = m. \tag{27}$$

3 Riemann-Roch on Neat Ball Quotient Surfaces

Following the basic ideas of Hirzebruch [Hi] and a generalization of Mumford [Mu77] to non-compact algebraic quotient varieties of symmetric domains by neat lattices we ex-plain the proportionality principle relating different Chern numbers of the toroidal com-pactification $X := \overline{\Gamma\diagdown\mathbb{B}}$ of ball quotient surfaces. For details we refer to [Ho97] IV.4. Consider pairs (E', h') of holomorphic G-vector bundles E' on $\mathbb{B}$ with G-invariant hermitian metric h' with $G = \mathbb{SU}((2,1),\mathbb{C})$. It can be canonically extended to a metrized holomorphic $\mathbb{S}l_3(\mathbb{C})$-invariant hermitian vector bundle $(\check{E}, \check{h})$ on $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$. Since Γ acts freely on $\mathbb{B}$, the Γ-equivariant pair (E', h') descends to a hermitian holomorphic vector bundle (E,h) on the quotient variety $X_f := \Gamma\diagdown\mathbb{B}$ ("finite part"). By a theorem of Mumford [Mu77] this pair can be extended in a unique manner to a hermitian vector bundle $(\bar{E}, \bar{h})$ on X. Around each boundary points $t \in X_\infty$ the sheaf of holomorphic sections of $\bar{E}$ consists of the sections of E around t (outside X_∞) underlying a logarithmic growth condition. More precisely, a holomorphic base field $e_1, ..., e_r$ of $\bar{E}$ over a small complex analytic coordinate neighbourhood U of t where X_∞ is defined by w = 0, has to satisfy the conditions

$$|h(e_i, e_j)|, |det(h(e_i, e_j))^{-1})| < C.(log|w|)^{2M}$$

outside X_∞ for suitable positive constants C, M. The connection of the Mumford extension $\bar{E}$ with the equivariant bundles we started with is illustrated in diagram (28).

$$\begin{array}{ccccccc} & restriction & & factorization & & extension & \\ \check{E} & \cdots\cdots\cdots & E' & \cdots\cdots\cdots\cdots & E & \cdots\cdots\cdots & \bar{E} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^2 & \longleftarrow & \mathbb{B} & \longrightarrow & X_f & \longrightarrow & X \end{array} \tag{28}$$

For instance, the construction applies to the G-equivariant hermitian tangent bundle $T_\mathbb{B}$ with Bergmann metric, but also to its dual (equivariant hermitian) cotangent bundle $(E', h') = (T^*_\mathbb{B}, h')$ and to the canonical bundle (K', k'), $K' = K_\mathbb{B} = T^*_\mathbb{B} \wedge T^*_\mathbb{B}$, $k' = h' \wedge h'$. Explicitly, the Bergmann metric $(g_{j\bar{k}})$ on $\mathbb{B}$ is defined by the Kaehler form

$$\omega = \frac{i}{2\pi}\sum g_{j\bar{k}}dz^j d\bar{z}^k = -\frac{i}{2\pi}\partial\bar{\partial}logN(z_1, z_2) \tag{29}$$

on $\mathbb{B}$, where $N(z_1, z_2) = 1 - |z_1|^2 - |z_2|$ is the distance function from the boundary of $\mathbb{B}$. The corresponding Ricci form

$$\gamma_1 = -\sum \frac{\partial^2 log\, det(g_{j\bar{k}})}{\partial z^j \partial\bar{z}^k} dz^j d\bar{z}^k \tag{30}$$

satisfies the Kaehler-Einstein condition $\gamma_1 = \lambda\omega$ with $\lambda = -3$. As volume form we will use the corresponding Euler form

$$\gamma_2 = 3\omega\wedge\omega = \frac{1}{3}\gamma_1^2 = \frac{1}{3}\gamma_1\wedge\gamma_1. \tag{31}$$

For these explicit details we refer to [BBH], Appendix B. Starting from $(E,h') = (T^*_{\mathbb{B}}, h')$ or $(K_{\mathbb{B}}, k')$ one gets as $\bar{E}, \bar{K}$ on $X = \Gamma\diagdown\mathbb{B}$ the logarithmic bundles corresponding to the sheaves

$$\begin{array}{rcl} \mathcal{O}(\bar{E}) &=& \Omega^1(logT) \quad or \quad \mathcal{O}(\bar{K}) = \Omega^2(logT), \\ T &=& X_\infty \; as\,(compactification)\; divisor, \end{array} \tag{32}$$

respectively, coinciding on X_f with Ω^1 or Ω^2, and are defined locally on U around $t \in X_\infty : w = 0$ by

$$\begin{array}{rcl} \Omega^1(logT)(U) &=& \{\frac{a(w,u)}{w}dw + b(w,u)du;\, a,\, b \in \mathcal{O}_X(U)\}, \\ \Omega^2(logT)(U) &=& \{\frac{a(w,u)}{w}(dw \wedge du); a \in \mathcal{O}_X(U)\}. \end{array}$$

The correspondence $(E', h') \mapsto \bar{E}$ described in diagram (28) is compatible with tensor products. Therefore to $(K^n_{\mathbb{B}}, k^n)$ is corresponded $(K_X \otimes \{T\})^n$, where the upper index n means tensor power, K_X is the canonical bundle on X and $\{T\}$ denotes the vector bundle corresponding to compactification divisor T. The corresponding sheaves (of holomorphic sections) are denoted by $\mathfrak{K}_X$ or $\mathfrak{T}$, respectively. An easy local coordinate calcultion along the first factorisation in diagram (21) $(z,u) \mapsto (w = exp(\frac{2\pi i}{q}z), u)$, with w = 0 as definig equation for the compactifying torus, shows that

$$[\Gamma, n] \cong H^0(X, (\mathfrak{K}_X \otimes \mathfrak{T})^n), \quad [\Gamma, n]_0 \cong H^0(X, \mathfrak{K}^n_X \otimes \mathfrak{T}^{n-1}), \tag{33}$$

compare with (7). These presentations have been first found by Hemperly [Hem]. One has only to know that dz and $\frac{1}{w}dw$ coincide up to a constant and to change from z,u- to w,u-coordinates in sections $f(z,u)(dz \wedge du)^n$ with Γ-automorphic form f(z,u) of weight n. Let F be a holomorphic vector bundle of rank r on X. In the cohomology groups on X with constant coefficients we consider the Chern classes $c_i(F) \in H^{2i}(X, \mathbb{R})$, see e.g. [Wel]. Together they form the total Chern class

$$1 + c_1(F) + c_2(F) \in H^{even}(X, \mathbb{R}). \tag{34}$$

the Chern classes $c_i(F)$ are represented by the closed real 2i-forms $\gamma_i(F)$ on X. Consider the Chern forms

$$\gamma_1^2(F) := \gamma_1(F) \wedge \gamma_1(F) \quad , \quad \gamma_2(F). \tag{35}$$

The latter is called the Euler-Chern form of F. The corresponding volumes

$$c_1^2(F)[X] := \int \gamma_1^2(F) \quad , \quad c_2(F)[X] := \int \gamma_2(F) \tag{36}$$

are called the Cern numbers of the bundle F. If $F = T_X$ is the tangent bundle on X, then one writes shortly $c_1^2(X), c_2(X)$ instead of $c_1^2(T_X)$ or $c_2(T_X)$, respectively and calls them the Chern numbers of X. Especially, $c_2(X)$ is called the Euler number of X. With the additional notations of diagram (28) there is a remarkable relations between the Chern numbers of of $\breve{E}$ and $\bar{E}$, for neat arithmetic groups Γ and bounded symmetric domains in general, the Hirzebruch-Mumford proportionality relations. Restricting our attention to our two-dimensional case one gets

Proposition 3.1 *(see [Ho97], IV.3). For two-dimensional neat ball quotient surfaces $X = \overline{\Gamma\diagdown\mathbb{B}}$ it holds that*

$$\begin{array}{rcccl} c_1^2(\bar{E}) & = & \frac{c_1^2(\breve{E})}{c_2(\mathbb{P}^2)} vol_{EB}(\Gamma) & = & \frac{1}{3} vol_{EB}(\Gamma) c_1^2(\breve{E}), \\ c_2(\bar{E}) & = & \frac{c_2(\breve{E})}{c_2(\mathbb{P}^2)} vol_{EB}(\Gamma) & = & \frac{1}{3} vol_{EB}(\Gamma) c_2(\breve{E}), \end{array}$$

where $vol_{EB}(\Gamma) := \int_{\mathcal{F}(\Gamma)} \gamma_2$ denotes the Euler-Bergmann volume of a Γ-fundamental domain $\mathcal{F}(\Gamma)$ on $\mathbb{B}$ defined by the Euler-Chern form γ_2 of the Bergmann metric on $\mathbb{B}$.

$\square$

Proposition 3.2 *([Ho97], IV.3). Let Γ be a neat ball lattice, $T = \sum_{\kappa mod \Gamma} T_\kappa$, $supp\, T = X_\infty$, the compactification divisor of $X_f = \Gamma\diagdown\mathbb{B}$ consisting of finitely many elliptic curves T_κ. Then the Chern numbers of $X = \overline{\Gamma\diagdown\mathbb{B}}$ are related with the Euler volume of a Γ-fundamental domain on $\mathbb{B}$ by*

$$c_2(X) = vol_{EB}(\Gamma) \quad , \quad c_1^2(X) = 3vol_{EB}(\Gamma) + (T^2).$$

$\square$

Since the arithmetic genus $\chi(X) = \sum_i (-1)^i dim H^i(X, \mathcal{O}_X)$ of smooth compact complex algebraic surfaces is connected with the Euler numbers by Noether's formula $\chi = \frac{1}{12}(c_1^2 + c_2)$ it follows immediately that

$$\chi(X) = \frac{1}{3} vol_{EB}(\Gamma) + \frac{1}{12}(T^2) \tag{37}$$

Remark 3.3 *The global volume parts of both Chern numbers are related as γ_2 and γ_1^2 in (31), which, together with $\frac{c_1^2(\mathbb{P}^2)}{c_2(\mathbb{P}^2)} = 3$, is the origin of this proportionality.*

Since T is a sum of disjoint elliptic curves resolving the cusp singularities, we have

$$(T^2) = \sum_{i=1}^{h(\Gamma)} (T_i^2) \tag{38}$$

By (24) and (25) the selfintersections (T_i^2) can be calculated by euclidean volumes of lattices sitting in the corresponding cusp lattices. This has been done partly at the end of the previous section for the natural congruence subgroups $\Gamma_K(m)$, $m > 2$, see (26), (27). Fixing K, these formulas enable us to determine the whole corresponding cusp contribution $(T(m)^2)$ of $X(m) = X_K(m) = \overline{\Gamma_K(m)\diagdown\mathbb{B}}$. Denote by $\bar{\Gamma}$ the projective arithmetic group $\mathbb{P}\Gamma_K)$ acting effectively on $\mathbb{B}$, and use analogous notations for the subgroups. $\bar{\Gamma}(m)$ and $\Gamma(m)$ and their subgroups will be identified. The factor group $\bar{\Gamma}/\Gamma(m)$ acts on the set $\hat{X}_\infty(m)$ of cusp points of the Baily-Borel compactification $\hat{X}(m)$ of $X_f(m) = \Gamma(m)\diagdown\mathbb{B}$.

Let $\hat{\kappa}$ be a fixed cusp point, $\kappa \in \partial_K \mathbb{B}$. The isotropy subgroup of this point is $\bar{\Gamma}_\kappa/\Gamma_\kappa(m)$. Therefore, the number of components of T(m) is

$$\#\hat{X}_\infty(m) = \frac{[\bar{\Gamma} : \Gamma(m)]}{[\bar{\Gamma}_\kappa : \Gamma_\kappa(m)]} h(\bar{\Gamma}) = \frac{[\bar{\Gamma} : \Gamma(m)] \cdot h(K)}{[\bar{\Gamma}_\kappa : \Gamma_{\kappa,u}] \cdot [\Gamma_{\kappa,u} : \Gamma_\kappa(m)]}$$

By a result of Feustel [Fe80] for all cusps the factor group $\bar{\Gamma}_\kappa/\Gamma_{\kappa,u}$ is the same (for fixed K). It is cyclic of finite order, more precisely it holds that

$$\bar{\Gamma}_\kappa/\Gamma_{\kappa,u} \cong \mathbb{Z}/C_K\mathbb{Z}, \quad C_K := \#\{unitrootsof K\} \in \{2, 4, 6\}.$$

Taking also into account the diagram before (26) we get together with obvious notations

$$\#\hat{X}_\infty(m) = \frac{[\bar{\Gamma} : \Gamma(m)] \cdot h(K)}{C_K \cdot [\Lambda_\kappa : \Lambda_\kappa(m)] \cdot [\Delta_\kappa : \Delta_\kappa(m)]}$$

On the other hand we have by (26)

$$(T_\kappa(m)^2) = \frac{[\Lambda_\kappa : \Lambda_\kappa(m)]}{[\Delta_\kappa : \Delta_\kappa(m)]}(T_\kappa^2).$$

Since all (T_κ^2) are the same, namely equal to $-\delta_K$ by Prop. 2.14, and $[\Delta_\kappa : \Delta_\kappa(m)] = m$ by (27), we get

Theorem 3.4 *([Ho80]). For the natural Picard modular congruence subgroup $\Gamma_K(m)$, $m > 2$, $of\, \bar{\Gamma}_K = \mathbb{P}\Gamma_K$ the selfintersection of the compactification divisor $T_K(m)$ on $X_K(m) = \overline{\Gamma_K(m)\diagdown\mathbb{B}}$ is determined by*

$$(T_K(m)^2) = -\frac{[\bar{\Gamma}_K : \Gamma_K(m)] \cdot h(K) \cdot \delta_K}{C_K m^2}.$$

Corollary 3.5

$$\begin{array}{lcl} c_2(X_K(m) & = & [\bar{\Gamma}_K : \Gamma_K(m)] vol_{EB}(\Gamma_K) \\ c_1^2(X_K(m) & = & [\bar{\Gamma}_K : \Gamma_K(m)] \cdot [vol_{EB}(\Gamma_K) - \frac{h(K)\cdot\delta_K}{C_K m^2}]. \end{array}$$

□

The rest of this section will be used to find the formulas for dimensions of cusp forms in the above style. Comparing with (33) we calculate first the arithmetic genus of the bundles $\mathfrak{K}_X^n \otimes \mathfrak{T}^{n-1}$ by the Riemann-Roch formula for bundles $\mathfrak{F}$ on surfaces X:

$$\chi(\mathfrak{F}) = \sum\nolimits_j (-1)^j h^j(X, \mathfrak{F}) = \frac{1}{2}(\mathfrak{F} \cdot \mathfrak{F} \otimes \mathfrak{K}_X^{-1})) + \chi(X)$$

We apply it to neat ball quotient surfaces $X = \overline{\Gamma\diagdown\mathbb{B}}$ with compactification divisor $T = \sum_i T_i$ as above and to $\mathfrak{F} = \mathfrak{F}_n = \mathfrak{K}_X^n \otimes \mathfrak{T}^{n-1}$. With a canonical divisor $K = K_X$ the Riemann-Roch formula translates to

$$\begin{array}{lcl} \chi(\mathfrak{F}_n) & = & \frac{1}{2}([nK + (n-1)T] \cdot [(n-1)K + (n-1)T]) + \chi(X) \\ & = & \frac{(n-1)}{2}([nK + (n-1)T] \cdot (K + T)) + \chi(X). \end{array}$$

The adjunction formula for curves on surfaces yields

$$0 = -e(T_i) = ((K + T_i) \cdot T_i)$$

because the Euler number of an elliptic curve vanishes. It follows that

$$(T \cdot (K + T)) = \sum\nolimits_i (T_i \cdot (K + \sum\nolimits_j T_j)) = \sum\nolimits_i (T_i \cdot (K + T_i)) = 0,$$

hence

$$\begin{array}{rcl} \chi(\mathfrak{F}_n) & = & \frac{(n-1)}{2}((nK + nT) \cdot (K + T)) + \chi(X) = \binom{n}{2}(((K + T)^2) + \chi(X) \\ & = & \binom{n}{2}(K \cdot (K + T)) + \chi(X) = \binom{n}{2}(c_1^2(X) - (T^2)) + \chi(X) \end{array}$$

Using $c_1^2 = 3c_2 + (T^2)$, see Proposition 3.2, and $\chi = \frac{1}{12}(c_1^2 + c_2) = \frac{1}{12}(4c_2 + (T^2))$ for our surface X, it follows that

$$\begin{array}{rcl} \chi(\mathfrak{F}_n) & = & 3\binom{n}{2}c_2(X) + \frac{1}{3}c_2(X) + \frac{1}{12}(T^2) \\ & = & [3\binom{n}{2} + \frac{1}{3}]c_2(X) + \frac{1}{12}(T^2) \\ & = & [3\binom{n}{2} + \frac{1}{3}]vol_{EB}(\Gamma) + \frac{1}{12}(T^2). \end{array}$$

Especially for $\Gamma = \Gamma_K(m)$, $m > 2$, and $n > 1$ one obtains with help of 3.4 and 3.5

$$\chi(\mathfrak{F}_n) = [\bar{\Gamma}_K : \Gamma_K(m)]\{[3\binom{n}{2} + \frac{1}{3}]vol_{EB}(\Gamma_K) - \frac{h(K) \cdot \delta_K}{12C_K m^2}\}. \qquad (39)$$

Proposition 3.6 *For a neat ball quotient surface $X = \overline{\Gamma \backslash \mathbb{B}}$ the invert ible sheaves $\mathfrak{F}_n = \mathfrak{K}_X^n \otimes \mathfrak{T}^{n-1}$, $n > 1$ are cohomologically trivial in the sense that the (higher) cohomology groups $H^j(X, \mathfrak{F}_n)$, $j > 0$, vanish.*

Proof. We have only to check the vanishing of H^1 and H^2. For the latter case we apply Serre duality to get

$$H^2(X, \mathfrak{F}_n) \cong H^0(X, \mathfrak{K}_X \otimes \mathfrak{F}^{-n}) = H^0(X, \mathfrak{K}_X \otimes \mathfrak{T}^{-(n-1)}) = 0.$$

Namely, assume that there exists a non-zero section. Then $-(n-1)(K_X + T)$ is linearly equivalent to an effective divisor D on X. But then the positive powers of $\mathfrak{K}_X \otimes \mathfrak{T}$ cannot have non-zero sections. This is a contradiction to the next proposition.

□

Proposition 3.7 *(Baily-Borel). For a suitable $N > 0$ the sheaf $(\mathfrak{K}_X \otimes \mathfrak{T})^N$ is generated by global sections. The corresponding morphism $X \longrightarrow \mathbb{P}^M$, $M + 1 = h^0((\mathfrak{K}_X \otimes \mathfrak{T})^N)$ factorizes through the Baily-Borel embedding $\hat{X} = \widehat{\Gamma \backslash \mathbb{B}} \longrightarrow \mathbb{P}^M$.*

□

For the vanishing of H^1 we need the following

Theorem 3.8 *(Kodaira-Mumford [Mu67]). Let V be a complete normal variety over a field of characteristic 0,* $\dim V \geqslant 2, \mathfrak{L}$ *an invertible* $\mathcal{O}_V$*-sheaf such that* $\mathfrak{L}^N$ *is generated by its global sections for a suitable N ¿ 0. Then it holds that*

$$H^1(V, \mathfrak{L}^{-n}) = 0 \quad \textit{for all } n \geqslant 1 \qquad \Leftrightarrow \qquad \dim \Phi_N(V) > 1,$$

where $\Phi_N : V \longrightarrow \mathbb{P}^M$ *denotes the morphism corresponding to a basis of the space of global* $\mathfrak{L}^N$*-sections.*

By Proposition 3.7 the vanishing conclusion for H^1 holds for $\mathfrak{L} = \mathfrak{K}_X \otimes \mathfrak{T}$. Via Serre duality we get

$$\begin{array}{rcl} H^1(X, \mathfrak{F}_n) & = & H^1(X, \mathfrak{K}_X^n \otimes \mathfrak{T}^{n-1}) \cong H^1(X, \mathfrak{K}_X \otimes (\mathfrak{K}_X^n \otimes \mathfrak{T}^{n-1})^{-1} \\ & = & H^1(X, (\mathfrak{K}_X \otimes \mathfrak{T})^{-(n-1)} = 0. \end{array}$$

□

Since $\chi = h^0 - h^1 + h^2$ one obtains

Corollary 3.9

$$h^0(X, \mathfrak{F}_n) = [3\binom{n}{2} + \frac{1}{3}]vol_{EB}(\Gamma) + \frac{1}{12}(T^2)$$

for $n \geqslant 2$.

□

Together with (33) we get finally the

Theorem 3.10 *For the natural Picard modular congruence subgroups of level* $m \geqslant 3$ *of an arbitrary imaginary quadratic number field K the dimension of the space of cusp forms of weight* $n \geqslant 2$ *is determined by*

$$dim[\Gamma_K(m), n]_0 = [\bar{\Gamma}_K : \Gamma_K(m)]\{[3\binom{n}{2} + \frac{1}{3}]vol_{EB}(\Gamma_K) - \frac{h(K) \cdot \delta_K}{12 C_K m^2}\}.$$

□

4 Index formulas for congruence subgroups

Finite hermitian modules and cardinality of their unitary groups

Let $\mathfrak{O} = \mathfrak{O}_K, K = \mathbb{Q}(\sqrt{-d}), D = D_{K/\mathbb{Q}}$, p a natural prime, Φ a hermitian form on $\mathfrak{O}^n$ represented by a hermitian matrix with coefficients in $\mathfrak{O}$ denoted by the same letter, descends to the hermitian module $(\mathfrak{O}/p^k\mathfrak{O})^n, \bar{\Phi})$ over the artinian ring $(\mathfrak{O}/p^k\mathfrak{O})^n$. Write Φ instead of $\bar{\Phi}$ and let also Ψ be such a hermitian form on $\mathfrak{O}^m$. Set

$$\begin{array}{rcl} A_{p^k}(\Phi,\Psi) & := & \#\{M \in Mat_{n,m}(\mathfrak{O}/p^k\mathfrak{O});\ {}^tM\Phi\bar{M} = \Psi\} \\ A_{p^k}(\Phi) & := & A_{p^k}(\Phi,\Phi) = \#\mathbb{U}(\Phi,\mathfrak{O}/p^k\mathfrak{O}). \end{array}$$

The *local densities*

$$\begin{array}{rcl} \alpha_p(\Phi,\Psi) & := & p^{km(m-2n)}A_{p^k}(\Phi,\Psi) \quad for \quad k >> 0, \\ \alpha_p(\Phi) & := & \alpha_p(\Phi,\Phi), \end{array}$$

are correctly defined, see [Bra].
The following recursion properties for diagonal forms are known:

$$\alpha_p(diag(d_1,...,d_n)) = \alpha_p(diag(d_1,...,d_n),d_i)\alpha_p(diag(d_1,..,\hat{d_i},..,d_n),$$

$$\text{if } p \nmid 2d_1 \cdot ... \cdot d_n D$$

$$\alpha_p(E_n) = \alpha_p(E_n,1)\alpha_p(E_{n-1}), \quad if \quad \nmid 2D,$$

where E_n denotes the unit matrix with n rows. The main role for our purposes plays $\Phi = diag(1,1,-1)$. Here we have the important local splittings

$$\alpha_p(\Phi) = \alpha_p((\Phi,-1)\alpha_p(E_2) = \alpha_p(\Phi,-1)\alpha_p(E_2,1)\alpha_p(1)$$

For the proof we used additionally the classification of hermitian lattices over local rings due to Jacobowitz [Jac].

Theorem 4.1 *([Ho97], Appendix of V). For $\Phi = diag(1,1,-1), \mathfrak{O} = \mathfrak{O}_K, D = D_{K/\mathbb{Q}}$ it holds that*

$$\#\mathbb{SU}(\Phi,\mathfrak{O}/p^k\mathfrak{O}) = p^{8k}(1-(\frac{D}{p})p^{-3})(1-p^{-2})$$

for all $k \geqslant 1$ if not $2 = p|D$ and for $k \geqslant 3$ in any case.

$\square$

Let $\mathbb{G}/\mathbb{Z}$ be an irreducible linear group scheme, p a natural prime, $\Gamma_p := \mathbb{G}(\mathbb{Z}_p)$. We introduce the following notations for (relative) reduction maps:

$$\begin{array}{rcll} red_m : \Gamma_p & \longrightarrow & \mathbb{G}(\mathbb{Z}_p/p^m\mathbb{Z}_p) & for \quad m \geqslant 0, \\ red_m^n : \mathbb{G}(\mathbb{Z}_p/p^n\mathbb{Z}_p) & \longrightarrow & \mathbb{G}(\mathbb{Z}_p/p^m\mathbb{Z}_p) & for \quad n \geqslant m. \end{array}$$

The kernel of red_m is denoted by $\Gamma_p(m)$. This is the *p^m − congruence subgroup* of Γ_p.

Theorem 4.2 *([Ho97], Appendix of V). There exist natural numbers $k_0 \geqslant 1, e \geqslant 0$, such that for all $k \geqslant k_0$ the left-exact commutative diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Gamma_p(k) & \longrightarrow & \Gamma_p & \to red_k(\Gamma_p) \to & \mathbb{G}(\mathbb{Z}_p/p^k\mathbb{Z}_p) \\ & & \uparrow & & \| & & \uparrow red_k^{k+e} \\ 1 & \longrightarrow & \Gamma_p(k+e) & \longrightarrow & \Gamma_p & \longrightarrow & \mathbb{G}(\mathbb{Z}_p/p^{k+e}\mathbb{Z}_p) \end{array}$$

has a commutative extension factorizing red_k^{k+e} through $red_k(\Gamma_p)$

□

Corollary 4.3

$$[\Gamma_p : \Gamma_p(k)] = \#red_k(\Gamma_p) = [\mathbb{G}(\mathbb{Z}_p/p^k\mathbb{Z}_p) : Ker(red_k^{k+e})].$$

□

Definition 4.4 *We call a pair (e, k_0) satisfying the properties of Theorem 4.2 a* Neron pair *of Γ_p.*

We look for minimal Neron pairs with respect to the lexicographical order. For given $\mathbb{G}/\mathbb{Z}$ as above the minimal Neron pairs of the local groups Γ_p depend on p. So it is precise to write e_p instead of e, if p is not clearly fixed. The notion is motivated by a more general existence theorem of Neron, [Ner] Prop.20, about polynomial maps over henselian discrete valuation rings R in characteristic 0 for certain pairs (e, k_0). We can restrict ourselves to the cases $R = \mathbb{Z}_p$. A polynomial map over R

$$F = (F_1, ..., F_m) : R^n \longrightarrow R^m, \quad x = (x_1, ..., x_n) \mapsto F(x)$$

is given by m polynomials $F_i \in R[X_1, ..., X_n]$. Neron's general theorem says that their exist pairs (e, k_0) of natural numbers such that for each $x \in R^n$ with $F(x) \equiv 0 \, mod \, p^{k+e}, k \geqslant k_0$, (approximative solution) one can find a solution $R^n \ni x' \equiv x \, mod \, p^k$ of $F(X) = 0$. For application to affine group schemes over $\mathbb{Z}_p$ one has to write the defining equations in the form $F(X) = 0$ with polynomial map of the above type.

Example 4.5 *The linear algebraic $\mathbb{Z}$-groups $\mathbb{G} = \mathbb{G}^{(d)}$ of the field $K = \mathbb{Q}(\sqrt{-d})$ are defined by $F : \mathfrak{O}^9 \longrightarrow \mathfrak{O}^{10}$, sending*

$$A \in Mat_3(\mathfrak{O}) \cong \mathfrak{O}^9 \qquad to \qquad ({}^t A\Phi\bar{A} - \Phi, detA - 1).$$

By means of a $\mathbb{Z}$-basis of $\mathfrak{O}$ it can be written as a polynomial map from $\mathbb{Z}^{18}$ to $\mathbb{Z}^{20}$ which extends for each prime p to a p-adic polynomial map $F_p : \mathbb{Z}_p^{18} \longrightarrow \mathbb{Z}_p^{20}$. The equation $F_p = 0$ describes the local group $\Gamma_p = \mathbb{G}(\mathbb{Z}_p)$. The elements of $\mathbb{G}(\mathbb{Z}_p/p^m\mathbb{Z}_p)$ are represented by the p^m-th approximative solutions of the equation $F_p(X) = 0$.

Now the transfer of Neron pairs of F to Neron pairs of Γ_p, whose existence is stated in 4.2, is obvious. In [Ho97] we succeeded to find fairly low Neron pairs expressed by highest exponents of $\mathbb{Z}_p$-polynomial maps. In order to define them, let $d_1|d_2|...|d_r \neq 0$ be a chain of elementar divisors of a matrix $0 \neq A \in Mat_{n,m}(\mathbb{Z}_p)$. Then we call e the *highest exponent* of A, iff p^e is the highest p-power dividing d_r. It is denoted by $e_p(A)$. The definition transfers correctly to polynomial maps between free finite $\mathbb{Z}_p$-modules and to polynomial maps $F : \mathbb{A}^n(\mathbb{Z}_p) \longrightarrow \mathbb{A}^m(\mathbb{Z}_p)$ representing rational (algebraic) morphisms of affine spaces $\mathbb{A}^n \longrightarrow \mathbb{A}^m$ defined over $\mathbb{Z}_p$. For the latter map at each point $Q \in \mathbb{A}^n(\mathbb{Z}_p)$ the differential map $d_Q F : \mathbb{Z}_p^n \longrightarrow \mathbb{Z}_p^m$ is defined (the linearization of F at Q). If, for instance, $F(0) = 0$, then $d_0 F$ is represented by the Jacobi matrix $(\partial F/\partial X)_0 = (\partial F_i(X)/\partial X_j)(0)$.

Definition 4.6 *With the above notations we call $e_{p,Q}(F) := e_p(d_Q(F)$ the* highest exponent of F at Q.

For the special unitary $\mathbb{Z}_p$-polynomial maps F_p of example 4.5 extending the $\mathbb{Z}$-polynomial map F we dispose now on highest exponents $e_{p,E}(F) = e_p(F_p)$ at the unit element E.

Proposition 4.7 *([Ho97], Appendix of V, section 2). Let $F = F^{(d)}$ be the special unitary $\mathbb{Z}$-polynomial map defining $\mathbb{G}^{(d)}/\mathbb{Z}$ as described in 4.5.*

(i) $$e = e_{p,E}(F^{(d)}) = \begin{cases} 1, & if \quad 2 = p|D \\ 0, & \quad otherwise \end{cases}$$

(ii) $(e, e+1)$ *is a Neron pair of the local group* $\Gamma_p = \mathbb{G}^{(d)}(\mathbb{Z}_p)$.

The proof of the first part is a simple calculation via Jacobian matrices. For (ii) one has to apply to $F = F^{(d)}$ and Q = E the following

Proposition 4.8 *(generalized Hensel lemma, [Ho97], Appendix of V, Prop. 2.9). Let $F : \mathbb{A}^n \longrightarrow \mathbb{A}^m$ be an algebraic morphism of affine spaces defined over $\mathbb{Z}_p$, $Q \in \mathbb{A}(\mathbb{Z}_p^n)$, $e = e_{p,Q}(F)$. If the residue of Q is a solution of $F(X) \equiv 0 \mod p^{k+e}$ and $k > e$, then there exists a solution $Q' \in \mathbb{A}^n(\mathbb{Z}_p)$ of $F(X) = 0$ such that $Q' \equiv Q \mod p^k$.*

By definition 4.4 the second part of Proposition 4.7 means that we have left-exact commutative diagrams

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Gamma_2(k) & \longrightarrow & \Gamma_2 & \rightarrow red_k(\Gamma_2) \rightarrow & \mathbb{G}(\mathbb{Z}_2/2^k\mathbb{Z}_2) \\ & & \uparrow & & \| & & \uparrow red_k^{k+e} \\ 1 & \longrightarrow & \Gamma_2(k+e) & \longrightarrow & \Gamma_2 & \longrightarrow & \mathbb{G}(\mathbb{Z}_2/2^{k+e}\mathbb{Z}_2) \end{array}$$

for all $k \geqslant 2$ factorizing red_k^{k+e} through $red_k(\Gamma_2$ in the exceptional cases $2|D$, and otherwise exact sequences

$$1 \longrightarrow \Gamma_p(k) \longrightarrow \Gamma_p \longrightarrow \mathbb{G}(\mathbb{Z}_p/p^k\mathbb{Z}_p) \longrightarrow 1$$

for all $k \geqslant 1$. Together with Corollary 4.3, Theorem 4.1 and a little calculation in the case $2 = p|D$ to determine $\#Ker(red_k^{k+1}) = 2^{10}$ for all $k \geqslant 1$ one gets finally

Theorem 4.9 *([Ho97], Appendix of V, Prop. 2.15). Let Γ_p be the local group $\mathbb{G}^{(d)}(\mathbb{Z}_p)$. For the congruence subgroups $\Gamma_p(k)$ holds*

$$[\Gamma_p : \Gamma_p(k)] = \frac{p^{8k}}{\delta_p}(1 - (\frac{D}{p})p^{-3})(1 - p^{-2})$$

for $k \geqslant 2$ in general and $k \geqslant 1$ if not $2 = p|D$.

$\square$

Global indices

Corollary 4.10 *(global index formula, [Ho97], Appendix of V, Prop. 2.22). For the special Picard modular groups $\Gamma = \mathbb{G}^{(d)}(\mathbb{Z})$ and all natural numbers m, except for $2|m, D$, $4 \nmid m$, it holds that*

$$[\Gamma : \Gamma(m)] = \frac{m^8}{\delta_m} \prod\nolimits_{p|m} (1 - p^{-2})(1 - (\frac{D}{p})p^{-3}).$$

For the proof one needs the strong approximation theorem for simple simply-connected algebraic groups $\mathbb{G}$ over $\mathbb{Q}$ with non-compact $\mathbb{G}(\mathbb{R})$ due to Platonov and others, see [Pla] and the references there. Our special unitary groups are simple, see [Hel], IX, §4, Lemma 4.4, and simply- connected. Strong approximation means that

4.11 For given primes p_i, natural numbers m_i and elements $A_i \in \mathbb{G}(\mathbb{Q}_{p_i})$, $i = 1, ..., r$, there exists $A \in \mathbb{G}(\mathbb{Q})$ such that $A \in \mathbb{G}(\mathbb{Z}_p)$ for all $p \neq p_1, ..., p_r$ and $A \equiv A_i \, mod \, p^{m_i}$ for $i = 1, ..., r$.

For algebraic groups defined over $\mathbb{Z}$ it is easy to show that from the strong approximation property one gets exact sequences for global and local congruence subgroups

$$1 \longrightarrow \mathbb{G}(\mathbb{Z})(m) \longrightarrow \mathbb{G}(\mathbb{Z}) \longrightarrow \prod_{i=1}^{r} \Gamma_{p_i}/\Gamma_{p_i}(m_i) \longrightarrow 1$$

for $m = p_1^{m_1} \cdot ... \cdot p_r^{m_r}$, see [Ho97], App. of V, Lemma 2.19. Now the corrollary follows immediately from Theorem 4.9.

$\square$

5 The Euler-Bergmann volume

Local volumina

The Lie algebra $\mathfrak{g}$ of the Lie group $G = \mathbb{G}(\mathbb{R}) = \mathbb{SU}((2,1), \mathbb{C})$ is a subalgebra of the Lie algebras $\mathfrak{sl}_3(\mathbb{C}) \subset \mathfrak{gl}_3(\mathbb{C}) \subset Mat_3(\mathbb{C})$. The imaginary quadratic number field K endows $\mathfrak{g}$ with a K-structure ($\mathbb{Q}$-structure) setting $\mathfrak{g}_K = \mathfrak{g} \cap Mat_3(K)$ containing $\mathfrak{g}_{\mathfrak{O}} := \mathfrak{g} \cap Mat_3(\mathfrak{O})$. Tensoring with $\mathbb{Q}_p$ we get the local Lie algebra $\mathfrak{g}_p = \mathfrak{g}_p^{(d)}$. For natural numbers m we define the local congruence subalgebras $\mathfrak{g}_p(m)$ consisting of all $A \in \mathfrak{g}_{\mathfrak{O}} \otimes \mathbb{Z}_p \subset Mat_3(\mathfrak{O}_p)$ with $A \equiv 0 \, mod \, m$. As a vector space, the Lie algebra $\mathfrak{g}_p$ of our special unitary local group $G_p = \mathbb{G}^{(d)}(\mathbb{Q}_p)$ is isomorphic to $\mathbb{Q}_p^8$. The exponential map

$$exp : \mathfrak{g}_p(m) \longrightarrow \Gamma_p(m) \ , \quad A \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

is a homeomorphism for $m \geqslant 2$ and even for $m \geqslant 1$ if $p \neq 2$ with respect to the usual local topologies, see [H097], App. of V, Lemma 3.4. The Haar measure on $\mathbb{Q}_p$ normalized by $\frac{1}{p}$ on $\mathbb{Z}_p$ extends to $\mathfrak{g}_p$ and transfers via the exponential map to a Haar measure on the topological group $\mathbb{G}(\mathbb{Q}_p)$, normalized with respect to the canonical $\mathbb{Z}$-lattice $\mathfrak{g}_{\mathfrak{O}} = \mathfrak{g}_{\mathfrak{O}}^{(d)}$ of $\mathfrak{g} = T_e\mathbb{G}(\mathbb{R})$. Together with a (normalized) invariant algebraic differential form ω on $\mathbb{G}$ of highest order defined over $\mathbb{Q}$ the local volumina $\omega_p(\Gamma_p)$ are well-defined as integrals over Γ_p. We normalize $\omega = \omega^{(d)}$ by the volume 1 on a $\mathfrak{g}_{\mathfrak{O}}$-fundamental domain of Γ. Then one gets ([Ho97])

$$\omega_p(\Gamma_p(m)) = p^{-8m}$$

after some differential geometric considerations of linearization and shiftings of the exponential map as described in [Hel], II, in terms of Taylor series. Then the local index formula of 4.9 yields the following

Proposition 5.1 *([Ho97], App. of V, Prop. 3.9).*

$$\omega_p(\Gamma_p) = (1 - p^{-2})(1 - (\frac{D}{p})p^{-3})/\delta_p.$$

$\square$

According to real differential geometry the volume form ω defines also a normalized Haar measure ω_∞ on $G = G_\infty = \mathbb{G}(\mathbb{R})$. We want to determine the ω-volume

$$\omega(\mathbb{G}(\mathbb{Z}) \diagdown G) := \omega_\infty(\mathbb{G}(\mathbb{Z}) \diagdown G)$$

of a $\mathbb{G}(\mathbb{Z})$-fundamental domain of G. This global volume and the local volumina come together on the adele group $\mathbb{G}(\mathbb{A}) \subset G_\infty \prod_p \mathbb{G}(\mathbb{Q}_p)$, where $\mathbb{A} = \mathbb{A}_\mathbb{Q} \subset \mathbb{R} \times \prod_p \mathbb{Q}_p$ denotes the adele ring of $\mathbb{Q}$. It is a locally compact topological group. Via infinite products the form ω defines a (normalized) Haar measure $\omega_\mathbb{A}$ on $\mathbb{G}(\mathbb{A})$, see e.g. [Kne], using the fact that the infinite product $\prod_p \omega_p(\mathbb{G}(\mathbb{Z}_p)$ converges. It is called the *Tamagawa measure* on $\mathbb{G}(\mathbb{A})$. The group $\mathbb{G}(\mathbb{Q})$ appears as discrete subgroup of $\mathbb{G}(\mathbb{A})$ by diagonal embedding. Consider the topological quotient space $\mathbb{G}(\mathbb{Q})\diagdown\mathbb{G}(\mathbb{A})$. Its volume with respect to $\omega_\mathbb{A}$ is called the *Tamagawa number* $\tau(\mathbb{G})$ of $\mathbb{G}$. It does not depend on the (normalization) choice of ω. It is not difficult to verify that the strong approximation property implies (generally for semisimple algebraic groups defined over $\mathbb{Z}$) the following

Lemma 5.2 *([Ho97], App. of V, 4.4).*

$$\omega(\mathbb{G}(\mathbb{Z})\diagdown G) = \tau(\mathbb{G}) \prod_p \omega_p(\mathbb{G}(\mathbb{Z}_p))^{-1}.$$

For the proof one has only to use the toplogical translation of the strong approximation property. It says that $G_\infty \cdot \mathbb{G}(\mathbb{Q})$ is dense in $\mathbb{G}(\mathbb{A})$, see [Pla]. It has the decomposing consequence $\mathbb{G}(\mathbb{A}_f) = \mathbb{G}(\mathbb{Q}) \cdot \prod_p \mathbb{G}(\mathbb{Z}_p)$, where $\mathbb{A}_f \subset \prod_p \mathbb{G}(\mathbb{Q}_p)$ is the ring of finite adeles. The rest is trivial.

□

The most striking fact is the following

Theorem 5.3 *(see [Kot]). The Tamagawa number $\tau(\mathbb{G})$ is equal to 1.*

□

Together with Lemma 5.2 and Proposition 5.1 we get the global volume

Theorem 5.4 *([Ho97], App. of V, 4.8)*

$$.\omega^{(d)}(\mathbb{G}(\mathbb{Z})\diagdown G) = \delta_D \prod_p (1-p^{-2})^{-1}(1-(\frac{D}{p})p^{-3})^{-1} = \delta_D \zeta(2)L(3,\chi).$$

□

In order to change to the Euler-Bergmann metric on the ball $\mathbb{B}$ we compare first the forms $\omega^{(d)}$ with a fixed one, say $\omega^{(1)}$. Being $\mathbb{G}$- invariant, they differ by a constant factor only. So one has only to compare the normalizing lattices $\mathfrak{g}_\mathfrak{O} forall \mathfrak{O} = \mathfrak{O}_{\mathbb{Q}(\sqrt{-d})}$. It turns out that

$$\omega^{(d)} = \frac{8\delta_D}{|D|^{5/2}}\omega^{(1)}. \tag{40}$$

In the next steps we set $\omega = \omega^{(1)}$ and $\mathfrak{O} = \mathbb{Z}[i]$ for abbraviaty. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, $\mathfrak{k} = Lie\,\mathbf{K}, \mathbf{K}$ the maximal compact subgroup $\mathbb{S}(\mathbb{U}(2)\times\mathbb{U}(1))$ of G, $\mathfrak{p} = T_0\mathbb{B}$. In both summands we defined in [Ho97] explicitly canonical $\mathbb{Z}$-lattices $\mathfrak{k}_{\mathfrak{O}}$ or $\mathfrak{p}_{\mathfrak{O}}$ with $\mathfrak{O}$-structure such that $\mathfrak{g}_{\mathfrak{O}} = \mathfrak{k}_{\mathfrak{O}} + \mathfrak{p}_{\mathfrak{O}}$. The volume form ω splits into $\omega = \kappa \cdot \lambda$, where κ is a $\mathbf{K}$-invariant and λ a $\mathbb{G}((R))$-invariant volume form on $\mathbb{B}$. Both forms are normalized by means of $\mathfrak{k}_{\mathfrak{O}}$ or $\mathfrak{p}_{\mathfrak{O}}$, respectively. By an elementary differential geometric calculation, mainly on a 3-dimensional sphere $S^3 \subset \mathbb{R}^4$ one gets the volume

$$\kappa(\mathbf{K}) = 4\pi^3.$$

For each ball lattice $\Gamma' \subset G$ acting freely on $\mathbb{B}$ the decomposition of ω yields via fibrewise integration ($G \longrightarrow G/\mathbf{K} = \mathbb{B}$) the splitting

$$\omega(\Gamma' \diagdown G) = \kappa(\Gamma') \cdot \lambda(\Gamma' \diagdown \mathbb{B}) = 4\pi^3 \cdot \lambda(\Gamma' \diagdown \mathbb{B}),$$

see [Hel], X.1, where the latter factor denotes the λ-volume of Γ'- fundamental domain on $\mathbb{B}$. The formula extends to ball lattices $\Gamma \subset G$ acting effectively on $\mathbb{B}$. One has to divide both sides by the index $[\Gamma : \Gamma']$ using a suitable torsionfree sublattices Γ' of Γ. For ball lattices $\Gamma \subset G$ not acting effectively on $\mathbb{B}$ one has to multiply additionally the left-hand side by the correcture factor $\#C(\Gamma)$, where $C(\Gamma)$ denotes the center of $\Gamma(= \Gamma \cap C(\mathbb{G}l_3(\mathbb{C}))$. Thus

$$\#C(\Gamma) \cdot \omega(\Gamma \diagdown G) = 4\pi^3 \cdot \lambda(\Gamma \diagdown \mathbb{B})$$

Together with (40) we get for all our unimodular ball lattices Γ

$$\#C(\Gamma) \cdot \omega^{(d)}(\Gamma \diagdown G) = \frac{32\delta\pi^3}{|D|^{5/2}} \lambda(\Gamma \diagdown \mathbb{B}).$$

The Lemma 5.6 below compares the G-invariant volume forms λ and γ_2, the Euler form of the Bergmann metric on $\mathbb{B}$, see (31). The corresponding substitution yields

$$\#C(\Gamma) \cdot \omega^{(d)}(\Gamma \diagdown G) = \frac{16\delta\pi^5}{3|D|^{5/2}} \gamma_2(\Gamma \diagdown \mathbb{B}).$$

The left hand-side has been determined in Theorem 5.4 for all special Picard modular groups $\Gamma = \Gamma^{(d)}$. It turns out that

$$\#C(\Gamma^{(d)}) \cdot \delta_D \zeta(2) L(3, \chi) = \frac{16\delta_D\pi^5}{3|D|^{5/2}} \gamma_2(\Gamma^{(d)} \diagdown \mathbb{B})$$

Remember that we used the notation vol_{EB} for the γ_2-volume of a fundamental domain and that the effective acting projective groups of Γ_K and $\Gamma^{(d)}$ are the same. Using also $\zeta(2) = \frac{\pi^2}{6}$, see e.g. [Ser], II, §4, one gets alltogether

Theorem 5.5 *([Ho97], App. of V).*

$$\begin{aligned} vol_{EB}(\Gamma_K) &= \tfrac{3}{16\pi^5} \#C(\Gamma^{(d)}) \cdot |D|^{5/2} \zeta(2) L(3, \chi_D) \\ &= \tfrac{\varepsilon_D \cdot |D|^{5/2}}{32\pi^3} L(3, \chi_D), \end{aligned}$$

with

$$\varepsilon_D = \#C(\Gamma^{(d)}) = \begin{cases} 3, & if\ D = -3\ (K\, is\, the\, field\, of\, Eisenstein\, numbers) \\ 1, & else \end{cases}$$

It remains to verify the following

Lemma 5.6 $\lambda = \frac{\pi^2}{6}\gamma_2.$

Proof (idea). Let $\theta = h^{-1}\partial h$ be the connection of a hermitian vector bundle (V,h) on a complex manifold and $\Theta = d\theta + \theta \wedge \theta$ its curvature tensor. Both are explicitly well-understood for the universal hermitian bundles $(\mathfrak{U}, h)$ on Grassmann varieties, see [Wel], III. Working with canonical coordinates at a canonical origin O, the curvature matrix there can be written as $\Theta(O) = {}^t d\bar{Z} \wedge dZ$. Especially, on the projective space $\mathbb{P}^n = Grass(1, n+1)$ one gets $\Theta(O) = (d\bar{z}_j \wedge dz_k)$ at the natural origin O of $\mathbb{A}^n(\mathbb{C}) \subset \mathbb{P}^n$. The Chern forms γ_i are determined by

$$1 + \gamma_1 + \gamma_2 + ... = det(E - \frac{1}{2\pi i}(\Theta_{jk}))$$

with corresponding unit matrix E. Therefore the first Chern form γ_1 is nothing but $\frac{i}{2\pi}$ times the trace of the curvature matrix. For instance, for $\mathbb{P}^n$ one gets

$$\gamma_1(O) = -\frac{i}{2\pi}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

The canonical and the universal bundle are related by $\mathfrak{K} \cong \mathfrak{U}^{n+1}$. The corresponding Chern forms are denoted by $\check{\gamma}_i$. Especially for the projective plane one gets

$$\begin{aligned} \check{\gamma}_1(O) &= -\tfrac{3i}{2\pi}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), \\ \check{\gamma}_1^2(O) &= -\tfrac{9}{2\pi} dz_1 d\bar{z}_1 dz_2 d\bar{z}_2. \end{aligned}$$

Up to sign the first Chern form of the canonical bundle and of the tangent bundle on $\mathbb{P}^2$ coincide. Using the relation $c_1^2 = 3c_2$ on $\mathbb{P}^2$ we get

$$\check{\gamma}_2(O) = -\frac{3}{2\pi} dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 = \gamma_2(O) \tag{41}$$

as second Chern form of the tangent bundle at O. At the same time (see e.g. [Ho97], end of section IV.3) this is the Euler-Chern form γ_2 of the Bergmann metric (of tangent bundle) on the two-ball $\mathbb{B}$ at O, we look for. Using the normalization of the volume form λ coming from normalization of κ and $\omega^{(1)} = \kappa \wedge \lambda$ and following also exercise 10.7 in [K-N], vol. 2, XI, §10, one gets

$$\lambda(O) = -\frac{1}{4} dz_1 d\bar{z}_1 dz_2 d\bar{z}_2.$$

Now compare with (41) and remember that both volume forms λ and γ_2 are equivariant.

□

6 Functional transfer

Assume that (m,n) is an admissible pair of natural numbers in the sense of the Main Theorem, see end of section 1. In our geometric dimension formula of Theorem 3.10

$$dim[\Gamma_K(m), n]_0 = [\bar{\Gamma}_K : \Gamma_K(m)] \left\{ \left(3\binom{n}{2} + \frac{1}{3} \right) vol_{EB}(\Gamma_K) - \frac{h(K) \cdot \delta_K}{12 C_K m^2} \right\}$$

we can substitute now, using the global index formula 4.10,

$$\begin{array}{rclcl} [\bar{\Gamma}_K : \Gamma_K(m)] & = & [\mathbb{P}\Gamma^{(d)} : \Gamma^{(d)}(m)] & = & \frac{[\Gamma^{(d)}:\Gamma^{(d)}(m)]}{\#C(\Gamma^{(d)})} \\ & = & [\Gamma^{(d)} : \Gamma^{(d)}(m)]/\varepsilon_D & = & \zeta^{(m)}(2)^{-1} L^{(m)}(3, \chi_D)^{-1} m^8 / \delta_m \varepsilon_D, \end{array}$$

but also the Euler-Bergmann volume expression of Theorem 5.5

$$vol_{EB}(\Gamma_K) = \frac{\varepsilon_D \cdot |D|^{5/2}}{32\pi^3} L(3, \chi_D).$$

We arrive at the arithmetic dimension formula

$$dim[\Gamma_K(m), n]_0 = \tag{42}$$
$$\frac{1}{\delta_m} \zeta^m(2)^{-1} L^m(3, \chi_D)^{-1} \left\{ \left(3\binom{n}{2} + \frac{1}{3} \right) \frac{|D|^{5/2}}{32\pi^3} L(3, \chi_D) - \frac{h(K) \cdot \delta_D}{12 \varepsilon_D C_K m^2} \right\} m^8.$$

We have to transfer the L-value and class number part inside of the braces to higher Bernoulli numbers. This will be done by means of the functional equation of L-series and some related classical relations. The corresponding formulas can be found in modern textbooks. We refer to [I-R], [B-S], [Lan].

Functional Equation 6.1 *for Dirichlet characters χ with Fuehrer N (smallest natural number such that χ factorizes through $\mathbb{Z}/N\mathbb{Z}$) :*

$$L(s, \chi) = L(1 - s, \bar{\chi}) (\frac{2\pi}{N})^s \frac{S(\chi)}{\Gamma(s)} \frac{e^{\pi i s/2} - \chi(-1) e^{-\pi i s/2}}{e^{\pi i s} - e^{-\pi i s}}$$

Here $\bar{\chi}$ denotes the complex conjugate character and

$$S(\chi) = \sum_{a=1}^{N} \chi(a) e^{2\pi i a/N} \quad (Gauss\ sum).$$

For $\chi = \chi_D$ it is clear that $\bar{\chi} = \chi$. Moreover, it is known that $N = |D|$, $\chi(-1) = -1, S(\chi) = i\sqrt{|D|}$ and

$$L(1 - n, \chi) = -\frac{B_{n,\chi}}{n} \qquad for \quad n \geqslant 1 \tag{43}$$

(see [Lan], XIV, Th. 2.3). So one calculates

$$L(3, \chi_D) = \frac{2i\pi^3}{|D|^3} S(\chi_D) L(1 - 3, \chi) = \frac{2\pi^3}{3|D|^{5/2}} B_{3,\chi}. \tag{44}$$

Furthermore, we dispose on the well-known class number formula

$$h(K) = \frac{C_K}{2\pi}\sqrt{|D|}\cdot L(1,\chi_D), \qquad (C_K = \#\{unit\,roots\,of\,K\})$$

which transfers by means of 6.1 and (43) to

$$h(K) = -\frac{C_K}{4}B_{1,\chi}. \tag{45}$$

With (44) and (45) the braces part of the starting formula of this section becomes

$$\left\{\tfrac{3\binom{n}{2}+\frac{1}{3}}{48}B_{3,\chi} + \tfrac{B_{1,\chi}\cdot\delta_D}{48\varepsilon_D m^2}\right\}m^8 =$$

$$\tfrac{1}{288}\left\{B_{3,\chi}(9n^2-9n+2)m^8 + \tfrac{6\delta_D}{\varepsilon_D}B_{1,\chi}m^6\right\}.$$

Thus we get the final dimension formula

$$dim[\Gamma_K(m),n]_0 =$$

$$\tfrac{1}{288\delta_m}\zeta^{(m)}(2)^{-1}L^{(m)}(3,\chi_D)^{-1}\left[B_{3,\chi}(9n^2-9n+2)m^8 + \tfrac{6\delta_D}{\varepsilon_D}B_{1,\chi}m^6\right]$$

as stated in the Main Theorem in section 1.

$\square$

Using (43) backwards one gets the L-value formula

$$dim[\Gamma_K(m),n]_0 =$$

$$-\tfrac{1}{96\delta_m}\zeta^{(m)}(2)^{-1}L^{(m)}(3,\chi)^{-1}\left[L(-2,\chi)(9n^2-9n+2)m^8 + \tfrac{2\delta}{\varepsilon}L(0,\chi)m^6\right]$$

and by (12) it can be written in pure zeta value terms:

$$dim[\Gamma_K(m),n]_0 =$$

$$-\tfrac{\zeta^{(m)}(3)}{96\delta_m\zeta^{(m)}(2)\zeta_K^{(m)}(3)}\left[\tfrac{\zeta_K(-2)}{\zeta(-2)}(9n^2-9n+2)m^8 + \tfrac{2\delta\zeta_K(0)}{\varepsilon\zeta(0)}m^6\right]$$

$\square$

By the way of proof we are also able to express the Euler-Bergmann volume of a Γ_K-fundamental domain explicitly as rational multiple of the third Bernoulli number combining the formulas of Theorem 5.5 and of the functional equation 6.1. One gets

$$vol_{EB}(\Gamma_K) = \frac{\varepsilon}{24}B_{3,\chi} = -\frac{\varepsilon}{8}L(-2,\chi)$$

$\square$

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