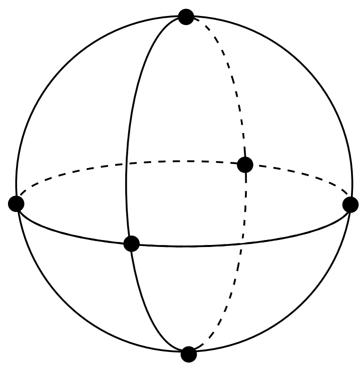
An Octahedral Galois-Reflection Tower of Picard Modular Congruence Subgroups

R.-P. Holzapfel, Humboldt-Univ. Berlin,
M. Petkova, ETH Zürich

09. 09. 09

Abstract

Between tradition (Hilbert's 12-th Problem) and actual challenges (coding theory) we attack infinite two-dimensional Galois theory. From a number theoretic point of view we work over $\mathbb{Q}(x)$. Geometrically, one has to do with towers of Shimura surfaces and Shimura curves on them. We construct and investigate a tower of rational Picard modular surfaces along a Galois group isomorphic to the (double) octahedron group and of their (orbitally) uniformizing arithmetic groups acting on the complex 2-dimensional unit ball $\mathbb{B}$.



Contents

1 Introduction

The main results are dedicated to a natural congruence subgroup $\Gamma(2)$ of the full Picard modular group Γ of Gauß numbers. From the number theoretic side it is interesting, that this infinite group is finitely generated by special elements of order two. More precisely we can choose as generator system a (finite) set of reflections. In number theory such elements are comparable with "inertia elements" generating inertia groups of a Galois covering. The proof is based on a strong geometric result: We need the fine classification of the (Baily-Borel compactified) quotient surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$. It turns out, that it is a nice blowing up of the projective plane at triple and quadruple points of the very classical harmonic configuration of lines. We mention that this is the first precise classification of a Picard modular surface of a natural congruence subgroup. Along an easy correspondence the harmonic configuration changes to the globe configuration with equator and two meridians meeting each other in six (elliptic) cusp singularities, see the picture at the end of section 6. On this way we visualized the octahedral action of the factor Galois group $\Gamma/\Gamma(2)$. In Galois towers between the surfaces $\widehat{\Gamma(2)\backslash\mathbb{B}}$ and $\widehat{\Gamma\backslash\mathbb{B}}$ we discover a classical orbital ball quotient surface of the PTDM-list (Picard, Terada, Mostow, Deligne), which was also published in Hirzebruch's (and other's) monograph [BHH]. On the one hand we need this del Pezzo surface for proving our results. On the other hand we found the arithmetic group uniformizing this orbital surface. It is a Picard modular congruence subgroup. The precise description is important for the further work with the Picard modular forms of this group found by H. Shiga and his team, see [KS], [Mat]. In the same manner we find also the uniformizing arithmetic group of the first surface (with a new line configuration) sitting in the infinite Galois-tower of orbital (plane) ball quotient surfaces constructed by Uludag [Ul]. It allows to work with algebraic equations for Shimura curves, which are important in coding theory.

2 Picard Modular Varieties and Galois-Reflection Towers

Let V be the vector space $\mathbb{C}^{n+1}$ endowed with hermitian metric $< .,. >$ of signature $(n, 1)$. Explicitly we will work with the diagonal representation

$$\begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot \\ & & \cdot & & \\ & & & \cdot & \\ \cdot & \cdot & & 1 & \\ \cdot & \cdot & & & -1 \end{pmatrix}.$$

For $v \in V$ we call $< v, v >$ the *norm* of v. The space of all vectors with negative (positive) norms is denoted by V^- (V^+). The image $\mathbb{P}V^-$ of V^- in $\mathbb{P}V = \mathbb{P}^n$ is the complex n-dimensional *unit ball* denoted by $\mathbb{B}^n$. The unitary group $\mathbb{U}((n, 1), \mathbb{C})$ acts transitively on it.

Now let K be an imaginary quadratic number field, $\mathcal{O}_K$ its ring of integers.

Definition 2.1. The arithmetic subgroup $\Gamma_K = \mathbb{U}((n, 1), \mathcal{O}_K)$ is called the *full Picard modular group (of K, of dimension n).* Each subgroup Γ of $\mathbb{U}((n, 1), \mathbb{C})$ commensurable with Γ_K is called *Picard modular group.*

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$, closed under complex conjugation. Then, over the finite factor ring $A = \mathcal{O}_K/\mathfrak{a}$, the finite unitary group $\Gamma_A = \mathbb{U}((n, 1), \mathcal{O}_K/\mathfrak{a})$ is well-defined together with the reduction (group) morphism $\rho_{\mathfrak{a}} : \Gamma_K \longrightarrow \Gamma_A$. The kernel of $\rho_{\mathfrak{a}}$ is denoted by $\Gamma_K(\mathfrak{a})$.

Definitions 2.2. This group is called the *congruence subgroup of the ideal $\mathfrak{a}$ in Γ_K.* A subgroup Γ of Γ_K is called a *(Picard modular) congruence subgroup,* iff it contains a congruence subgroup $\Gamma_K(\mathfrak{a})$. If $\mathfrak{a}$ is a principal ideal (α), then we get a *principal congruence subgroup* $\Gamma_K(\alpha)$. For any natural number a we call $\Gamma_K(a)$ a *natural congruence subgroup of* Γ_K. Intersecting the above subgroups with a given Picard modular group Γ, we get *(principal, natural) congruence subgroups* $\Gamma(\mathfrak{a})$*,* $\Gamma(\alpha)$*,* $\Gamma(a)$ *of* Γ*.*

Remark 2.3. *The full Picard modular group appears also as* $\Gamma_K(1)$ *now. More generally, we have to identify the groups* $\Gamma(1)$ *and* Γ*.*

The ball quotients $\Gamma\backslash\mathbb{B}^n$ are quasiprojective. They have a minimal algebraic compactification $\widehat{\Gamma\backslash\mathbb{B}^n}$ constructed by Baily and Borel in [BB]. The authors proved that these compactifications are normal projective complex varieties. We call them *Baily-Borel compactifications.* In the Picard modular cases the Baily-Borel compactifications consist of finitely many points, called *cusp singularities* or *cusp points.* It may happen that such point is a regular one.

The Picard modular groups of a fixed imaginary quadratic number field K act also on the hermitian $\mathcal{O}_K$-lattice $\Lambda = (\mathcal{O}_K)^{n+1} \subset V$.

Definition 2.4. Let $c \in \Lambda$ be a primitive positive vector and $c^\perp$ its orthogonal complement in V. It is a hermitian subspace of V of signature $(n - 1, 1)$. The intersection

$$\mathbb{D}_c := \mathbb{P}c^\perp \cap \mathbb{B}^n$$

is isomorphic to $\mathbb{B}^{n-1}$. We call it an *arithmetic hyperball of* $\mathbb{B}^n$. Arithmetic hyperballs of $\mathbb{B}^2$ are called *arithmetic subdiscs.*

Take all elements of Γ acting on $\mathbb{D}_c$:

$$\Gamma_c := \{\gamma \in \Gamma;\ \gamma(\mathbb{D}_c) = \mathbb{D}_c\}$$

This is an arithmetic group. The image $p(\mathbb{D}_c)$ along the quotient projection $p:\ \mathbb{B}^n \longrightarrow \Gamma\backslash\mathbb{B}^n$ is an algebraic subvariety H_c of $\Gamma\backslash\mathbb{B}^n$ of codimension 1.

Definition 2.5. The algebraic subvarieties H_c are called *arithmetic hypersurfaces* of the Picard modular variety $\Gamma\backslash\mathbb{B}^n$. The same notion is used for the compactifications. The *norm* $n(H_c)$ *of* H_c is defined as $n(c)$.

The analytic closure of H_c on the Baily-Borel compactification $\widehat{\Gamma\backslash\mathbb{B}^n}$ is denoted by $\hat{H}_c$. Around general points the quotient variety $\Gamma_c\backslash\mathbb{D}_c$ coincides with $H_c = \Gamma\backslash\mathbb{D}_c$. More precisely, we have normalizations

$$\begin{array}{ll} \Gamma_c\backslash\mathbb{D}_c & \longrightarrow \Gamma\backslash\mathbb{D}_c = H_c \\ \widehat{\Gamma_c\backslash\mathbb{D}_c} & \longrightarrow \widehat{\Gamma\backslash\mathbb{D}_c} = \widehat{H}_c \end{array}$$

For the proof we refer to [BSA] IV.4, where it is given for the surface case $n = 2$. It is easily seen, that it works also in general dimensions n.

Definition 2.6. A non-trivial element of finite order $\sigma \in \mathbb{U}((n,1),\mathbb{C})$ is called a *reflection* iff there is a positive vector $c \in V$ such that $V_c := c^\perp$ is the eigenspace of σ of eigenvalue 1. If σ belongs to the Picard modular group Γ, then we call it a Γ-*reflection.*

Remark 2.7. Some authors call them "quasi reflections". Only in the order 2 cases they omit "quasi".

Looking at the characteristic polynomial of σ we see that the eigenvector c belongs to K^{n+1} in the Picard case in 2.6. We can and will choose c primitive in $\Lambda = \mathcal{O}^{n+1}$. Then it is clear that σ acts identically on the arithmetic hyperball

$$\mathbb{D}_\sigma := \mathbb{D}_c = \mathbb{P}V_c \cap \mathbb{B}^n$$

of $\mathbb{B}^n$. We call such $\mathbb{D}_c$ a Γ-*reflection subball* of $\mathbb{B}^n$, or a Γ-*reflection disc* in the surface case $n = 2$.

Definition 2.8. The hypersurface H_c of the primitive eigenvector $c = c(\sigma)$ of a Γ-reflection σ is called a Γ-*reflection hypersurface.* In the two-dimensional case we call it Γ-*reflection curve.*

Fact. The irreducible hypersurface components of the branch locus of the quotient projection $p:\ \mathbb{B}^n \to \Gamma\backslash\mathbb{B}^n$ are precisely the Γ-reflection hypersurfaces.

Let Γ' be a normal subgroup of finite index of the Picard modular group Γ. We do not change notations, if such lattices doesn't act effectively on $\mathbb{B}^n$. We keep the effectivization (= projectivization) in mind. We do the same for the Galois group $G := \Gamma/\Gamma'$ of the covering

$$\Gamma'\backslash\mathbb{B} \longrightarrow \Gamma\backslash\mathbb{B} \tag{1}$$

Definition 2.9. This finite morphism (1) is called a *Galois-Reflection covering* iff G is generated by Γ'-cosets of some Γ-reflections. We call G in this case a *Galois-Reflection group.*

In pure ball lattice terms this means that

$$\Gamma = < \Gamma', \sigma_1, ..., \sigma_k > \tag{2}$$

for suitable reflections σ_i, i=1,...,k.

We want to prove

Proposition 2.10. *If $\Gamma\backslash\mathbb{B}$ is simply-connected and smooth, then (1) is a Galois-Reflection covering for each normal sublattice Γ' of Γ.*

This can be easily deduced from the following

Theorem 2.11. *If $\Gamma\backslash\mathbb{B}$ is simply-connected, then Γ is generated by finitely many elements of finite order (*torsion elements*). If, moreover, the Picard modular variety $\Gamma\backslash\mathbb{B}$ is smooth, then Γ is generated by finitely many reflections.*

For the proof we need first the following

Theorem 2.12. *(Armstrong, [Ar] 1968). Let G be a discrete group of homeomorphisms acting on a path-wise connected, simply-connected, locally compact metric space X and H the (normal) subgroup generated by the stabilizer groups G_x of all points $x \in X$. Then G/H is the fundamental group of the (topological) quotient space X/G.*

Proof of Theorem 2.11. We substitute $\Gamma, \mathbb{B}, Tor\,\Gamma$ for G, X, H in Armstrong's Theorem. It follows that $\Gamma/Tor\,\Gamma$ is the fundamental group of the quotient variety $\Gamma\backslash\mathbb{B}$. If it is 1, then $\Gamma/Tor\,\Gamma = 1$. This means that Γ is generated by all its torsion elements. These elements are finite order. Now we remember that each arithmetic group is finitely generated, by a theorem of Borel [Bo]. All generators are products of finitely many torsion elements. So we can generate Γ by finitely many torsion elements. This proves the first part of Theorem 2.11. For the second part, we look at the stabilizers Γ_x, $x \in \mathbb{B}^n$. These are finite groups. Claude Chevalley proved in [Ch] that the image point $p(x) \in \Gamma\backslash\mathbb{B}^n$ is regular, if and only if Γ_x is generated by reflections. On the other hand, each torsion element of Γ has a fixed point $x \in \mathbb{B}^n$. Therefore Tor Γ is generated by reflections, if $\Gamma\backslash\mathbb{B}^n$ is smooth. So the second part of Theorem 2.11 follows now from the first.

$\square$

Definition 2.13. Let

$$\Gamma_N \lhd .... \lhd \Gamma_{i+1} \lhd \Gamma_i \lhd .... \lhd \Gamma_1 \subseteq \Gamma \tag{3}$$

be a normal series of subgroups of finite index of the Picard modular group Γ. We call it a Γ-*reflection series,* if Γ_i is generated by Γ_{i+1} and finitely many reflections for each in (3) occuring pair $(i+1, i)$. The corresponding Galois tower of finite Galois coverings

$$\Gamma_N\backslash\mathbb{B}^n \to .... \to \Gamma_{i+1}\backslash\mathbb{B}^n \longrightarrow \Gamma_i\backslash\mathbb{B}^n \to .... \to \Gamma_1\backslash\mathbb{B}^n, \tag{4}$$

with the normal factors Γ_i/Γ_{i+1} as Galois groups, is then called a *Galois-Reflection tower* (attached to the normal series (3)).

In this case each map of the sequence is a Galois-Reflection covering with the normal factors Γ_i/Γ_{i+1} as Galois groups. The extension of the definition to (Baily-Borel or other) compactificatons should be clear. It is left to the reader.

Theorem 2.14. *If all members, except for $\Gamma_N\backslash\mathbb{B}^n$, in the covering tower* (4) *attached to* (3) *are simply-connected smooth varieties, then it is a Galois-Reflection tower.*

Proof. We have to show that each covering of the tower has the Galois-Reflection property. We refer to Proposition 2.10.

$\square$

Moreover, we call an infinite tower

$$\mathbb{B}^n \to .... \to \Gamma_{i+1}\backslash\mathbb{B}^n \longrightarrow \Gamma_i\backslash\mathbb{B}^n \to .... \to \Gamma_1\backslash\mathbb{B}^n, \tag{5}$$

a *Galois-Reflection tower*, if all occuring ball lattices Γ_i are generated by reflections.

Example 2.15. Uludag constructed in [Ul] an infinite tower

$$.... \to \mathbb{P}^2 \to \mathbb{P}^2 \to .... \to \mathbb{P}^2 \to \mathbb{P}^2 \tag{6}$$

of ball quotient planes $\mathbb{P}^2 = \widehat{\Gamma_i\backslash\mathbb{B}^2}$. It is not clear until now that the Γ_i's can be choosen as infinite normal series. We know only the existence of the ball lattices Γ_i, $i = 1, 2, 3, ...$, and that the successive coverings in (6) have the Klein's 4-group $Z_2 \times Z_2$ as Galois group. The last member is the orbital $\mathbb{P}^2 = \Gamma(\widehat{1-i)\backslash}\mathbb{B}$ with "Apollonius divisor", supported by a quadric and three tangents as orbital branch divisor of the ball covering. We refer to [HPV] or [BMG], first appearence of the Appolonius picture in [SY]. In [HPV], [BMG] we proved that the congruence subgroup $\Gamma(1-i)$ is the uniformizing ball lattice, with the *full Picard-Gauß lattice* $\Gamma = \Gamma(1) := \mathbb{SU}((2,1), \mathbb{Z}[i])$. By Theorem 2.11 it is true that all ball lattices Γ_i in this example are generated by reflections.

We consider a Γ-reflection covering as in 2.9. We want to construct a set of reflections whose Γ'-cosets generate the Galois group $G = \Gamma/\Gamma'$. For this purpose we consider all *K-arithmetic subballs* $\mathbb{D}$ of $\mathbb{B}^n$. By definition, these are the arithmetic subballs for our fixed imaginary-quadratic field K, see Def. 2.4. Such $\mathbb{D}$ is a Γ-reflection if and only if the finite cyclic group

$$Z_\Gamma(\mathbb{D}) = \{\sigma \in \Gamma; \sigma|_{\mathbb{D}} = id_{\mathbb{D}}\},$$

called *centralizer group of* Γ *at* $\mathbb{D}$, is not trivial. In this case the image H of $\mathbb{D}$ on $\Gamma\backslash\mathbb{B}^n$ belongs to the branch divisor, and the ramification index there coincides with $\#Z_\Gamma(\mathbb{D})$.

Now let Γ' be a subgroup of finite index of Γ. Then we dispose on a commutative diagramm

$$\begin{tikzcd}
\mathbb{B}^n \arrow[r, "="] \arrow[d, "p'"'] & \mathbb{B}^n \arrow[d, "p"] \\
\Gamma'\backslash\mathbb{B}^n \arrow[r, "f"'] & \Gamma\backslash\mathbb{B}^n
\end{tikzcd}$$

of analytic maps, where f is finite, and the verticals are locally finite. With $H' := p'(\mathbb{D})$, it restricts to

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "="] \arrow[d] & \mathbb{D} \arrow[d] \\
H' \arrow[r] & H
\end{tikzcd}$$

The covering f is branched along H, if and only if $Z' := Z_{\Gamma'}(\mathbb{D})$ is a honest (cyclic) subgroup of Z. The ramification order of f at H' is equal to the index $[Z : Z']$.

Now we see a practical way to get generating reflection elements σ_i of the Galois group f, if it is a Galois-reflection covering as described in (2). We have to know the components H of the branch divisor of f. Then we must find a reflection subball $\mathbb{D} = \mathbb{D}_\sigma \subset \mathbb{B}^n$ projecting onto H along p as above. Then σ is one of the generating σ_i you look for. Now we change to the next branch divisor component to find the next of the generating reflections. It is helpful to know the order of the Galois group G of f. Then one can compare group orders of $G = \Gamma/\Gamma'$ (assumed to be known) and of $G'_i := \Gamma/ < \Gamma', \sigma_1, ..., \sigma_i >$ using all the reflections already found. One has to stop the procedure, if both group orders coincide. If $\Gamma' = \Gamma(\mathfrak{a})$ is a congruence subgroup of Γ, then we calculate the orders of G'_i modulo the ideal $\mathfrak{a}$ by a computer program, e.g. MAPLE.

3 The Level 2 Reflection Tower

From now on we restrict ourselves to the second (complex) dimension $n = 2$. We write $\mathbb{B}$ for the complex 2-dimensional unit ball $\mathbb{B}^2$. Moreover we concentrate our attention to the Gauß number field $K = \mathbb{Q}(i)$.

A) **The Galois-Reflection covering of** $\Gamma(1-i) \subset \Gamma$

For $\Gamma = \mathbb{SU}((2,1), \mathbb{Z}[i])$ we want to construct reflection generators of

(7) $$\Gamma(1)/\Gamma(1-i) \subseteq \mathbb{O}(3, \mathbb{F}_2) \cong S_3,$$

where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ denotes the primitive field of characteristic 2. We take two primitive elements of $\Lambda = \mathbb{Z}[i]^3$ of norm 2, namely

$$a = (1+i, 1, 1),\ b = (1, i, 0).$$

We look for a reflection with eigenvector a of eigenvalue -1. It sends each $z \in V = \mathbb{C}^3$ to $z - < z, a > a$. For explicit Γ-representations we refer to the appendix section 7. It turns out that both reflections generate a subgroup Σ_3 of $\mathbb{SU}((2,1), \mathbb{Z}[i])$ isomorphic to S_3. Especially, the inclusion in (7) is an equality.

It is easy to find $\mathbb{C}$-bases of the orthogonal complements $a^\perp$ or $b^\perp$ in V, respectively. Via projectivization we get explicitly the Γ-reflection discs

$$\mathbb{D}_a = \mathbb{P}a^\perp \cap \mathbb{B}\ ,\ \mathbb{D}_b = \mathbb{P}b^\perp \cap \mathbb{B}.$$

These linear discs go through $(1 : 0 : 1-i)$ or $(0 : 0 : 1)$ in $\mathbb{B} \subset \mathbb{P}^2$, respectively, and intersect each other in $P = (i : 1 : 2+i)$. This is the common fixed point of Σ_3. Conversely, Σ_3 is the isotropy group of Γ at P.

The Baily-Borel compactification $\widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is $\mathbb{P}^2$. It has been determined in [HPV], [BMG]. More precisely, this *orbital* quotient surface is a pair $(\mathbb{P}^2; 4C_0 + ... + 4C_3)$, where C_0 is an S_3-invariant quadric, and C_1, C_2, C_3 are three of its tangent lines. The three (Baily-Borel) compactifying cusp points are the touch points of the tangents and the quadric. Look at picture 5 in the later section 5. The coefficients 4 denote the branch indices of each curve C_i along the locally finite quotient covering $\mathbb{B} \to \mathbb{P}^2 \setminus \{3 \text{ points}\}$. Especially, $\Gamma(1-i)\backslash\mathbb{B}$ is smooth. From Theorem 2.11 it follows now that $\Gamma(1-i)$ is generated by finitely many reflections. Together with 7 and the above reflection representation of S_3-generators, we see altogether that Γ itself is generated by finitely many reflections. This doesn't follow directly from Theorem 2.11, because $\widehat{\Gamma\backslash\mathbb{B}}$ has a surface singularity, namely the image point of $P = (i : 1 : 2+i) \in \mathbb{B}$ on the quotient surface. This is the only singularity there, see [BSA], ch. V, 5.3 (especially, point P_2 in Figure 5.3.7). This shows that surface smoothness is not necessary for the existence of finitely many reflections generating the corresponding ball lattice.

B) **The Galois-Reflection covering of** $\Gamma(2) \lhd \Gamma(1-i)$

We continue the above Γ-example with the consideration of the natural congruence subgroup $\Gamma(2)$. In [HPV], Theorem 7.2 we proved that all torsion elements of $\Gamma(2)$ have order 2. Moreover, they all are squares of $\Gamma(1-i)$-elements of order 4. Each isotropy group of $\Gamma(1-i)$- elliptic points is generated by two $\Gamma(1-i)$-reflections of order 4. Each non-reflection torsion element $\tau \in \Gamma(1-i)$ of order 4 fixes a(n *elliptic*) point, say $Q \in \mathbb{B}$. It turns out that τ is the product of two $\Gamma(1-i)_Q$-generating reflections. So we have

$$\Gamma(1-i)_Q \cong Z_4 \times Z_4, \text{ with } Z_d := (\mathbb{Z}/d\mathbb{Z}, +).$$

Conversely, all squares of order 4 elements belong to $\Gamma(2)$. In [HPV], Prop. 8.3, we determined the index as $[\Gamma(1-i) : \Gamma(2)] = 8$. The diagonal reflections

$$\sigma_1 := diag(i, 1, 1), \; \sigma_2 := diag(1, i, 1)$$

have the *coordinate reflection discs* $\mathbb{D}_2 : z_2 = 0$ or $\mathbb{D}_1 : z_1 = 0$, respectively. They generate the isotropy group $\Gamma(1-i)_O$, O the zero coordinate point. Reduction mod $(1-i)$ yields the exact sequence

$$1 \longrightarrow Z_2 \times Z_2 = \Gamma(2)_O \longrightarrow Z_4 \times Z_4 = \Gamma(1-i)_O \longrightarrow \Gamma(1-i)/\Gamma(2).$$

The image group on the right has the same structure as the kernel, namely

$$K4 := Z_2 \times Z_2 \subset \Gamma(1-i)/\Gamma(2) \text{ (Klein's Vierer-Gruppe).}$$

Observe that the norm 1 vectors, whose ortho-complements determine the coordinate reflection discs, are $\mathfrak{n}_1 = (0,1,0)$ or $\mathfrak{n}_2 = (1,0,0)$, respectively. We determine a third reflection σ_0, which is incongruent mod 2 to the elements of $< \sigma_1, \sigma_2 >$. For this purpose we take the norm 1 vector $\mathfrak{n}_0 := (1,1,1)$. Then σ_0 is the (order 4) reflection corresponding

$$V = \mathbb{C}^3 \ni v \mapsto v \,-\, (1-i) < v, \mathfrak{n}_0 > \mathfrak{n}_0. \tag{8}$$

For its Γ-representation we refer again to the appendix section 7. The orthogonal reflection disc $\mathbb{D}_0 \subset \mathbb{B}$ has the linear equation $z_1 + z_2 = 1$.

The disc $\mathbb{D}_0$ projects along the quotient projection $\mathbb{B} \to \mathbb{P}^2$ to the quadric C_0, and $\mathbb{D}_1, \mathbb{D}_2$ to the tangents C_1, C_2 of the Apollonius configuration. For more details we refer to [HPV], [BMG].

The reflections σ_0, σ_1, σ_2 generate mod 2 a subgroup of order 8 in $\Gamma(1-i)/\Gamma(2)$, which has the same order. Therefore we found the Galois group together with Galois-Reflection generators of the covering $\Gamma(2)\backslash\mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B}$:

$$Z_2 \times K4 = < \bar{\sigma}_0, \bar{\sigma}_1, \bar{\sigma}_2 > = \Gamma(1-i)/\Gamma(2). \tag{9}$$

In the next section we look for fine Kodaira classification of $\widehat{\Gamma(2)\backslash\mathbb{B}}$. This will be managed step by step along Galois- Reflection coverings/towers along the ball lattices in the following commutative diagram of inclusions:

(10)
$$\begin{tikzcd} \Gamma(2) \arrow[r] \arrow[d] & <\Gamma(2),\sigma_1,\sigma_2> =: \Gamma'' \arrow[d] \\ \Gamma' := <\Gamma(2),\sigma_0> \arrow[r] & \Gamma(1-i) \end{tikzcd}$$

It reduces mod $\Gamma(2)$ to the Galois group diagram of finite Galois coverings (on the right):

(11)
$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & K4 \arrow[d] \\ Z_2 \arrow[r] & Z_2 \times K4 \end{tikzcd} \quad , \quad \begin{tikzcd} \widehat{\Gamma(2)\backslash\mathbb{B}} \arrow[r] \arrow[d] & \widehat{\Gamma''\backslash\mathbb{B}} \arrow[d] \\ \widehat{\Gamma'\backslash\mathbb{B}} \arrow[r] & \widehat{\Gamma(1-i)\backslash\mathbb{B}} \end{tikzcd}$$

C) **The Galois-Reflection tower of** $\Gamma(2) \subset \Gamma$

Composing A) and B) we have the normal series

$$\Gamma(2) \lhd \Gamma'' \lhd \Gamma(1-i) \lhd \Gamma(1) = \Gamma = \mathbb{SU}((2,1),\mathbb{Z}[i])$$

We can and will also Γ'' substitute by Γ'.

Proposition 3.1.

i) *The full Picard lattice Γ is generated by finitely many reflections.*

ii) *The quotient morphism $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma\backslash\mathbb{B}}$ is a Galois-Reflection covering.*

iii) *The Galois group $\Gamma/\Gamma(2)$ is isomorphic to $Z_2 \times S_4$, where S_4 is the symmetric group of 4 elements.*

iv) *Altogether we dispose on the normal Galois-Reflection series*

$$\Gamma(2) \lhd \Gamma' \lhd \Gamma(1-i) \lhd \Gamma$$

of the Galois-Reflection (covering) tower

$$\Gamma(2)\backslash\mathbb{B} \longrightarrow \Gamma'\backslash\mathbb{B} \longrightarrow \Gamma(1-i)\backslash\mathbb{B} \longrightarrow \Gamma\backslash\mathbb{B}$$

with normal factors (Galois groups) Z_2, $K4$, S_3, or of compositions:

$$Z_2 \times K4 \cong Gal(\Gamma(2)\backslash\mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B}) \quad , \quad S_4 \cong Gal(\Gamma'\backslash\mathbb{B} \to \Gamma\backslash\mathbb{B}).$$

Proof.

i) We know that $\Gamma(1-i)\backslash\mathbb{B}$ is smooth as open part of $\mathbb{P}^2$. Then, from Theorem 2.11 follows that $\Gamma(1-i)$ is generated by finitely many reflections, say $\rho_1, ..., \rho_k$. With A) we get Γ, if we add (generators of) Σ to $\Gamma(1-i)$. With the notations of A) we receive

$$\Gamma = < \rho_1, ...\rho_k, \sigma_a, \sigma_b > .$$

ii) Abstractly, this follows immediately from i). Explicitly we dispose on the presentation

$$\Gamma/\Gamma(2) = < \bar{\sigma_0}, \bar{\sigma_1}, \bar{\sigma_2}, \bar{\sigma_a}, \bar{\sigma_b} > \tag{12}$$

where $\bar{\sigma}$ denotes the $\Gamma(2)$-coset of σ, and we use the reflections defined in A) and B).

iii) By direct computation using the explicit representations in appendix section 7 one checks first that $\bar{\sigma}_0$ commutes with all the other four generators in (12). Further direct computations yield isomorphic short exact sequences, where $K4$ below denotes the normal subgroup of all products of two disjoint transpositions in the symmetric group S_4.

$$\begin{tikzcd} < \bar{\sigma_1}, \bar{\sigma_2} > \arrow[r] \arrow[d, "||"] & < \bar{\sigma_1}, \bar{\sigma_2}, \bar{\sigma_a}, \bar{\sigma_b} > \arrow[r] \arrow[d, "\sim"] & < \bar{\sigma_a}, \bar{\sigma_b} > \arrow[d, "\sim"] \\ K4 \arrow[r] & S_4 \arrow[r] & S_3 \end{tikzcd} \tag{13}$$

iv) For the S_3-part look back to A), (7) with proven isomorphy. The $Z_2 \times K4$ -part one can find in B), especially (11).

□

For the next corollary we need a further reflection, namely the orthogonal reflection of the norm-1 vector $\mathfrak{n}_3 = (1+i, 0, 1)$. We find the corresponding order-4 reflection σ_3 in a similar manner as σ_0 in B). Its Γ-representation you can find in the appendix section 7 again.

Remark 3.2. The symmetric group S_4 has a well-known representation as motion group $\mathbb{O}$ of the octahedron. With a 3-dimensionally drawn curve configuration in section 6 it will be geometrically visible.

Corollary 3.3.

1) The following two sets coincide:

$$\{\Gamma(1-i) - \textit{reflection discs}\} = \{\mathbb{D}_v;\ v \in \Lambda \textit{ a primitive norm-}1 \textit{ vector}\}.$$

2) The set of $\Gamma(1-i)$*-reflection discs on* $\mathbb{B}$ *coincide with the set of* $\Gamma(2)$*-reflection discs.*

3) Each $\Gamma(2)$*-reflection is a squares of a* $\Gamma(1-i)$ *reflection of order 4.*

4) The reflection disc $\mathbb{D}_0$ *of* σ_0 *projects to the Apollonius quadric* C_0 *along* $p:\ \mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B}$*.*

5) For $i = 1,2,3$ *the reflection discs* $\mathbb{D}_i$ *of* σ_i *project to the 3 Apollonius tangent lines* C_1, C_2, C_3*, respectively, along* p*.*

6) The branch curve of the Galois covering

$$\hat{f}:\ \widehat{\Gamma(2)\backslash\mathbb{B}} \to \Gamma(\widehat{1-i)}\backslash\mathbb{B} = \mathbb{P}^2$$

is the Appollonius curve $C_0+C_1+C_2+C_3$*. The covering has ramification index* 2 *over each component* C_i*.*

For a visualization we refer to picture 5 in section 5 again. The key of proof is the following statement presented in [HPV],[BMG]:

Theorem 3.4. *The Apollonius curve* $C_0 + C_1 + C_2 + C_3$ *is the (Baily-Borel compactified) branch curve of* p*. More precisely,* $4C_0 + 4C_1 + 4C_2 + 4C_3$ *is the orbital branch divisor of* p*. This means that the branch order is* 4 *over all components* C_i*. All reflections in* $\Gamma \setminus \Gamma(1-i)$ *have order* 2*. Each of them is* Γ*-conjugated to one of the three reflections of* Σ_3*.*

□

Proof (of Corollary 3.3).

1) $\subseteq$: If $\mathbb{D}$ is a $\Gamma(1-i)$-reflection, then it belongs, by definition, to the ramification locus of p on $\mathbb{B}$. This means, that its image C belongs to the branch locus. But then, by Theorem 3.4, it is one of the above C_j, $j \in \{1,..,4\}$. It follows that $\mathbb{D} = \mathbb{D}_v$ belongs to the $\Gamma(1-i)$-orbit of the reflection disc $\mathbb{D}_j$ of σ_j. Then the normal vector v of $\mathbb{D}$ belongs to the orbit $\Gamma(1-i)\mathfrak{n}_j$. We conclude that $norm(v) = norm(\mathfrak{n}_j) = 1$.
$\supseteq$: If we start with a reflection disc $\mathbb{D}_{\mathfrak{n}}$ of a norm-1 vector $\mathfrak{n} \in \Lambda$, then we can construct the order-4 reflection $\sigma_{\mathfrak{n}}$ as we did in (8) for σ_0. It belongs to $\Gamma(1-i)$ because $\Gamma \setminus \Gamma(1-i)$ contains only order-2 reflections.

2) $\subseteq$: A $\Gamma(1-i)$-reflection disc $\mathbb{D}$ has a generating reflection σ of order 4. Its square belongs to $\Gamma(2)$ (easy congruence calculation with a Γ-representation). Therefore $\mathbb{D}$ is also a $\Gamma(2)$-reflection disc.
$\supseteq$: Obviously, by inclusion $\Gamma(2) \subset \Gamma(1-i)$.

3) Let s be a $\Gamma(2)$-reflection with reflection disc $\mathbb{D}$. Since it is a $\Gamma(1-i)$-reflection disc, its reflection group has, by the proof of 1), a generating element σ of order 4. Therefore $s = \sigma^2$.

4) The reflection disc $\mathbb{D}_0$ with p-image C_0 has been constructed in [HPV], see also [BMG].

5) The three other order-4 reflection discs $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ are neither $\Gamma(1-i)$-equivalent to $\mathbb{D}_0$ nor to each other, because their ortho-vectors $\mathfrak{n}_i$ are not. You can check it simply with modulo 2 calculations. Therefore their p-images are C_1, C_2, C_3, respectively, for a suitable umeration. Namely, by the Theorem 3.4, there is no other possibility.

6) We omit the cusp points and decompose p in

$$\begin{tikzcd}
\mathbb{B} \arrow[d, "p'"'] \arrow[dr, "p"] & \\
\Gamma(2)\backslash\mathbb{B} \arrow[r, "f"'] & \Gamma(1-i)\backslash\mathbb{B}
\end{tikzcd} .$$

The quotient maps p' and p have the same ramification locus joining all reflection discs of $\Gamma(1-i)$. Let $\mathbb{D}$ be one of them, $C' = p'(\mathbb{D})$, $C = p(\mathbb{D})$. The ramification orders of p' and p at $\mathbb{D}$ coincide with the order of a generating $\Gamma(2)$- or $\Gamma(1-i)$-reflection at $\mathbb{D}$, respectively. The former order is 2, the latter equal to 4; both by 3) and Theorem 3.4, which restricts the maximal $\Gamma(1-i)$-reflection order to 4. Ramification indices v behave multiplicatively along compositions of coverings. Especially, we have

$$4 = v(\mathbb{D} \to C) = v(\mathbb{D} \to C') \cdot v(C' \to C) = 2 \cdot v(C' \to C).$$

Now it is clear that $v(C' \to C) = 2$. This happens iff C belongs to branch locus of p. This branch locus coincides with $C_0 + C_1 + C_2 + C_3$.

The Corollary is proved.

$\square$

4 The Harmonic Model of $\widehat{\Gamma(2)\backslash\mathbb{B}}$

Our next goal is to obtain a fine Kodaira classification of the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$, based on results of the previous two sections and from the works of K. Matsumoto [Mat], T. Riedel [Ri] and M. Uludag [Ul].

In [Mat] and [Ri], Matsumoto and Riedel study a ball quotient surface $\widehat{\Gamma_M\backslash\mathbb{B}}$, where Γ_M is a subgroup of index 2 of $\Gamma(1-i)$ and the degree 2 covering $\widehat{\Gamma_M\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is ramified exactl over the Apollonius' quadric C_0. On the other hand $\Gamma'' =< \Gamma(2), \sigma_1, \sigma_2 >$, Diagram (10), is also an index 2 group of $\Gamma(1-i)$ and $\widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ has C_0 as branch locus, Cor. 3.3. Therefore, according to the Cyclic Cover Theorem, [EPD], the two coverings $\widehat{\Gamma_M\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ and $\widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$, being both of degree 2 with branch locus C_0, are the same, hence $\Gamma_M = \Gamma''$.

The next ball quotient surface we are interesting in is $\widehat{\Gamma_U\backslash\mathbb{B}}$. In [Ul], M. Uludag has constructed an infinite tower of finite coverings of ball quotient surfaces, all of them equal to $\mathbb{P}^2$. This particular surface, which we call *Uludag's surface*, is a part of the tower and is defined as a degree four covering of the Apollonius' $\mathbb{P}^2$, ramified over the three tangent lines C_1, C_2, C_3. We consider again the group $\Gamma' =< \Gamma(2), \sigma_0 >$ of index four in $\Gamma(1-i)$, Diagram (10). By Cor. 3.3, $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is a degree four covering with branch locus C_1, C_2, C_3. According to the Extension Theorem of Grauert and Remmert, [GR], the two coverings $\widehat{G_U\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ and $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$, both of degree four with the same unramified (affine) part and the same branch locus, are equal, wherefrom $G_U = \Gamma'$.

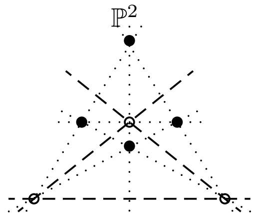
Following results from the previous sections there are two ways to construct $\widehat{\Gamma(2)\backslash\mathbb{B}}$ from $\widehat{\Gamma(1-i)\backslash\mathbb{B}}$: as a degree four covering of $\widehat{\Gamma''\backslash\mathbb{B}}$, or as degree two covering of the surface $\widehat{\Gamma'\backslash\mathbb{B}}$. The two lifts of the Apollonius $\mathbb{P}^2$ are compositions of coverings of degree 8, with corresponding Galois group for the whole covering in each one of the cases $Z_2 \times Z_2 \times Z_2$, and are ramified exactly over the Apollonius configuration. The Galois group $\Gamma(1-i)/\Gamma(2)$ is generated by $\overline{\sigma}_0, \overline{\sigma}_1, \overline{\sigma}_2$. The surface covering $\widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is of degree 2 with Galois group generated by $\overline{\sigma}_0$ and ramified over C_0, and $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma''\backslash\mathbb{B}}$ is of degree 4 with corresponding Galois group generated by $\overline{\sigma}_1, \overline{\sigma}_2$ and ramified over the preimages of the curves C_1, C_2, C_3 on $\widehat{\Gamma''\backslash\mathbb{B}}$. On the other hand the covering $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is of degree 4, ramified over C_1, C_2, C_3, with Galois group generated by $\overline{\sigma}_1, \overline{\sigma}_2$, and that of $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma'\backslash\mathbb{B}}$ is generated by $\overline{\sigma}_0$ and the map is ramified over the preimage of C_0 on $\widehat{\Gamma'\backslash\mathbb{B}}$. Hence both paths lift the Apollonius $\mathbb{P}^2$ to the surface $\widehat{\mathbb{B}/\Gamma(2)}$ as visualized by the following diagram:

$$\begin{array}{cccccc} & \widehat{\Gamma(2)\backslash\mathbb{B}} & \to & \widehat{\Gamma''\backslash\mathbb{B}} & (Matsumoto) \\ & \downarrow & & \downarrow & \\ (Uludag) & \widehat{\Gamma'\backslash\mathbb{B}} & \to & \widehat{\Gamma(1-i)\backslash\mathbb{B}} & (Apollonius) \end{array}$$

In order to obtain the Kodaira classification of the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$, we need a non singular model which can be obtained by the blow up of the cusps, and which we denote with $(\Gamma(2)\backslash\mathbb{B})'$. The aim is by series of blow downs to obtain from the smooth model a minimal model for the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$.

In this way we come to the minimal rational surface $\mathbb{P}^2$ together with a line arrangement called the *harmonic configuration*, which is the image of the branch divisor of $(\Gamma(2)\backslash\mathbb{B})'$ with respect to the ball uniformization map. The harmonic configuration is an highly symmetric arrangement, consisting of 9 lines through 7 points. It can be used for the construction of a quadruple of harmonic points in $\mathbb{P}^2$, well studied in the classical projective geometry, as an example in [Har2].

Picture 1



Harmonic Configuration

To show that $\widehat{\Gamma(2)\backslash\mathbb{B}}$ is a rational surface we use the following technical tools:

1. The Extension Theorem of Grauert and Remmert, [GR], Thm. 8, which we apply in the following situation, where all varieties we consider are complex and normal:
   Let $W^\circ \to V^\circ$ be a finite covering and V be a compactification, then there exists an unique extension of $W^\circ \to V^\circ$ to a finite covering $W \to V$.

$$\begin{array}{ccc} W^\circ & \hookrightarrow & W \\ \downarrow & & \downarrow \\ V^\circ & \hookrightarrow & V \end{array}$$

2. Compatibility of finite coverings and blow ups.
   This property of surface coverings, that finite coverings and blow ups commute, follows from a celebrated theorem of Stein, Stein Factorization Theorem, which can be found in [Har1], p. 280.

Next we come back to our particular surfaces and we consider the tower of finite coverings

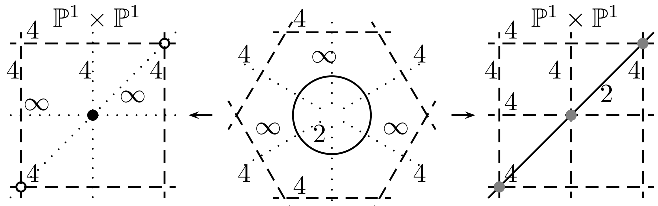
$$\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma''\backslash\mathbb{B}} \to \Gamma\widehat{(1-i)\backslash\mathbb{B}},$$

corresponding to the Galois-Reflection tower of $\Gamma(2)\triangleleft\Gamma(1-i)$ (Diagr. (10),(11)). The Galois groups are $\Gamma(1-i)/\Gamma'' = Z_2$ and $\Gamma''/\Gamma(2) = K4$, as shown in the last chapter, and the branch locus for the composition covering $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \Gamma\widehat{(1-i)\backslash\mathbb{B}}$ is the Apollonius curve (Cor. 3.3).

The ball quotient $\widehat{\Gamma''\backslash\mathbb{B}}$, as shown by Matsumoto and Riedel, is the orbital surface $M = (\mathbb{P}^1 \times \mathbb{P}^1, 4V_1' + 4V_2' + 4V_3' + 4H_1' + 4H_2' + 4H_3' + 2D')$ with three cusp points, which are intersection of more than two lines from the orbital divisor. If we blow up the cusps we obtain the surface X'. According to Yoshida, [Yo], (p.139), this is a projective algebraic surface, which can be also realized by a blow up of four points of $\mathbb{P}^2$ in general position, hence is the del Pezzo surface of degree 5. Considered as a blow up of four points of $\mathbb{P}^2$, X' has been also studied by Bartels, Hirzebruch and Höfer in [BHH]. There they have shown, by proving the proportionality law, that it is a Baily-Borel compactification of a ball quotient surface (number 20 in their list, (p. 201)). The branch configuration on X' with respect to the natural ball projection is given by a configuration of ten lines, six of them with branch index 4, one with 2, and three with ∞.

If we blow down 3 curves from X', two with branch index 4 and one with 2, we obtain [Yo] the orbital surface $X = (\mathbb{P}^1 \times \mathbb{P}^1, 4V_1 + 4V_2 + 4H_1 + 4H_2)$, where $V_i,\ H_i\ i = 1, 2$ denote vertical resp. horizontal lines. Therefore, X is birationally equivalent to the surface $\widehat{\Gamma''\backslash\mathbb{B}}$.

Picture 2



$X \qquad X' \qquad M$

Let $(\Gamma(2)\backslash\mathbb{B})'$ be the surface obtained from $\widehat{\Gamma(2)\backslash\mathbb{B}}$ by blow up of the cusp points. With *cusp curves* we denote the irreducible exceptional curves plugged in for the cusp points, see [BSA].

Lemma 4.1. *The covering $(\Gamma(2)\backslash\mathbb{B})' \to X'$ is unramified over the cusp curves in the Hirzebruch's orbital del Pezzo surface X'.*

Proof. According to [Fe] the surface $\widehat{\Gamma(1)\backslash\mathbb{B}}$ has only one cusp, so the Galois Group $\Gamma(1)/\Gamma(2)$ acts transitively on the cusps set of $\widehat{\Gamma(2)\backslash\mathbb{B}}$, and transforms small neighborhoods of a cusp in a neighborhood of a cusp again. Hence it is enough to consider only the ball cusp point $\kappa = (1 : 0 : 1)$. The canonical homomorphism $\phi : \Gamma(1-i) \to G = \Gamma(1-i)/\Gamma(2)$ induces for each point P on $\mathbb{B}$ a surjective homomorphism of isotropy groups $\phi_P : \Gamma(1-i)_P \to G_{P'}$, where P' is the image of P on $\widehat{\Gamma(2)\backslash\mathbb{B}}$ [BSA], (4.6.2). The Galois group $\Gamma(1-i)/\Gamma(2)$ is generated by $\overline{\sigma}_0, \overline{\sigma}_1, \overline{\sigma}_2$ (see (9)). The preimages of the $\overline{\sigma}_0,\ \overline{\sigma}_1,\ \overline{\sigma}_2$ act on κ as $\sigma_0(\kappa) = \kappa$, $\sigma_1(\kappa) = (i : 0 : 1)$, and $\sigma_2(\kappa) = \kappa$. The two cusp points $\kappa = (1 : 0 : 1)$ and $(i : 0 : 1)$ are non equivalent modulo 2. Hence the image point κ' of the cusp κ on $\widehat{\Gamma(2)\backslash\mathbb{B}}$ has an isotropy group $< \overline{\sigma}_0, \overline{\sigma}_2 > \cong Z_2 \times Z_2$.

Following [BSA], (4.5.3), the cusp curve $L_{\kappa'}$ is a rational curve, because the cusp group $\Gamma(2)_\kappa$ is not torsion free, i.e. it contains a reflection , e.g. σ_2^2.

We consider the covering tower

$$(\Gamma(2)\backslash\mathbb{B})' \to (\Gamma'\backslash\mathbb{B})' \to (\Gamma(1-i)\backslash\mathbb{B})',$$

and especially its restriction to the cusp curve $L_{\kappa'}$, in order to show that it is not a ramification curve. For this we study the action of the isotropy group of κ' on $L_{\kappa'}$. $C_0 + C_1 + C_2 + C_3$ is the branch divisor of p, (see Thm. 3.4), and $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is a degree 4 covering branched along $C_1,\ C_2,\ C_3$ [Ul]. According to [Ul] the quadric C_0 has exactly 4 lines as preimages by the whole covering p, and 2 of them intersect $L_{\kappa'}$ in different points. $\overline{\sigma}_0$ acts identically on the preimages of C_0 on $(\Gamma(2)\backslash\mathbb{B})'$, but the extension of the action of $\overline{\sigma}_0$ in the tangential space of the intersection points implies different reflections directions, so $\overline{\sigma}_0$ is not the id on $L_{\kappa'}$.

The group $K4 =< \overline{\sigma}_1, \overline{\sigma}_2 >$ (see Prop. 2.1) acts transitively on the preimages of C_0 on $\widehat{\Gamma'\backslash\mathbb{B}}$. $\overline{\sigma}_0$ fixes the intersection points of these curves with L, where L is the corresponding to κ exceptional curve on $(\Gamma'\backslash\mathbb{B})'$, and $\overline{\sigma}_2$ interchanges these intersection points, so does the composition $\overline{\sigma}_0\overline{\sigma}_2$. The same is true for the preimages of the intersection points on $(\Gamma(2)\backslash\mathbb{B})'$. Hence $L_{\kappa'}$ is not fixed by $\overline{\sigma}_0$, $\overline{\sigma}_2$ or their composition and is not a ramification curve, for the whole covering $(\Gamma(2)\backslash\mathbb{B})' \to \Gamma(1-i)\backslash\mathbb{B})'$ and for every part extension.

$\square$

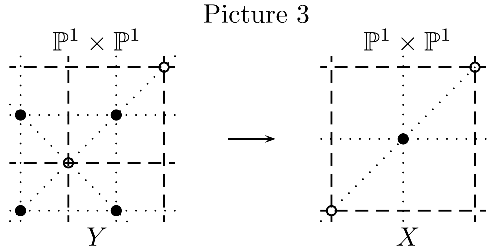
Now, it is clear that the orbital branch locus on $X = \mathbb{P}^1 \times \mathbb{P}^1$, transfered from X', sits on fibres (see above picture 2). In opposite to the orbital surfaces X' and M it is easy now to find the $K4$-covering of X with prescribed weighted branch curves. For this purpose we consider a rational quadric Q with $Q \to \mathbb{P}^1$ of degree 2, branched over 0 and ∞. The product $Q \times Q \to \mathbb{P}^1 \times \mathbb{P}^1$ is a degree four covering with Galois group $K4$, generated by $g \times id$ and $id \times g$, where $< g >$ is the Galois group of $Q \to \mathbb{P}^1$. Because Q is birationally equivalent to the projective line, the above covering is birationally equivalent to $\mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^1\times\mathbb{P}^1$. The branch locus is the orbital divisor $4V_1 + 4V_2 + 4H_1 + 4H_2$ and is lifted as $2\overline{V}_0 + 2\overline{V}_\infty + 2\overline{H}_0 + 2\overline{H}_\infty$ with vertical lines $\overline{V}_0$ and $\overline{V}_\infty$ through 0 and ∞, and the corresponding horizontal lines $\overline{H}_0$ and $\overline{H}_\infty$.

Conversely if we consider a $K4$ quotient of the surface $Q\times Q$ we obtain again the surface X.

$$(Q \times Q)/K4 = (Q/ < g >) \times (Q/ < g >) \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

This $K4$-covering of X is denoted with Y.

Picture 3



We denote with Y', the surface obtained after a blow up of the 6 points, which are intersection of more than 2 lines on Y, as shown in Pic. 3.

Proposition 4.2. $\widehat{\Gamma(2)\backslash\mathbb{B}}$ *is birationally equivalent to* Y.

Proof. Consider the following diagram:

$$\begin{tikzcd} Y \arrow[d] \arrow[dr] & \\ X & X' \arrow[l] \end{tikzcd}$$

Let Y° be the surface Y without the line arrangement of 4 dashed and 6 dotted lines and X° the surface obtained from X by removing the 4 dashed and 3 dotted lines, or from X' again by removing the configuration of 10 curves. From the fact that X' is a compactification of X° it follows by the Extension Theorem of Grauert and Remmert that the finite covering $Y^\circ \to X^\circ$ can be extended in an unique way (up to isomorphism) to the $K4$-covering $Y'' \to X'$. Therefore, $Y'' \to X'$ is the unique extension of the finite covering $Y \to X$, which completes the above diagram.

Because of the compatibility of finite coverings with blow ups, the map $Y \leftarrow Y''$ is exactly the blow up of those points on Y, which lie over the 3 thick points of X, blown up by the map $X \leftarrow X'$. This is exactly the definition of Y', hence $Y'' = Y'$, wherefrom we obtain that Y' is a $K4$-covering of the Hirzebruch's surface X'.

On the other hand let us consider the following diagram:

$$\begin{tikzcd} (\Gamma(2)\backslash\mathbb{B})' \arrow[d] \arrow[dr] & \\ X & M \arrow[l] \end{tikzcd}$$

The Hirzebruch's list, [BSA], p. 201, gives the branch locus for the $K4$-covering $(\Gamma(2)\backslash\mathbb{B})' \to X'$, consisting of 7 lines, 6 dashed and 1 black, as represented in Pic. 2., all of ramification index 2. The 3 dotted lines, which complete the picture are not branch curves according to Lemma 4.1.

Let X° be as above X' without the line configuration of 10 curves and M° be M without the 7 curves (6 dashed and 1 black, Pic. 2.), then $X^\circ = M^\circ$. By the Extension Theorem there exists an unique extension of $Y^\circ \to X^\circ$ to a $K4$-covering $Y' \to X'$. On the other hand $(\Gamma(2)\backslash\mathbb{B})^\circ \to X'$, where $(\Gamma(2)\backslash\mathbb{B})^\circ$ is $(\Gamma(2)\backslash\mathbb{B})'$ without the line arrangement obtained by the $K4$-lift of the curve configuration on X', is again an extension of $Y^\circ \to M^\circ = X^\circ$, hence the both extensions are the same, i.e. $Y' = (\Gamma(2)\backslash\mathbb{B})'$.

As a consequence we obtain the following commutative diagram of surfaces, where the vertical maps are $K4$ coverings and the horizontal are birational transformations:

$$\begin{array}{ccccc} Y & \dashleftarrow & (\Gamma(2)\backslash\mathbb{B})' & \dashrightarrow & \widehat{\Gamma(2)\backslash\mathbb{B}} \\ \downarrow & & \downarrow & & \downarrow \\ X & \dashleftarrow & X' & \dashrightarrow & M \end{array}$$

Therefore, the surfaces Y and $\widehat{\Gamma(2)\backslash\mathbb{B}}$ are birationally equivalent. The line configuration of 10 curves on X' is lifted as the arrangement of 16 lines, four (black) of weight 2, six (dashed) of weight 2, six (dotted) of weight ∞, which come after blow up of the cusp of $\widehat{\Gamma(2)\backslash\mathbb{B}}$. ☐

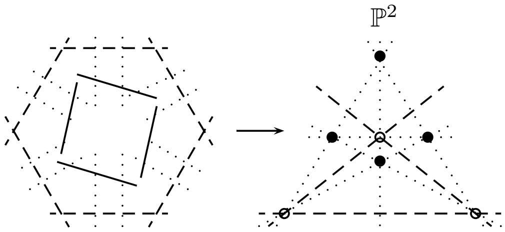
With the results of the former proposition now we are able to prove the following statement.

Theorem 4.3. $(\Gamma(2)\backslash\mathbb{B})'$ *is the surface obtained as a blow up of seven points on* $\mathbb{P}^2$. *The line arrangement on* $(\Gamma(2)\backslash\mathbb{B})'$ *is the preimage of the harmonic configuration.*

Proof. The surface $(\Gamma(2)\backslash\mathbb{B})'$ can be obtained from Y by blow up of the six points, which are intersection of at least three lines.

Y itself is a model of $\widehat{\Gamma(2)\backslash\mathbb{B}}$ given by $\mathbb{P}^1 \times \mathbb{P}^1$ together with the line configuration $2\overline{V}_0 + 2\overline{V}_\infty + 2\overline{H}_0 + 2\overline{H}_\infty$. By blow up of the intersection point of two dashed lines and one dotted, in the line arrangement on Y, and afterwards blow down of the dashed lines $\overline{V}_\infty$ and $\overline{H}_\infty$ going through this point one obtains the projective plane. Hence $(\Gamma(2)\backslash\mathbb{B})'$ can be constructed from $\mathbb{P}^2$ by blowing up the 7 thick points of the harmonic line configuration on $\mathbb{P}^2$ as represented in the following picture.

Picture 4



$(\Gamma(2)\backslash\mathbb{B})'$ Harmonic Configuration

☐

At the end of this section we want to remark that the detailed study of the Galois groups of the towers of surface coverings $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma''\backslash\mathbb{B}} \to \Gamma(\widehat{1-i)\backslash\mathbb{B}}$ as well as $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma'\backslash\mathbb{B}} \to \Gamma(\widehat{1-i)\backslash\mathbb{B}}$ proves that the natural congruence subgroup $\Gamma(2)$ is contained in the groups Γ', studied by Hirzebruch, Matsumoto and Riedel, and Γ'', corresponding to the Uludag's surface, which leads to the following result:

Corollary 4.4. *The two groups* Γ' *and* Γ'' *are Picard congruence subgroups.*

Corollary 4.5. *The natural Picard congruence subgroup* $\Gamma(2)$ *is generated by finitely many order-2 reflections.*

Proof. By Theorem 4.3 the quotient surface $\Gamma(2)\backslash\mathbb{B}$ is simply-connected. It is also smooth. Now we apply the second statement of Theorem 2.11 to see that our group is generated by finitely many reflections. At the begin of B) in section 3 we already remarked that $\Gamma(2)$ contains only reflections of order 2. This finnishes the proof.

$\square$

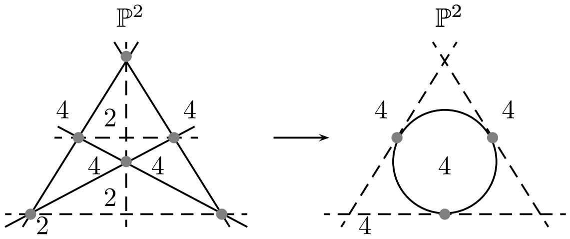
5 Numerical Space Model

In this section we would like to compute a numerical model for $\widehat{\Gamma(2)\backslash\mathbb{B}}$. For this we consider the covering

$$\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma'\backslash\mathbb{B}} \to \Gamma(\widehat{1-i)\backslash\mathbb{B}},$$

from Diagram (11), with Galois groups $\Gamma'/\Gamma(2) = Z_2$ and $\Gamma(1-i)/\Gamma' = K4$ (Diagram (10)).
$\Gamma(\widehat{1-i)\backslash\mathbb{B}}$ is the orbital surface $(\mathbb{P}^2, 4C_0+4C_1+4C_2+4C_3)$. The three tangents $C_1,\ C_2,\ C_3$ can be given for example by the equations $x'=0,\ y'=0,\ z'=0$ and the quadric C_0 by $(x'+y'-z')^2-4x'y'=0$. The Uludag's surface $\widehat{\Gamma'\backslash\mathbb{B}}$ is the orbital surface $(\mathbb{P}^2, 4G_1+4G_2+4G_3+4G_4+2B_1+2B_2+2B_3)$. It is a degree four covering of the Apollonius $\mathbb{P}^2$, ramified along the tangents. C_0 is lifted by this covering as the curve $(x+y-z)(x+y+z)(x-y+z)(x-y-z)=0$, where each irreducible component is of branch index 4. The tangents, defining the branch locus, are lifted as lines of branch index 2.

Picture 5



Uludag's Configuration   Apollonius Configuration

The Picard group of $\mathbb{P}^2$ is generated by a line, hence the divisor class of the four lines $G_1+G_2+G_3+G_4$ is divisible by 2 in $Pic(\mathbb{P}^2)$. Then according to the cyclic cover theorem, see e.g. [EPD], there exists exactly one degree two covering of the Uludag's surface, ramified along these lines and this surface is exactly $\widehat{\Gamma(2)\backslash\mathbb{B}}$.
The covering $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \mathbb{P}^2$-Uludag's is cyclic with Galois group Z_2. The surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$ is obtained as a normalisation of $\mathbb{P}^2$ along the function fields extensions $\mathbb{C}(\mathbb{P}^2) \subset \mathbb{C}(\widehat{\Gamma(2)\backslash\mathbb{B}})$. Using Kummer extensions theory [Ne] we obtain $\mathbb{C}(\widehat{\Gamma(2)\backslash\mathbb{B}}) = \mathbb{C}(x,y)(\sqrt{\delta})$, where $\delta = (x+y-1)(x+y+1)(x-y+1)(x-y-1)$ is the

affine divisor corresponding to the branch divisor of the covering $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \mathbb{P}^2$-Uludag's. If we set $u^2 = \delta$, we obtain by projectivisation for the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$ the following numerical model:

$$\widehat{\Gamma(2)\backslash\mathbb{B}}: \; t^2u^2 + 2x^2t^2 + 2x^2y^2 + 2y^2t^2 - t^4 - x^4 - y^4 = 0.$$

This space model enables the computation of explicit equations for various Shimura curves, important for the coding theory. In the central part of her doctoral thesis [Pet] the second author connects towers of such curves inside of our octahedral Picard surface tower. They are constructed as quotients of "arithmetic subdiscs" of the 2-ball.

6 The Octahedral Configuration of Norm-1 Curves

We call an orbital ball quotient surface $\Gamma\backslash\mathbb{B}$ (also its compactification) *neat*, if the ball lattice Γ is neat. In this case $\mathbb{B} \to \Gamma\backslash\mathbb{B}$ is a universal covering.

From Hirzebruch's work in the 1980-s, see e.g. [Hi], and a systematic study in [Ho04] we know that there exist coabelian neat ball lattices Γ. *Coabelian* means that the quotient surface $\Gamma\backslash\mathbb{B}$ has an abelian surface as model. We found the following general situation:

Let A be an abelian surface, $T = T_1 + ... + T_k$ a sum of elliptic curves T_i on A with pairwise normal crossings at intersection points. We denote by s the number $\#Sing(T)$ of curve singularities of T and set

$$S_i := Sing(T) \cap T_i \,, \; s_i := \#S_i; \; i = 1, ..., k.$$

By the adjunction formula for curves on smooth surfaces, it is easy to see that the selfintersection indices of elliptic curves on abelian surfaces vanishes. We assume, that $S_i \neq \emptyset$ for all i. If we blow up each curve singularty of T, we get a surface A' with s exceptional lines of first kind. The proper transforms of the T_i on A' we denote by the same symbol. They do not intersect each other and have negative selfintersections. Therefore we can contract them all to elliptic singularities. On this way we get a surface $\hat{A}$ with k singularities $\hat{\kappa}_i$. We put together the whole construction in the following diagram:

(14)
$$\begin{tikzcd} A & A' \arrow[l] \arrow[r] & \hat{A} \\ T_i \arrow[u] \arrow[r, "="] & T_i \arrow[u] \arrow[r] & \hat{\kappa}_i \arrow[u] \end{tikzcd}$$

with vertical inclusions. We proved

Theorem 6.1. *([Ho04], Theorem 2.5) With the above notations/assumptons, $\hat{A}$ is a neat ball quotient surface $\widehat{\Gamma\backslash\mathbb{B}}$ with cusp singularties $\hat{\kappa}_i$, if and only if the relation*

$$4s = s_1 + ... + s_k \tag{15}$$

is satisfied.

Now we start again from the biproduct $\mathbb{P}^1 \times \mathbb{P}^1$, endowed with three horizontal lines and three verticals as drawn in picture 3 of section 4 (on the right, without diagonal). We consider the (unique) 4-cyclic cover of $\mathbb{P}^1$ branched over three points: namely the elliptic CM-curve $E = \mathbb{C}/\mathbb{Z}[i]$ with cyclic automorphism group Z_4 of order 4 generated by the i-multiplication. The corresponding Galois covering (with intermediate step)

$$E \longrightarrow E/ < -id_E > = \mathbb{P}^1 \longrightarrow E/Z_4 = \mathbb{P}^1$$

is ramified at the 2-torsion points $Q_0 = O$, Q_2 of ramification order 4 and Q_1, Q_3 of ramification order 2. Their image points on $\mathbb{P}^1$ are denoted by P_0, P_2 or P_1, respectively, preserving indices. Taking bi-products we get a Galois covering of surfaces with Galois group $Z_4 \times Z_4$

$$E \times E \longrightarrow (E \times E)/(Z_4 \times Z_4) = E/Z_4 \times E/Z_4 = \mathbb{P}^1 \times \mathbb{P}^1$$

with ramification curves $Q_i \times E$, $E \times Q_j$, $i, j = 0, .., 3$, and branch curves $P_i \times \mathbb{P}^1$, $\mathbb{P}^1 \times P_j$, $i, j = 0, .., 2$. More precsely, the orbital branch divisor is

$$4 \cdot P_0 \times \mathbb{P}^1 + 4 \cdot P_2 \times \mathbb{P}^1 + 4 \cdot \mathbb{P}^1 \times P_0 + 4 \cdot \mathbb{P}^1 \times P_2 + 2 \cdot P_1 \times \mathbb{P}^1 + 2 \cdot \mathbb{P}^1 \times P_2$$

The diagonal curve D of $\mathbb{P}^1 \times \mathbb{P}^1$ has 4 irreducible preimage curves $D_1, .., D_4$ on $E \times E$. These are ellptic curves. So the whole divisor

$$T := D_1 + D_2 + D_3 + D_4 + Q_1 \times E + Q_3 \times E + E \times Q_1 + E \times Q_3$$

is a sum of 8 elliptic curves with

$$Sing(T) = \{O, Q_2 \times Q_2, Q_1 \times Q_1, Q_1 \times Q_3, Q_3 \times Q_1, Q_3 \times Q_3\}.$$

We count $s = 6$ singular points, 4 of them on each T-component D_i and 2 on each horizontal and vertical component. Altogether we see that the relation (15) is satisfied:

$$4 \cdot 6 = 4 + 4 + 4 + 4 + 2 + 2 + 2 + 2.$$

For more calculation details we refer to [Ho04], Example 4.6. It follows from Theorem 6.1 that $E \times E$ is an abelian model of a neat ball quotient surface of a lattice Γ_E with smooth compactication $(E \times E)' = (\Gamma_E\backslash\mathbb{B})'$ received by blowing up the six points of $Sing(T) \subset E \times E$. Altogether we have the commutative Galois-covering diagram of blow-ups/contractions:

$$\begin{tikzcd}
E \times E & (E \times E)' \arrow[l] \arrow[r] \arrow[d, "Z_2 \times Z_2"] & \widehat{E \times E} \arrow[d] \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[d] \arrow[from=1-1, "<-id>\times<-id>"] \arrow[from=1-1, r, phantom, "\cong"] & (\Gamma(2)\backslash\mathbb{B})' \arrow[l] \arrow[r] \arrow[d, "K4"'] & \widehat{\Gamma(2)\backslash\mathbb{B}} \arrow[d] \\
\mathbb{P}^1 \times \mathbb{P}^1 & (\Gamma(1-i)\backslash\mathbb{B})' \arrow[l] \arrow[r] & \widehat{\Gamma(1-i)\backslash\mathbb{B}}
\end{tikzcd}$$

The upper row comes, as already mentioned, from Theorem 6.1. The partial diagram of middle and bottom rows was constructed in section 4. Both parts are joined as drawn, because the blown-up points of Sing(T) have as image along $< -id > \times < -id >$ the six image points blown-up in the middle row to get $(\Gamma(2)\backslash\mathbb{B})'$.

Altogether we have a Galois-Reflection tower

$$\Gamma_E\backslash\mathbb{B} \to \Gamma(2)\backslash\mathbb{B} \to \Gamma_M\backslash\mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B} \to \Gamma\backslash\mathbb{B}$$

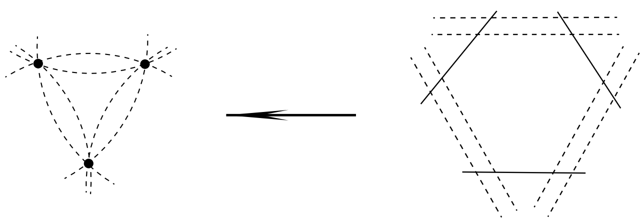
of Picard modular surfaces, which starts with a neat one of abelian type.

Let t be the translation automorphism of $E \times E$ adding to each point $Q \times Q$ the 2-torsion point $Q_1 \times Q_1$. We consider the isogeny

$$E \times E \to (E \times E)/ < t > =: \ B.$$

It is easy to see that t doesn't move the divisor T and the intersection points of their components collected in $Sing(T)$. The image of the latter points on the abelian surface B consists of three points. The image of T on B consists of 3 elliptic curve pairs. Each of the three points is intersection point of the 4 components of two such pairs. We blow them up, and denote the arising surface by B'. We visualize the transfer of the 6 (here black dotted) elliptic curves along the birational morphism $B \leftarrow B'$:

Picture 6

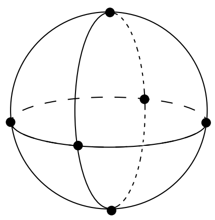


On this way we get the

Globe configuration on the abelian surface model
$\hat{B} = \widehat{\Gamma_B \backslash \mathbb{B}}$:

With $s = 3$ and $s_i = 2$, $i = 1, .., 6$ we see that the relation (15) is satisfied again. Therefore, after blowing up the 3 intersection points, we get a neat ball quotient surface compactified by the 6 elliptic curves. Contracting them we get a surface $\hat{B}$ with six cusp singularities painted as black points in picture 7. Thereby we arrange the (transfers of the) 3 (black) exceptional lines of this picture 3-dimensionally as crossing circles on a globe, reflecting precisely their intersection behaviour. Obvously, the six cusp points span a regular octahedron.

Picture 7



Excercise 6.2. *Find with help of next section the octahedron motion group representations (on $\mathbb{R}^3$) of our Galois-Reflection groups extending $\Gamma(2)$.*

Remark 6.3. The above globe curve configuration is (along our coverings and modifications) a transformation of the Apollonius configuration (consisting of a quadric and 3 tangent lines). By Corollary 3.3, the Apollonius curves are (all) norm-1 curves on $\widehat{\Gamma(1-i)\backslash\mathbb{B}} = \mathbb{P}^2$, defined as quotients of norm-1 subdiscs of $\mathbb{B}$. The latter property doesn't change along correspondence transformations. Therefore the two meridians and the equator on the above globe represent norm-1 curves on $\hat{B}$.

7 Appendix: Explicit Unitary Representations

For $\Gamma = \Gamma(1) = \mathbb{SU}((2,1), \mathbb{Z}[i])$ we remember to the sequence of normal group extensions by reflections well-defined in sections 3, 4.

(16)

$$\begin{aligned}
\Gamma' = \Gamma_U &=< \Gamma(2), \sigma_0 >, \quad \text{(recognized as Uludag's);} \\
\Gamma'' = \Gamma_M &=< \Gamma(2), \sigma_1, \sigma_2 >, \quad \text{(rec. as Matsumoto's, Hirzebruch's);} \\
\Gamma(1-i) &=< \Gamma(2), \sigma_1, \sigma_2; \sigma_0 >; \\
\Gamma &=< \Gamma(2), \sigma_1, \sigma_2, \sigma_0; \sigma_a, \sigma_b >;
\end{aligned}$$

with small abelian factor groups

$$\Gamma'/\Gamma(2) \cong Z_2,\ \Gamma''/\Gamma(2) \cong Z_2 \times Z_2;$$
$$\Gamma(1-i)/\Gamma(2) \cong Z_2 \times Z_2 \times Z_2,\ \Gamma/\Gamma(1-i) \cong S_3.$$

As promised we give the special unitary representations of the reflections. One has only to apply their explicit definitions to the canonical basis of $\mathbb{C}^3$:

$$\begin{aligned}
&\sigma_0 = -i \cdot \begin{pmatrix} i & -1+i & 1-i \\ -1+i & i & 1-i \\ -1+i & -1+i & 2-i \end{pmatrix};\\
&\sigma_1 = i \cdot \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = i \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix};\\
&\sigma_a = \begin{pmatrix} -1 & -1-i & 1+i \\ -1+i & 0 & 1 \\ -1+i & -1 & 2 \end{pmatrix}, \sigma_b = -\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{17}$$

Proposition 7.1. *The factor group* $\Gamma(1)/\Gamma'$ *is isomorphic to the motion group* $\mathbb{O}$ *of the octahedron. The factor group* $\Gamma(1)/\Gamma(2)$ *is (isomorphic to) the* double octahedron group $Z_2 \times \mathbb{O} \cong Z_2 \times S_4$.

For the proof one uses a presentation of S_4. The corresponding relations are easily checked by the unitary representation of the generating elements (17). The calcultions $\mod {}^{\times}\Gamma(2)$ are left to the reader.

Problem. Find explicitly 2-reflections generating $\Gamma(2)$.

Hint. Matsumoto found in [Mat] explicit generators of $\Gamma'' = \Gamma_M$ using the monodromy of a curve family. Try to present them as products of reflections. This is a finite problem. Then take squares of the order-4 reflection among the factors.

The solution of the problem is important for modular function tests for all arithmetic lattices in (16). In [Mat], or better now in [KS], generating modular forms for Γ_M are explicitely known. The interaction with the octahedron group is very interesting, especially for construction of class fields, see [Ri].

References

[Ar] Armstrong, P., *The fundamental group of the orbit space of a discontineous group*, Proc. Cambridge Philos. Soc. **64** No. 2 (1968), 299-301

[BB] Baily, W.L., Borel, A., *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966), 442-528

[BHH] Barthel, G., Hirzebruch, F., Höfer, T., *Geradenkonfigurationen und Algebraische Flächen*, Aspects of Mathematics D 4, Vieweg, Braunschweig, 1986

[BMG] , Holzapfel, R.-P., Vladov, N., *Quadric-line configurations degenerating plane Picard-Einstein metrics I-II*, Sitzungsber. d. Berliner Math. Ges. 1997-2000, Berlin (2001), 79 - 142

[Bo] Borel, A., *Compact Clifford-Klein forms of symmetric spaces*, Topologie, **2** (1963), 111-122

[BSA] Holzapfel, R.-P., *Ball and Surface Arithmetic*, Vieweg, Braunschweig, 1998

[Ch] Chevalley, C., *Invariants of finite groups generated by reflections*, Am. Journ. Math. **77** (1955), 778-782

[EPD] Holzapfel, R.-P., *Geometry and Arithmetic Around Euler Partial Differential Equations*, VEB Deutscher Verlag der Wissenschaft Berlin & Reidel Publ. Company, Dordrecht, 1986

[Fe] Feustel, J., *Über die Spitzen von Modulflächen zur Zweidimensionalen Komplexen Einheitskugel*, Preprint Serie der Akademie der Wissenschaften der DDR, Reposrt 03/77, 1977

[GR] Grauert, H., Remmert, R., *Komplexe Räume*, Math. Ann. **136** (1958), 245-318

[Har1] Hartshorne, R., *Algebraic Geometry*, Springer, Berlin, 2000

[Har2] Hartshorne, R., *Foundations of Projective Geometry*, Lecture notes, Harvard University, 1967

[Hi] Hirzebruch, F., *Chern numbers of algebraic surfaces - an example*, Math. Ann. **266** (1984), 351-356

[HPV] Holzapfel, R.-P., Pineiro, A., Vladov, N., *Picard-Einstein Metrics and Class Fields Connected with Apollonius Cycle*, HU-Preprint, **98-15** 1998; see also [BMG]

[Ho04] Holzapfel, R.-P., *Complex hyperbolic surfaces of abelian type*, Serdica Math. J. **30** (2004), 207-238

[HUY] , Holzapfel, R.-P., Uludag, M., Yoshida, M. (ed.), *Arithmetic and Geometry Around Hypergeometric Functions*, Progr. in Math. **260**, Birkhäuser, Basel, 2007

[KS] Koike, K. Shiga, S., *An extended Gauß AGM and corresponding Picard Modular Forms*, Journ. of Number Theory **128** (2008) 2097-2126

[Mat] Matsumoto, K. *On modular Functions in " Variables Attached to a Family of Hyperelliptic Curves of Genus 3*, Annale della Scola Normale Superiore di Pisa - Classe di Scienze, Ser. IV, vol. **XVI**, no.4 (1989), 557-578

[Na] Namba, M., *On Finite Galois Coverings Germs*, Osaka Mathematical Journal, **28** (1991) , 27-35

[Ne] Neukirch, J. *Algebraische Zahlentheorie*, Springer, Berlin, 2002

[Pet] Petkova, M., *Families of Algebraic Curves with Application in Coding Theory and Cryptography*, doctoral thesis, Humboldt-Univ. Berlin, 2009

[Ri] Riedel, T., *Ringe von Modulformen zu einer Familie von Kurve mit* $\mathbb{Q}(i)-$*Multiplikation*, Diplomarbeit, 2004; Main results in [HUY]: *On the Construction of Class Fields by Picard Modular Forms*, 273-285

[SY] Sakurai, K., Yoshida, M., *Fuchsian systems associated with the* $\mathbb{P}^2(\mathbb{F}^2)$-*arrangement*, Siam J. Math. Anal. **20**, No. 6 (1989), 1490-1499

[Ul] Uludag, M. *Covering Relations Between Ball Quotient Orbifolds*, Mathematische Annalen **308** no. 3 (2004), 503-523

[Yo] Yoshida, M., *Fuchsian differential equations*, Vieweg, Aspects of Mathematics E 11, Brauschweig, 1987