Mobile History of Algebraic Equations[0]: The Visions

Rolf-Peter Holzapfel
e-mail: holzapfl@mathematik.hu-berlin.de
Homepage:
www.math.hu-berlin.de/∼zyska/Holzapfel/welcome.html

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> It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain
>
> Pierre de Fermat[1]

Timetable I

- 1799 Fundamental Theorem of Algebra: Each (non-constant) algebraic equation has a (complex number) solution. Proved in the celebrated thesis of **C.F. Gauß** .
- 1828 Invited by Alexander v. **Humboldt** (1769-1859) Gauß travels to Berlin for fruitful communications with the great universal scientist.
- 2005 The German bestseller "Measuring the World" (dedicated to the travel of Gauß to Berlin) written by Daniel **Kehlmann** appears. In the meantime (2013) film and DVD with same titles have been produced.
- 2012 Announcement in a newspaper (November): First time the schools of a German destrict (Barnim) has been completely equipped with electronic (smart) tables.

[0] Translated from my German essay "Mobile Gleichungsgeschichte(n)" on my homepage, submitted to Forum of Berliner Mathematische Gesellschaft, to appear 2014

[1] Around 1630, Fermat wrote his Last Theorem in the margin of his exemplar of Diophantos' 6-th book on Arithmetics. Unfortunately, the original has been lost. But Fermat's son Samuel reproduced the margin in his issue of Fermat's work published in 1670, see [Si], [CF] S. 208.

Contents

1 Introduction

In the beginning: The **VISION**

By analogy with Faust I (Goethe)

The history of mathematics includes a lot of visions appearing first nebulously. Then, after many calculations, conjectures of formulas and proof attempts the contures get clear. The useful results were held tight in documents, in books which partly got famous and popular. The visions have been further bequeathed to the next generations as sequences of pictures. Since last century spectacular applications you can look in films and videos.

A studying scientist must know that films/videos are helpful, but induce in any case only a passive learning. Especially in Mathematics there's a long way from resarch results to visible applications. Therefore it is important to get to know the original visions of eminent authorities. The new mathematical softwares enable us to reconstruct in a simple way such basic master ideas by ourselves as digital animations.

Twenty years after André Wiles proof of the millenium result (Fermat's Last Theorem) the time is high to track down the most important course settings in the history of mathematics concerning algebraic equations. A combination of this article with facebook is possible. It allows immediately to consider the pictures and animations described in the text. You can find them on the side announced on my homepage (choose: "Rolf-Peter Holzapfel" on Google). The list of all can be found at the end of the article before the references. Also on my homepage you can find all MAPLE orders I have used for calculations and animations. A lot of time was necessary to produce them. But, if they are already there, then it is easy for the reader to use them writing creatively an own version. For example, we recommend the reader to fill into the kit of our optical procedure (with drones) for finding all solutions of a special equation, an own equation (see Chapter 6). Moreover, our homepage kits can be also considered as instructive foundations for further constructions of digital animation clips. This allows an active and playful intervention into the learning process. That's what we need in the future.

2 Conica

Let's go back to ancient times. Regard a rotary double cone. It is easy to visualize it on each notebook with popular mathematical software (Mathematica, Maple). Additionally, we consider sections with moving planes. The oldest preserved work on conical sections is "Conica" due to Apollonius of Perge (ca. 262 - ca. 190 b.C.). The names of the sectional curves appeared there: Ellipse, parabola, hyperbola. The Animation [A10] shows the rotary motion of a plane around a fixed axis together with the moving sections with a rigid double cone.

Of course, in the software blocks for calculating the animations we use cartesian coordinates. So we jump for a moment near to the Modern Times of mathematics beginning with Newton (1643-1727) and Leibniz (1646-1716). A proof without coordinates characterizing the three types of the sectional curves by focus properties was managed by the French officer and mathematician Germinal P. Dandelin (1794-1847), [Da]. Instead of setting a ball into a cannon, he put it into a cone touching simultaneously its coat and an intersecting plane, [B4]. First knowledge I gained as student in a beautiful presentation in Section IX (Kegelschnitte) of the old German book [Bo]. Much later I observed that the name of Dandelin was nowhere mentioned in the book. It remembers to times when leaders of two countries regarded each other as enemies. They propagated "Erbfeindschaft" for the preparation of terrible wars. Such hard nationalism against other countries should never happen again.

Let's do a further jump: into the 20-th century. The most significant mathematician of this era was David Hilbert (1862-1943). He describes the practical application of hyperbolas in mechanics. Two rotary hyperboloids touching each other along a scew straight line transport rotation around one axis to another.

We refer to the Animations [A7],[A6],[A5].

"Thus we receive a crushing (Abschrotung) of both surfaces, putting them together in such a way that they touch each other along a straight line, and spinning them around their rotation axes with suitable speed ratio. From this results a technically applicable method of a cogwheel transmission between scew lines. Since during the mutual sliding the material wears out, one has to restrict the hyperboloids to congruent ones. Such a transmissen is shown in Picture 269."
[HC], ch. 5 (Kinematics), 43 (Motion in 3-space), Abb. 269 (Mechanique model with handle).

3 Cubic Algebraic Equations

Now we reduce the number of variables to only one, but we admit higher degrees. More precisely, we will from now on consider mainly algebraic equations

$$x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0, \quad a_i \in \mathbb{Q},\ i = 0...n-1,$$

Even in the case of quadratic equations ($n = 2$) it is not possible to solve each of them in terms of real numbers:

$$\begin{aligned} input: &\quad solve(x^2 + 2px + q, x), \\ output: &\quad -p \pm \sqrt{p^2 - q} \end{aligned}$$

Already in school the solution formula was explained, if $p^2 \geq q$. The contrary case has been supressed tacitly or declared as "unsolvable". G. Cardano (1601-1676) is detectably the first who mentions solutions of seemingly unsolvable types (captious quantities, lat. "quantitas sophistica"). In his book [Ca], Ch. 39, he interpreted the equation $x(10 - x) = 40$ as task for finding a rectangle with area 40 and scope 20. A geometrical solution doesn't exist. Numerically the equation can be solved by means of "imaginary numbers". The application of such numbers has been denoted as "useless gimmik" by Cardano himself, cp. [FH]. René Descartes (1596-1650) shaped the term "imaginary root". The introduction of the "imaginary unit $i = \sqrt{-1}$" is attributed to Leonhard Euler (1707-1783), see Google/Wikipedia: Complex Numbers.

The extension of the story to cubic polynomials

$$x^3 + ax^2 + bx + c$$

became dramatically. By the method of cubic completing it is easily seen that the search for solutions can be reduced to the equation type

$$x^3 + 3px - 2q = 0$$

Now the following table will be given ahead:

Timetable II

- **Leonardo da Vinci** (1452 - 1519): His friend, mathematician and Franciscan monk L. Pacciolli calculated the bronze amount for an equestrian statue.
- **Pacioli** (1445 - 1514): States that there are cubic equations $x^3+bx+c=0$, which are not solvable. He presented some explicit examples.
- **del Ferro** (1500) solves all these examples. He announces the solution way to his student **Fior**.
- **Tartaglia** (1535) also states to be able to solve the cubic equations. He demonstrated this to Fior, who gave him a list of examples.
- **Cardano** was introduced by Tartaglia into his method after promising to keep it secret. Breaking oath Cardano published the procedure in his book "Ars magna" (1545).
- Public dispute duel Tartaglia - Cardano (represented by his student **Ferrari** together with two roughnecks).

At his time Luca Pacciolli's statement about general unsolubility of cubic equations had (for short time) a trendsetting effect. In contrast to the monks believe Scipione del Ferro (1465 - 1526) found solutions for all examples of Pacciolli. Del Ferro transmitted his method to some friends, especially to Antonio Maria Fior (end of 15-th century - middle of the 16-th). After Niccol Tartaglia's (1499 - 1557) independent discovery a competence meeting with Fior and Tartaglia was organized, which ended successfully for the latter mathematician, who solved in acceptable time 30 cubic equations choosen by Fior. Geronimo Cardano (1501 - 1576) asks Tarataglia for his computational secret. After several requests the latter revealed his approach to the former in terms of misty verses. Mathematically disrobed they include the following instruction:

$$x^3+3px=2q, \quad y-z=2q, \quad y\cdot z=p^3, \quad x=\sqrt[3]{y}-\sqrt[3]{z},$$

It leads to the solution

$$x=\sqrt[3]{\sqrt{p^3+q^2}+q}-\sqrt[3]{\sqrt{p^3+q^2}-q}$$

One year after Cardano's sneaky publication Tartaglia (see Timetable) complained the breach of promise in "Questiti et inventione diverse". Now Cardano arranged his pupil Ludovico Ferrari (1522 - 1565) to accuse Tartaglia the (made-up) theft of ideas from del Ferro. In the year 1548 the public debate of Tartaglia and Ferrari (as agent of Cardano) had been organized. Because of Ferrari's powerful bodyguards Tartaglia prefered a piqued retraction.

A littlebit later Ferrari discovered how to solve algebraic equations of degree 4 (quartic). He reduced the problem to the resolution of a cubic equation, the resolvent of the given quartic equation, see e.g. [QF], [MY]. In the Internet one can find several articles with different instructions for solving equations of the above types. A method of Euler is described in [Ni].

4 Newton's Dynamization in the Plane

In the mean time the coordinate method had been developed for the study of geometric problems. Worth to read is René Descartes' book "La Geometrica" printed in 1637. Already 1627/28 Descartes sketched how a quartic equation can be solved intersecting a parabola with a circle.

On this R. Descartes (1596 - 1656) founded mainly the Analytic Geometry bringing together Algebra and Geometry. However, rectengular coordinates (called "cartesian") didn't appear in his work. Originally, "cartesian" means: "introduced by Descates". But as creators of these coordinates should be counted Apollonius of Perge, Nikolaus of Oresme (1330-1382), Pierre de Fermat (1601-1665) and/or Jan de Witt (1625-1675).

In accordance with cubic equations none other than Isaac Newton (1643 - 1727) dealt with plane cubic curves in some detail. Following dynamical considerations he divided them in five types looking at their normal form equations

$$y^2 = x^3 + ax^2 + bx + c \in \mathbb{R}[x] \tag{1}$$

A German description of the Newtonian world of ideas can be found in [Wi]. It's a nice explanation of Newton's Latin original work in [Ne]. The Animation [A8] then illustrates dynamically Newton's cubic curve family. For self construction the corresponding digital building kit you can find on my homepage. Degenerated curves appear in the family. They consist of cubics with a singularity. Two types of plane curve singularities have been discovered graphically on this way by Newton: Double point and cusp. The curves on the right-hand side have only one connected component, those on the left-hand have two of them: An oval and a branch.

Descartes (as well Albert Girard, 1595 - 1632) begun to decompose polynomials into linear factors. The types of cubics could be understood by the kind of zeros of the corresponding cubic polynomial. Geometrically the real zeros appear as intersection points of the curve (1) with the x-axis.

5 Quartic Examples: Fermat Quartic and Egg of Columbus

Increasing the degree from 3 to 4 leads us to the world of plane quadrics. We consider only two examples, namely the Fermat quartic (below) and the "ovoid" [A3] with equation

$$(x^2 + y^2)^2 - x^3 = 0,$$

([Fi], S. 32). With our digital kit (homepage) we draw this curve, then we rotate it around the x-axis. For this purpose solving the equation for y is necessary. Namely, the rotation command needs a function graph. With our Mathe-notebook (MAPLE) we find

$$\begin{aligned} &input: \quad solve((x^2 + y^2)^2 - x^3, y), \\ &output: \quad \sqrt{-x^2 + x^{3/2}}, -\sqrt{-x^2 + x^{3/2}}, \sqrt{-x^2 - x^{3/2}}, -\sqrt{-x^2 - x^{3/2}} \end{aligned}$$

We choose the function $\sqrt{-x^2+x^{3/2}}$ over the intervall $[-1,0]$. After rotation of the function curve the arising egg will be colored, illuminated and trimmed at the top. Finally, you can regard a dynamical version of the "Egg of Columbus".

We are interested in rational solutions of the quartic ovoid equation, t.m. in points with rational coordinates sitting on the curve. In general, a plane equation of degree 4 has only finitely many $\mathbb{Q}$-solutions. This was shown by Faltings in a celebrated theorem proved in 1983: Smooth curves of genus $g > 1$ defined by equations with rational coefficients have at most finitely many rational points. Since the 19-th century one knows that almost all plane curves of degree 4 have genus 3. More precisely, the smooth ones have this genus, the others not.

Gerd Faltings received for his theorem the highest mathematical prize, the Fields Medal, comparable with the Nobel Prize (not existing for mathematics). Faltings' Theorem was already conjectured in 1922 by the british mathematician (born in the USA) Louis J. Mordell (1888-1972). But nobody could prove it before 1983. Until then at international centers of mathematics high technics in Algebraic and Arithmetic Geometry were developed. After all, the proof of the Mordell conjecture has been accomplished. A littlebit more impression about the methods you find in the closing remark of Section 12.

Looking again at the ovoid equation we check its rational solution set. First one ascertain that the curve is not smooth finding out its singularities:

$$\begin{aligned} \mathit{input}: &\quad f := x^4 + 2x^2y^2 + y^4 - x^3,\ \mathit{singularities}(f,x,y) \\ \mathit{output}: &\quad (0,0) \end{aligned}$$

Therefore it has exactly one singularity: It sits in the coordinate origin being the top point of the ovoid. The determination of the curve genus is managed by the following MAPLE command:

$$\begin{aligned} \mathit{input}: &\quad \mathit{genus}(f,x,y) \\ \mathit{output}: &\quad 0 \end{aligned}$$

Now we conclude that there must exist infinite many rational points on the ovoid. Indeed, by a simple order one gets a parametrization of them:

$$\begin{aligned} \mathit{input}: &\quad \mathit{parametrization}(f,x,y,t) \\ \mathit{output}: &\quad \left(\frac{1}{1+2\cdot t^2+t^4}, \frac{t}{1+2\cdot t^2+t^4})\right) \end{aligned}$$

These rational points lie dense on the ovoid.

In contrast to the ovoid, the Fermat quartic $x^4 + y^4 = 1$ [B3] is smooth, has therefore genus 3. Thus it contains only finitely many rational points by Mordell-Faltings. Already Fermat knew all of them, namely $(0, \pm 1)$ and $(\pm 1, 0)$. For the proof method see Section 9.

6 The Solution Navigator

We start the section with the following

Timetable III

- 1895 **H.Weber**: First explicit presentation of an (by its radicals[2]) unsolvable algebraic equation (in a textbook): $x^5 + 5x + 5 = 0$.
- 1783 **L.Euler** was convinced, till the end of his life, that all algebraic equations of 5-th degree can be resolved by radicals.
- 1799 **P.Ruffini** states the existence of radically unsolvable equations of 5-th degree. But his publications contain some ambiguities. However, his "proof" was not accepted by the leading mathematicians of his and later times.
- 1799 **C.F.Gauß** proved in his dissertation that all (non-constant) algebraic equations are solvable with the help of imaginary numbers.
- 1824 **N.H.Abel** provided an exact proof for the existence of pentagonal equations unsolvable in terms of radicals.
- 1832 **E. Galois** found for arbitrary algebraic equations a precise solvibility criterion.

As already mentioned in the Timetable I, Carl-Friedrich Gauß (1777 - 1855) submitted his thesis [GD] in 1799. Not only the result was a big bang for the mathematical science around 1800, but also the skillful arithmetic - geometric application of imaginary numbers, which were generally outlawed until then. The domain of complex numbers $\mathbb{C} = \mathbb{R} + \mathbb{R} \cdot i$ was not only necessary, but was also proved to be sufficient for solving algebraic equations. In order to avoid difficulties with his referees, Gauß avoided to mention imaginary numbers in his main result: It was presented as "Decomposition Theorem for real polynomials into linear and quadratic factors".

Only in 1931 Gauß introduced the name "complex numbers" in his article about the biquadratic reciprocity law. The geometric interpretation in the plane together with the description of the four basic arithmetical operations he presented carefully. In his paper (see Gauß' Works **10**, No. 1) he remarked:

"With all the considerations are the imaginary quantities, as long as their fundament was only a fiction, not naturalized, rather considered as suffered, They remained a lot of time far away from elevation to one level with the real quantities. Now, there is no further reason for such a resetting, after the metaphysics of imaginary quantities has been putted in its true light, and since it has been proved that the imaginary numbers have an objective meaning as the negative ones."

[2]Radicals of a polynomial are numbers generated by means of the four basic arithmetic and root operations starting from the coefficients of the polynomial (assumed to be rational throughout this article.

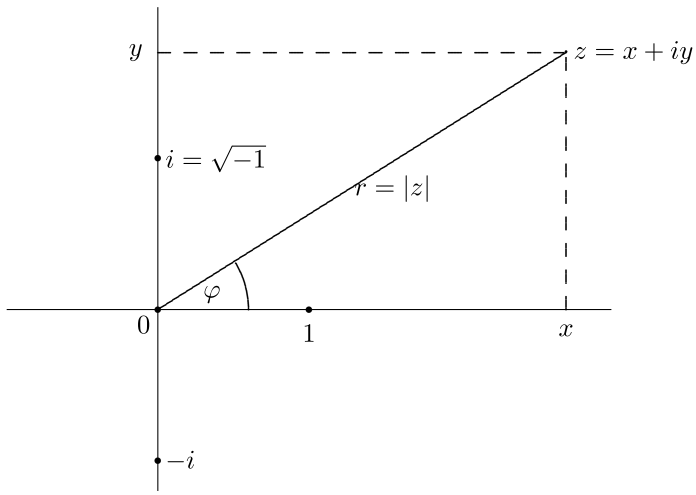
By virtue of the geometric interpretation as plane vectors, the complex numbers fitted into the well developed analytic geometry. Therefore the plane geometry together with trigonometric functions was available. We use the well-known notations for polar and cartesien coordinates for the complex numbers $z = x + yi \in \mathbb{C}$:

polar coordinates :

absolute value $r = |z| = \sqrt{x^2 + y^2}$, argument $\varphi = arg(z) = \arcsin(x/|z|)$,

cartesian coordinates :

real part $x = Re(z) = r \cdot \cos(\varphi)$, imaginary part $y = Im(z) = r \cdot \sin(\varphi)$.



Now we want to tinker a "solving navigator" for algebraic equations, which tracks down the zeros of any polynomial in a visible manner. More precisely, for a given polynomial some animations allow us to read off the polar coordinates of each root. On this way you can see in each case an optical proof of Gauß' solution theorem (Fundamental Theorem of Algebra). Let $P(x)$ be the given poloynomial with rational coefficients, say of degree $d > 0$. It provides a mapping $z \mapsto P(z)$ of the (Gauß) z-plane to the image w-plane.

We let run (in mind) a wave of circles $C_r : |z| = r$ with increasing radius r. Our first animation shows the image wave $O_r := \{P(z);\ |z| = r\}$ on the w-plane. It crosses ν times, $1 \leq \nu \leq d$, the origin of the coordinate plane. We stop the wave at one of the ν moments. The ticker on the picture reveals the preimage radius r. It is the absolute value of a zero lying on the ν-th circle of the z-plane. We fix the image curve O_r and use it as route for a drone, which starts at the intersection point of O_r with the positive part of the x-axis. This is the image point of $(r, 0)$ on the z-plane. Observe that O_r consists of all image points of (r, φ), $0 \leq \varphi < 2\pi$. We move the argument φ from 0 away till the image point of (r, φ) flying (like a drone) along the route O_r lands at the origin point $(0, 0)$ of the w-plane. The argument φ can be read from the ticker in circular measure. The pair (r, φ) is one of the zeros of P(x) in polar coordinates. Along the ν orbital routes you find all zeros of the polynomial in the same manner. Our watching precision of 3 digits can be enlarged to e.g. 20 digits by a simple button pressure activating a short procedure based on Newton's approximation. By our eyes we find roughly the five zeros (in polar coordinates):

$$\begin{aligned} z_1 &= (0.89, -\pi) \\ z_2 &= (1.43, 2.18) \\ z_3 &= (1.655, 0.72) \\ z_4 &= (1.43, -2.18) \\ z_5 &= (1.655, -0.72) \end{aligned}$$

With the commands $Newton(x^5 + 5x + 5, z_i, 9)$, $i = 1..5$, we get the zeros precisely up to a milliardth $= \frac{1}{10^{-9}}$ (after conversion in Gauß coordinates):

$$\begin{aligned} \zeta_1 &= -0.8889660370 & & \\ \zeta_2 &= -0.8026283841 & &+1.185081859i \\ \zeta_3 &= -0.8026283841 & &-1.185081859i \\ \zeta_4 &= +1.247111403 & &+1.090967675i \\ \zeta_5 &= +1.247111403 & &-1.090967675i \end{aligned}$$

In the MAPLE instructions one finds another (rigid) visual location of zeros by means of the absolute value surface. Five pigots in 3-space are directed to the zeros in the Gauß botton plane, see [A10]. The precision is low: It's difficult to read the numbers in an accurate manner. It's only a sketch, and no proof idea is visible.

We recommand the reader to apply the zero navigator (route finder with drone flights) to the polynomial function

$$P(z) = z^6 - 4z^5 + 2z^4 + 22z^3 - 89z^2 + 126z - 90$$

We reveal that all zeros are Gauß integers (belonging to $\mathbb{Z} + \mathbb{Z} \cdot i$). So one has an easy control by polynomial factor command

$$evala(Factor(P(z), I))$$

decomposing P(z) in linear factors in the polynomial ring $\mathbb{C}[z]$ over the field of complex numbers.

7 A View to Galois Theory

The zeros of any polynomial of degree ≤ 4 can be expressed by means of the four (elementary) basic operations $+, -, \cdot, :$ and the root operations, starting from the coefficients. Generally, for higher degrees, this not possible. Finally, after a half century of turmoils around equations of degree 5, the existence theorem for unsolvable ones (by radicals) was proved precisely by the young norwegian mathematician Niels Henrik Abel (1802 - 1829) in 1824. He worked with a polynomial factorization with indeterminated zeros. By some manipulations with them he could conclude that a randomly chosen polynomial is unsolvable in terms of its radicals. Unfortunately, one could not recognize in the first half of the 19-th century an explicit polynomial with this unsolvable property. For a concretely choosen pentagonal polynomial without obvious zeros one cannot be sure whether it is a radically unsolvable or not. For the decision a criterion was necessary. It was found some years later by the genial French mathematician Evariste Galois (1811 - 1832), see below.

At the age of almost 27 years Abel's life ended tragically. As a result of hard scientific work in poverty his body expired more and more during his last year of life. Until then his work found already highest respect among others by C.G. Jacobi (1804 - 1851), Legendre (1752 - 1833) und Gauß. Strong support came from A.L. Crelle (1780-1855). Only two days after the death of the young Norwegian mathematician Crelle held the commitment to the vocation for a chair at the University of Berlin (today Humboldt University) in his hands. It came too late for one of the significantest mathematical geniuses of the world history, see [Wu], 357-365.

The equally gifted French mathematician Evariste Galois found after Abel's work a necessary and sufficient criterion for the radical-solvibility of each algebraic equation. He considered substitutions of the abstract zeros among each others. Moreover, he organized them in a new structure, later called "group", more precisely, dealing with zeros of polynomials, "Galois group". This was the birth of group theory, today applied successfully in many branches of mathematics.

Proposition (Galois Criterion). *An irreducible polynom is solvable if and only if the its Galois group is solvable.*

Thereby an irreducible polynomial is called *solvable*, if all zeros are radicals of it. The notion of *solvable group* is explained in each algebra textbook, which includes an introduction to group theory, see also Google/Wikipedia.

Also tragically ended the life of E. Galois. Political circumstances leaded to the suicide of his father. Based on the background of such painful experience Galois took part on demonstrations against King Louis Philippe. His provocative appearance had been registered by the Secret Service. Probably with help of a whore Galois has been engaged in a duel he didn't survive. In the night before he wrote down his last important mathematical results. This manuscript has been known in the world as his "Mathematical Testament". Evariste Galois was only 20 years old. The physicist Leopold Infeld (1898 - 1868), a temporary colleague of Albert Einstein (1870 - 1955), wrote an interesting Galois Biography [In] allowing a deep insight into the French school system around 1830 embedded in the political circumstances.

To find an explicit simply built polynomial not solvable in radicals is by no means an easy undertaking. The German mathematician Issai Schur (1875 -

1941) found almost hundred years after the polynomial

$$Q = 5! \cdot \sum_{k=0}^{5} \frac{x^k}{k!} = x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120 \tag{2}$$

with unsolvable Galois group S_5 (permutation group of five elements), see Russian Encyclopedia [ME], I, S. 849. Immediately visible is the irreducibility of Q. It comes from Eisenstein's criterion for normalized polynomials with integral coefficients:

There is a prime number p dividing - except for the highest - all coefficients, but p^2 does not divide the last (the constant term).

Obviously, $p = 5$ satisfies the condition for Schur's polynomial. Also Weber's polynomial in the introduction satisfies obviously the above irreducibility criterion.

A speedy entry into Galois Theory, especially to line up the Galois group of an irreducible polynomial, one finds in M.S. Milne's [MG]. In the Appendix A-35 is announced the formally simple example $X^5 - 6X^4 + 3$ with Galois group S_5. We recommend the reader to determinate its zeros by means of our navigator drones.

An unsolved problem until now is to find for any given finite group G a polynomial with rational coefficients and Galois group G, or only to prove its existence. This is the *Inverse Problem* of Galois Theory. Restricting to some special types of groups or admitting another number field for coefficients the problem has been solved. For the general cases the number theorists will further work hardly.

For the actual arithmetic-geometric level of Galois Theory we refer to [AG].

8 From Complex Logarithm to Riemann Surface

A historical jump from dynamical elements of small graduated equations to the proof of Fermat's Last Theorem by Andrew Wiles at the end of last century leads us to Euler's investigations of the complex logarithm. The exponential function is explained by extension of the well-known series expansion - together with the necessary convergence check - from real to complex numbers:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

In contrast, the familiar series expansion of the (natural) logarithm

$$\ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - + ...$$

converges only for absolutely small real numbers. The extension from real to complex values came with some labor pains. Leonhard Euler wrote in [Eu], Kap. II:

"Although the theory of logarithms is rather firmly established, such that truths contained in it seem to be proven as strong as for basic facts of geometry, several mathematicians have a very different view to the nature of logarithms of negative and imaginary numbers."

A special occassion to this remark gave the attempt of well-known mathematicians to determine the value of negative and purely imaginary (Gauß) integers. Johann Bernoulli (1667 - 1748) stated that for any real number a should hold the identity $\ln(-a) = \ln(a)$. In his correspondence with Euler he offered weak arguments for his opinion. Leibniz, however, in his correspondental dispute with Euler, believed that logarithms of negative numbers are (purely) imaginary. Obviously, the above logarithm series is divergent for $x = -2$, e.g. For this reason it was audacious trying to solve the equation

$$e^y = -1$$

for detecting the value $y = ln(-1)$. Only 35 years after the above correspondences L. Euler clarified the situation. He gave four arguments supporting Bernoulli's belief and three for that of Leibniz. Subsequently he setted six respectively three objections against, in order to formulate and prove the following

Proposition (Euler, 1747, [Eu], II). *There is always an infinity of logarithms, which correspond to a given number in the same manner, or in other words, if the logarithm of the number x is denoted by y, then I claim that y includes an infinite number of different values.*

At the end of the essay one finds the formula

$$e^{f+g\cdot\sqrt{-1}} = e^f \cdot e^{g\cdot\sqrt{-1}} = e^f \cdot (\cos g + \sqrt{-1}\cdot \sin g)$$

(with real numbers f, g), from where the trigonometric periodicity is readable. We visualize this relation by an infinite spring. The latter represents the twisted imaginary axis of the Gauß plane. The exponential map appears as animation pressing the spring down to the unit circle of (complex) image plane.

The above proposition remained obscure for most of the mathematicians of the 18-th and 19-th century: an obstruction for accepting imaginary numbers. Only since the middle of the 19-th century a broader understanding of imaginary sizes prevailed. Multivalued functions could been better understood with the construction of "Riemann surfaces". The spring opens the door for illustrating the later uses of such surfaces: Imagine our spring widened to a spiral staircase. If you want to get all logarithms of a number x inside of the unit circular area, then you have only to start from one logarithm - it sits on the string staircase - and turn rounds up and down on our stair, siehe Animation [A4].

More precisely, let's move in the basic plane on a circle around 0 starting at x. We denote by y one logarithm value of x. It sits on the stair. Moving x

on the circle induces a ride of y on the spring staircase over the basic points. after one circulation of x the corresponding y's run through a bow on our stair. For several circulations the corresponding bows ly one over each other on the spring stair case. This principle of moving along a bow on a Riemann surface of a multivalued function over a bow in the argument plane is called *analytic extension* or *monodromy*. On this way we can, for example, real functions, given by series expansions, extend into a complex domain. Even if a real series is only locally convergent, it happens that we can extend the function (multivalued) around the critical point to a punctured complex neighbouhood. A first example was the logarithm function.

Euler's pioneering work is a launch platform for the building of modern complex analysis around such essential notions as *fundamental groups*, *universal coverings*, *fundamental domains*. Concerning logarithms, the map $\exp : \mathbb{C} \to \mathbb{C}^*$ appears as universal covering of $\mathbb{C}^*$. Thereby, the whole complex plane $\mathbb{C}$ is the universal cover of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The fundamental group is $\mathbb{Z}$ (integers). It measures the multivalence of the inverse map of the universal covering, in our special case: of the logarithm. A memorable one-dimensional (real) example yields the restriction of the exponential map to the imaginary axes. It is visualized by the above spring we started with. It has each bow covering biunivoquely the unit circle in the image plane as fundamental domain (fundamental bow in this case), look at [A4] again.

9 Numerical Prehistory of the Fermat Theorem

Theorem of Wiles (Fermat's Conjecture, Fermat's Last Theorem). *It is not possible to find three integers $a, b, c \neq 0$ and a natural number $n > 2$, such that*

$$a^n + b^n = c^n.$$

Generations of mathematicians worked hardly 360 years long on a proof of Fermat's conjecture, but without success. Fermat himself was able to prove the statement for $n = 4$ by a new method called "infinite descent": If there exists a positive solution tripel a, b, c, then one can construct from it a smaller positive one. The contradiction to the existence assumption follows obviously. Translated to geometry it means that the plane curve (Fermat quartic)

$$x^4 + y^4 = 1$$

supports no rational point except for the trivial ones $(0, 1), (1, 0)$.

The case $n = 3$ was treated successfully by L. Euler (ca. 1770). Independently confirmed P.G.L. Dirichlet (1805-1859) and A.M. Legendre (1752 - 1833) the conjecture for $n = 5$ in 1825. Both relied on the preliminary work of the female French mathematician Sophie Germain (1776-1831). She found a simultaneous approach for infinitely many special exponents n, see [Si], p. 128 ff. With a new conceptual approach E.E. Kummer (1810-1893) shifted important points to further developments of number theory. At a stroke he proved

Fermat's conjecture for all prime exponents $5 \leq n = p \leq 43,\ p \neq 37$. With help of strong computers the conjecture could be acknowledged till 1993 for all exponents $n < 4000000$, see [BCEM].

10 Non-Euclidean Preparations

Now we turn back again to Newton and Gauß .Henceforth, we allow in a cubic equation complex solutions. The complex zero set $E_{\mathbb{C}}$ appears as surface with complex structure embedded in the 3-space $\mathbb{R}^3$. If there is no singularity, then, after suitable choice of embedding, the image is a torus. The universal cover of $E_{\mathbb{C}}$ is the complex plane $\mathbb{C}$. The universal covering $\mathbb{C} \to E_{\mathbb{C}}$ can be realized by a double periodic complex function, a so called *elliptic function*. As fundamental domain F in $\mathbb{C}$ one can choose a parallogram with the two periods as putting up sides. We can translate F in both period directions such that the complex plane $\mathbb{C}$ is (not overlapping) covered by fundamental domains congruent to F. The shifts form an infinite group called *group of deck transformations*. It is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. It is the fundamental group of our biperiodic universal covering.

From F (assumed to be rectangular for simplicity) we can construct $E_{\mathbb{C}}$ in two steps: First put together two opposite sides of F to get a cylinder. The latter will be twisted such that the opposite boundary circles come together and we get a tyre (torus), see the pictures in [A2].

Before we come back to the non-euclidean geometry, we want to remember the foresight of Gauß on this place. In a letter to Schumacher (see [GS]) he wrote on the 17-th of Septembre 1808:

"To me, there is little intest in that aspect of integral calculus where we use substitutions, transformations, etc. - merely clever mechanical tricks - in order to reduce integrals to algebraic, logarithmic or trigonometric forms, as compared with the deeper study of those transcendental functions which cannot be so reduced. We are as familiar with circular and logarithmic functions as with one times one, but the magnificient goldmine which contains the secret of higher functions is still almost completely unknown territory. I have, formerly, done a lot of work in this area and intend to devote a substantial treatise to it, of which I have given a glimpse in my Disquisitiones Arithmeticae p. 593, Art. 335. One cannot help but be astounded at the great richness of the new and extremely interesting results and relations which these functions exhibit (the functions associated with rectification of the ellipse and hyperbola being included among them)."

The announced great work developed to become a neverending story. To work on this program remained reserved to generations of subsequent mathematicians of the 19-th and 20-th century and didn't stop until now. We register again the enormous thrust revealed in the above letter of Gauß .

Also in the 19-th century the non-euclidean geometry has been discovered. Gauß again had a substantial portion on the first fundamental steps. For the moment consider the following *modular figure*, see e.g. [B1]: It consists of infinitely many arc-limited triangles covering the unit disc, the closer to the boundary the smaller they are. In the hyperbolic geometry on the unit circular area $\mathbb{D}$ all these triangles have the same area size. They are congruent in the non-euclidean (hyperbolic) sense. Indeed, there is - comparable with parallelograms in $\mathbb{C}$ - a group moving each triangle to any other one. It consists of non-euclidean isometries transforming the circle area in itself (automorphisms). In their entirety we have to deal with the *modular group*

$$\mathbb{S}l_2(\mathbb{Z}) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),\ a,b,c,d \in \mathbb{Z},\ det(\gamma) = 1\}$$

It is easy to find subgroups Γ of finite index, which - for a suitable natural number $r > 0$ - act as fundamental group of a r-times punctured elliptic curve $E^*_{\mathbb{C}} = E_{\mathbb{C}} \setminus \{r\ Punkte\}$. As fundamental domain of the universal covering $\mathbb{D} \to E^*_{\mathbb{C}}$ appears a polygon area covered by $m := [\mathbb{S}l_2(\mathbb{Z}) : \Gamma]$ triangles of the modular figure[3].

An important role play - for natural numbers $N > 0$ - the (special congruence) subgroups of the modular group:

$$\Gamma_0(N) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{S}l_2(\mathbb{Z}),\ c \equiv 0\, mod\, N\}.$$

The elements act on the unit disc $\mathbb{D}$ as fractional linear transformations. The (non-compact) Riemann (quotient) surface $X^*_0(N) := \mathbb{D}/\Gamma_0(N)$ has the quotient map $\mathbb{D} \to X^*_0(N)$ as universal covering. By adding finitely many points (compactification) one gets the compact Riemann surface $X_0(N)_{\mathbb{C}}$. It can be visualized as multi pretzel, that means a pretzel with $g = g(N) \geq 0$ holes (instead of two). To them belong tori ($g = 1$) and the Riemann sphere ($g = 0$). In our phantasy we break out a connected field F of finitely many triangles from our modular disk triangulation and turn them together to a closed surface with some holes (multi pretzel). In [Sh], IX, 3.3 is visually remembered the classical construction with special attention to the formation of holes. Independently from our imagination the universal covering $\mathbb{D} \to X^*_0(N)$ can be realized by suitable F-periodic complex-analytic functions on $\mathbb{D}$ called *modular functions*. The field $F \subset \mathbb{D}$ of triangles is a fundamental domain of this map.

11 Jump into the 20-th Century

Already in the exponential series the sequence of coefficients is arithmetically striking. Extending the view to the series of periodic modular functions opened a large playground for outstanding mathematicians of the last two centuries. Leopold Kronecker (1823-1891) experienced his *liebsten Jugendtraum* (dearest dream of youth) intertwining deeply number and function theory. This has been taken by David Hilbert (1862-1943) in his celebrated problem presentation at the International Math. Congress in Paris, 1900. The central 12-th Problem was a far-reaching generalization. Starting from Kronecker's dream he saw a futural development:

[3] We assume that $diag(-1,-1)$ belongs to Γ. Otherwise take $m/2$ instead of m.

”It will be seen that in the problem just scetched the three fundamental branches of mathematics, number theory, algebra and function theory, come into closest touch with one another, and I am certain that the theory of analytic functions of several variables in particular would be notably enriched if one should succeed in finding and discussing those functions which play the part for any algebraic number field corresponding to that of the exponential function in the field of rational numbers and of the eliptic modular functions in the imaginary quadratic number field.”

Through a century the further development of mathematics had been influenced intensively by Hilbert's problems. In the first half of the last century the affirmative solution of Kronecker's problem had been clarified essentially. After World War II the Japanese mathematicians G. Shimura (born 1930) und Y. Taniyama (1927-1958) advanced to higher dimensions. Still in the foreword of their much observed book [ST] they announced Hilbert's 12-th Problem as a guiding star, although there cannot exist a closed solution of its higher dimensional second part. We have to deal with a broad extension of Gauß' infinite story. Taniyama wrested a pregnant conjecture from there, see below.

12 The Great Fusion

Taniyama Conjecture, Modularity Theorem (1956, z.B. [MF], 11.22, p.112). *Let E be an elliptic curve over $\mathbb{Q}$ wth geometric conductor*[4] *N. Then E is modular of level N, this means that there is a non-constant $\mathbb{Q}$-algebraic map $X_0(N) \to E$.*

$E/\mathbb{Q}$ (to read: E over $\mathbb{Q}$) will be an abbreviation for the property of E to be defined by an equation with rational coefficients. $\mathbb{Q}$-algebraic means that the map is defined by polynomials with rational coefficients.

For illustration we consider the complexification $X_0(N)_{\mathbb{C}} \to E_{\mathbb{C}}$ of the $\mathbb{Q}$-algebraic map as in the above conjecture. We have to deal with a finite covering of Riemann surfaces. Let's call back to our minds the triangulized torus. Namely, the surface $X_0(N)_{\mathbb{C}}$ is also the image of the non-euclidean unit disc. Along the complex-analytic composition map $\mathbb{D} \to X_0(N)_{\mathbb{C}} \to E_{\mathbb{C}}$ the triangulation net of $\mathbb{D}$ is projected onto $E_{\mathbb{C}}$ via $X_0(N)_{\mathbb{C}}$. In [B2] you find a rough geometric visualization (a finitely triangulated torus) of the Taniyama Conjecture[5].

Over several decades, mathematical luminaries tried to prove the conjecture. They were successful only for some special cases of elliptic curves. The inte-

[4]The *geometric conductor* of an elliptic curve $E/\mathbb{Q}$ is a squarefree natural number. The precise general definition (see e.g. [MG]) is not needed in this top.

[5]In order to spiritualize the non-euclidean structure on the elliptic curves (tori) of the Taniyama Conjecture we turn back to the lifetime of Jules Verne: Imagine a temperature distribution on the disc $\mathbb{D}$: It's warm at the center but becomes colder and colder near to the boundary with absolute small degree of cold there. This contribution is transfered to the triangulated torus by means of the fundamental triangle field F. Now we dispose on a location dependent continous measure on the triangulated torus. At some triangle vertices it's absolutely cold.

rest diminished more and more until the German mathematician Gerhard Frey discovered a connection between elliptic curves and Fermat's conjecture:

Let a, b, c with $abc \neq 0$ be a coprime integer triple solving the Fermat equation $x^l + y^l = z^l$, $l > 2$. Without loss of generality (easy to see) we can assume that the exponent l can be assumed to be a prime number. Frey considered now the elliptic curve

$$E_{a,b,c}: \ y^2 = x(x - a^l)(x + b^l) \tag{3}$$

(Frey curve, should be a phantom). Its geometric conductor $N_{a,b,c}$ is a squarefree factor of von abc.

Suddenly, after break downs of untiring experts of arithmetic, algebraic and analytic geometry, the interest awoke once again. The following deep result was a consequence of the new power:

Ribet's Descent Theorem (Ribet [RT], 1990). *Is the Frey-curve $E_{a,b,c}$ modular of conductor level $N_{a,b,c}$, then for any prime p of this level the curve is also modular of the smaller conductor level $N_{a,b,c}/p$, if $v_p(a^{2l}b^{2l}c^{2l}/2^8)$ is divisible by l.*[6]

In the mean time the British number theorist Andrew Wiles worked six years secluded: He recognized a connection between Taniyama's and Fermat's conjecture, and he felt the time ripe enough to prove both with all the already created techniques and some to be added by his own work. Wiles left the collegues around completely in the dark about his intentions. At the end he succeeded to verify Taniyama's conjecture for a broad class of elliptic curves, namely for *semistable* ones including those of Frey (if such exist). There was a gap which could be removed a year later in cooperation with his collegue R. Taylor (1995). The famous consequence was Fermat's Last Theorem, the change from conjecture to a proven fact. Before we draw it, let us indicate that the Modularity Theorem (Taniyama Conjecture) had been proven completely a littlebit later, that means for **all** elliptic curves $E/\mathbb{Q}$, at turn of the millenium (see [BCDT]).

Significant step of proof: From Taniyama to Wiles (Fermat).

Assume that there is a non-trivial solution triple a, b, c of a Fermat equation and therefore exists a Frey curve as described in (3). We know already that the geometric conductor $N_{a,b,c}$ is a squarefree product of prime divisors of $N_{a,b,c}$. Now it is a simple exercise to prove with the help of Ribet's Descent Theorem that one can go down to the modular level 2, if 2 is divisor of the above conductor (even to 1, if 2 is not a divisor). Therefore, in any case there exists a complex-analytic covering map $X_0(2)_{\mathbb{C}} \to E_{\mathbb{C}}$ for $E = E_{a,b,c}$. It is known that $X_0(2)_{\mathbb{C}}$ is nothing else but the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ (also known as complex projective line). In first lessons about Riemann surfaces one learns that there can never be a covering map $\mathbb{P}^1_{\mathbb{C}} \to E_{\mathbb{C}}$ for any elliptic curve E. Hence Ribet's Descent

[6] $a^{2l}b^{2l}c^{2l}/2^8$ denotes the (generally for elliptic curves defined) *minimal discriminant* of the Frey curve, where $v_p(M)$ denotes the maximal exponent m, satisfying $p^m|M$.

Theorem leads us to emptyness. Therefore the Frey curve must be a phantom. Then the non-trivial solution triple of a Fermat curve is a Fata Morgana, too. In other words there is no non-trivial $\mathbb{Z}$-solution of any Fermat equation

$$x^n + y^n = z^n,\ n > 2.$$

Fermat's Last Theorem is proved.

Closing Remarks All the proofs required an intimate fusion of (sometimes infinite) Galois theory of number fields, the (analytic-arithmetic) theory of L-series, theory of modular forms and of algebraic geometry. This synthesis had been advanced some decades before by leading experts of arithmetic geometry in the world centers of mathematics. Already the Theorem of Faltings (Mordell Conjecture) marked a high level of it. We dispense here with the appointment of further involved mathematicians, in order to remain understandable for a wide audience. For further deepening of knowledge we refer the reader to the Google posts "Taniyama-Shimura Conjecture - from Wolfram MathWorld" and "Modularitätssatz, Wikipedia".

A nice emotional access to the dramatic development of ideas leading at long last to the spectacular proof of Fermat's Great Theorem one can find in the book [Si] of Simon Singh. Only basic mathematic knowledge (of highschool level) is necessary to follow the exciting presentations of the author.

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www.facebook.com/Rolfpeter.holzapfel

[A1] Non-euclidian triangulated torus / Rad

[A2] Euklidian Torus

[A3] Egg of the Columbus

[A4] Logarithm spirale / spiral staircase

[A5] Dance on the hyperboloid

[A6] Oppositely twisted cylinders

[A7] Hyperbolic gear

[A8] Newton curves

[A9] Solution drones of an equation (visualizing Gauß́thesis)

[A10] Digitale cone sections

[A11] Dynamical solution spigots of the equation

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[B1] Non-euclidean (infinitely) triangulated disc

[B2] Non-euclidean triangulted torus

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