International Research Training Group GRK 870

Arithmetic and Geometry

Orbital Surfaces, A Mathematical Wave Through Centuries

D. Hilbert, Paris 1900:

Wie wir sehen, treten in dem eben gekennzeichneten Problem die drei grundlegenden Disziplinen der Mathematik, nämlich Zahlentheorie, Algebra und Funktionentheorie in die innigste gegenseitige Berührung,...

L. Euler's discovery of second order partial differential equations for wave motion of gas, Berlin/Petersburg 1769

$$\frac{\partial^2}{\partial z_1 \partial z_2}F + \frac{1}{z_2 - z_1}\left(a\frac{\partial}{\partial z_1} - b\frac{\partial}{\partial z_1}\right)F = 0,\ F = F(z_1, z_2).$$

B. Riemann, Göttingen 1860:

Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite, Abh.d.Kgl.Ges.d.Wiss.Göttingen **8, 3-25.**
Contains solution-base (idea) by means of modular forms.

C.F. Gauss, Göttingen 1813

Hypergeometric Differential Equations/Functions:

$$z(1-z)f'' + (c - (a+b+1))f' - abf = 0,\ f = f(z).$$

H.A. Schwarz, from Weierstrass School, Berlin/Göttingen/Zürich, Crelle Journal, 1873

Discovery of "Inverse Integral Conditions" for nice monodromic solutions, with inverse automorphic functions on the upper half plane $\mathbb{H}$, of (all possible) triangle groups/lattices:

$$(IIC\,1) \qquad 1 - c = \frac{1}{l}\ ,\ c - a - b = \frac{1}{m}\ ,\ a - b = \frac{1}{n}$$

É. Picard, from Hermite School, Paris, $\sim$ 1885

Inverse Integer Conditions for nice monodromic solutions with automorphic inverse map on the complex 2-ball $\mathbb{B}$, e.g. hypergeometric integrals (take pairs of quotients):

$F(z_1, z_2) = \int u^{\lambda_1 - 1}(u-1)^{\lambda_2 - 1}(u - z_1)^{\lambda_3 - 1}(u - z_2)^{\lambda_4 - 1}du$

$$(IIC\,2) \qquad \begin{cases} \lambda_i + \lambda_j & = \frac{1}{m_{ij}}, \quad i \neq j \\ 2 - \sum\limits_{j \neq i} \lambda_j & = \frac{1}{n_j}\ , \quad i = 1,..,4. \end{cases}$$

G.D. Mostow, P. Deligne, USA, $\sim$ 1985:

Proved with modern methods (e.g. Gauss-Manin connection) that $(IIC\ 2)$ is also sufficient for the existence of monodromic/automorphic solutions (hypergeometric functions/series) w.r.t. a ball lattice $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$. Moreover, $\Gamma\backslash\mathbb{B}$ leads back (up to birationality) to the monodromy starting plane. Euler-Picard system of DE's: R.-P. Holzapfel, 1983/86.

F. Hirzebruch (with Barthel, Höfer), Bonn, 1985:

Observed that $(IIC2)$ is a system of diophantine equations solved already completely by the doctorand Le Vavasseur of Picard in 1896 (thesis). Hirzebruch found the following Geometrization: The solutions represent branch divisors, supported by exceptional lines, of the Γ-quotient maps $\mathbb{B} \to F$, where F is the del Pezzo surface of degree 5.

G. Shimura, 1963

Domain/Modular group classification of (todays) Shimura-Varieties. Shortly before, together with Taniyama: Class field constructions by means of abelian CM-varieties and corresponding moduli points on the Shimura varieties.

1980-s: Birth of Picard Modular Varieties/Groups/Forms

as special cases (unit ball) of Shimura varieties. Rough and first fine classification of Picard modular surfaces (R.-P. Holzapfel/ J.-M. Feustel, Berlin).

Recognition of ball representations of PTDM-(Picard, Terada, Deligne, Mostow) mononodromy groups, also of Hirzebruch's uniformizing ball lattices, as explicite Picard modular groups:

1983, 1986, 2004, 2009 (R.-P. Holzapfel).

Yau, Miyaoka, Mumford 1977; Hirzebruch 1987

Proportionality formulas for ball quotient surfaces of neat ball lattices: $c_1^2 = 3c_2$, or branch-modified $\mathbb{P}^2$-line-arrangement-version, respectively (geometric version of Inverse Integral Conditions).

General Prop Formulas for smooth ball quotients

$\Gamma\backslash\mathbb{B}$ with normal crossing model X' and branch divisor transform $\mathbf{vC}' := v_1C_1' + ... + v_nC_n'$ on X' (Holzapfel, 2009):

$$(Prop\,1) \quad \mathbf{e} - (\mathbf{1} - \mathbf{v}^{-1}) \cdot S\ -\ \mathbf{h} = 2S \circ \mathbf{v}$$

$\mathbf{v}$, $\mathbf{h}$, $\mathbf{e}$, (branch-, cusp-, Euler vector) , $\mathbf{1}$, $\mathbf{v}^{-1} \in \mathbb{Q}^n$ ($\mathbb{Q}$-algebra with componentwise multiplication $\circ$); $S + D$, (D): (self) intersection matrix of $\mathbf{C}' = \sum C_i$.

$$(Prop\,2)$$
$$e(X') - 2H_0 - 2(\mathbf{1} - \mathbf{v}^{-1}) \cdot D \cdot {}^t\,\mathbf{v}^{-1}$$
$$-\frac{1}{2}(\mathbf{1} - \mathbf{v}^{-1}) \cdot S \cdot {}^t\,(\mathbf{1} - \mathbf{v}^{-1})$$
$$= 3\tau(X') - (\mathbf{v} - \mathbf{v}^{-1}) \cdot D \cdot {}^t\,\mathbf{v}^{-1} - (T^2)$$

$e(X'), \tau(X')$: Euler number, signatur of X'; T: compactification divisor; H_0: number of rational cusps.

Remark. $(Prop\,1)$ consists of n diophantine equations; with $(Prop\,2)$ we have altogether $n+1$ of them.