The Orbital Principle

Orbital Surfaces, (smooth, normal crossing versions):

Pairs $(X', \mathbf{v}\mathbf{C}') =: \mathbf{X}'$ as in $(Prop\,1), (Prop\,2)$.
Category with "(orbital) finite coverings".

Orbital Invariants:

Contravariant functors into rings R; most important into $R = \mathbb{Q}$ or $\mathbb{R}$ (numerical orbital invariants), for instance

Orbital Chern Invariants:

$\mathbf{Eul}(\mathbf{X}')$ Orbital Euler Number (left side of $(Prop\,2)$),
$\mathbf{Sig}(\mathbf{X}')$ Orbital Signature (right side of $(Prop\,2)$).

On the subcategory of orbital quotient surfaces of natural congruence subgroups $\Gamma_K(N)$ of Picard modular groups,
$\Gamma_K = \mathbb{SU}((2,1), \mathfrak{O}_K$, $\mathfrak{O}_K$ ring of integers of any imaginary quadratic number field K, we discovered by deep number theory also the following

Arithmetic Orbital Invariant:

$$X' = (\mathbf{\Gamma_K(N)}\backslash\mathbb{B})' \mapsto \frac{N^8\cdot L(3,\chi_K)}{\prod_{p|N}(1-\frac{1}{p^2})^{-1}\cdot\prod_{p|N}(1-\frac{\chi_K(p)}{p^3})^{-1}},$$

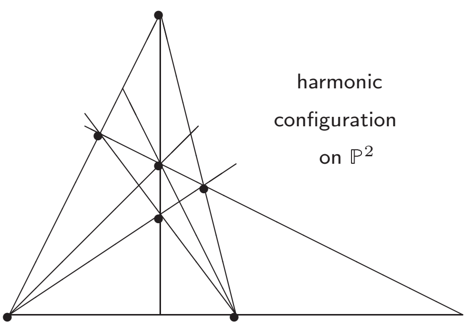
where $L(s,\chi_K)$ is the L-series of the Dirichlet character χ_K.

$$(Orb\,2) \quad \mathbf{Eul}(\mathbf{X}') = q_{N,\chi}\cdot L(-2,\chi) = 3\cdot\mathbf{Sig}(\mathbf{X}')$$

with explicitly known $q_{N,\chi} \in \mathbb{Q}$, closely connected with the above $\mathbb{Q}$-factor of $L(3,\chi)$ and well-calculable $L(-2,\chi)$ (via higher Bernoulli numbers). Notice, that we used the functional equation for Dirichlet L-series in order to change from the transcendental 3-value to the rational value at -2.
For the proof of $(Orb\ 2)$ one uses 3 integrals $\int_{\mathfrak{F}(\Gamma)}\mu$ with Lie-invariant volume forms μ on $\mathbb{B}$, where $\mathfrak{F}(\Gamma)$ denotes a Γ-fundamental domain of Γ, and compares them.

Remark. This is a splitting of the diophantine equation $(Prop\,2)$ into two of them. Knowing enough about $\Gamma_K(N)$ and some geometry of the corresponding orbital surface (also hidden in $(Prop\,1)$, which could be analogously doubled to a system $(Orb\,1)$ with L-values at $s = 2$ by a result of J.-M. Feustel), enables to solve $(Orb\ 2)$ uniquely and to classify the surface together with the branch curves.
R.-P. Holzapfel, 2009

Application:



harmonic configuration on $\mathbb{P}^2$

Blow up the 7 marked points, then blow down the 6 lines supporting 3 of them to surface singularities.

Then you get the Baily-Borel compactification $\widehat{\Gamma_{\mathbf{Q}(i)}(2)\backslash\mathbb{B}}$. It has six cusp singularities; they coincide with the surface singularities above.

This is the door for finding explicit Shimura curves together with infinite Galois towers of them, an explicit door for Coding Theory, for Th. Zink's (abstract) approach to optimal codes.

See M. Petkova, Thesis, 2009. / Background:

Th. Zink (with Tsfasman, Vladut), 1982

Break of Varshanov-Gilbert bound in coding theory by means of infinite towers of quaternionic Shimura curves.

Outlook to Picard modular forms:

Working with moduli surfaces of genus-3 curves one gets explicitly symplectic embeddings $\Omega : \mathbb{B} \hookrightarrow \mathbb{H}_3$. K. Matsumoto and H. Shiga (1988/2006) found in the case of Gauss numbers a theta-constant presentation for $\Gamma_{\mathbf{Q}(i)}(2)$-modular forms. For ball points $(u,v) \in \mathbb{B}$ define first:

$$\vartheta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right]((u,v)) := \sum_{\mathbf{n}\in\mathbf{Z}^3} \exp\left[\pi i(\mathbf{n}+\frac{\mathbf{p}}{2})\Omega(u,v)^t(\mathbf{n}+\frac{\mathbf{p}}{2}) + \pi i(\mathbf{n}+\frac{\mathbf{p}}{2})^t\mathbf{q}\right]$$

with known characteristics $\mathbf{p}/2, \mathbf{q}/2 \in \{0, 1/2\}^3$. It holds that

Fourth powers of them are $\Gamma_{\mathbf{Q}(i)}(2)$-modular forms. There are enough for a projective embedding.

A similar result (with third theta-powers) holds for the most classical Picard case of the field $K = \mathbb{Q}(\sqrt{-3})$ of Eisenstein numbers (Shiga, publ. 1988). Here also the Fourier expansion has been calculated (see also a 1998-publ. of T. Finis).

In the frame work of Hilbert's 12-th Problem
we obtain explicit class field contructions
with special values of the modular forms above.

R.-P. Holzapfel, ETH-lectures 89/90, publ. 1995;
Th. Riedel, Ar. Geo. Conf., Istanbul 2005.
In the same Conf. you find the 1-dimensional modular form

$$\sum_{N=0}^{\infty}\left((\tfrac{3N}{2}-\tfrac{1}{8})a_2(N) + 3\sum_{m=1}^{N}\sigma(m)a_2(N-m)\right)q^N$$

$q = \exp(2\pi i\tau)$, $\tau \in \mathbb{H}$, $\sigma(m)$: Summe der Teiler von m; $a_2(k)$: Number of $\mathbb{Z}$-solutions of $x^2 + y^2 = k$.

Theorem. The (Fourier-)coefficients count the number of Shimura curves of given norm N on a certain Picard modular plane (of Gauss-numbers):

Special case of extension from neat work of S. Kudla, J.W. Cogdell, 1983, to all ball lattices. (R.-P. Holzapfel, 2003)