On Smooth Picard Modular Orbifaces

R.-P. Holzapfel

In memory to my academy collegue J.-M. Feustel, [1]
17. Jan. 1951 - 15. Jan. 2009

1 Preface

An important guideline for our common work at the Berlin Academy were the words of D. Hilbert: "Wie wir sehen, treten in dem eben gekennzeichneten Problem die drei grundlegenden Disziplinen der Mathematik, nämlich Zahlentheorie, Algebra und Funktionentheorie in die innigste gegenseitige Berührung,..." They were given during his talk 1900 in Paris reflecting a deep understanding of the mathematical developments in the 18-th and 19-th century with a view to the future. Recently you can find the intimate touch in [HUY 07], indeed.

What I found in the 1970-th was a ditch between the highly developed monodromy theory of complex analysis and Shimura's very arithmetic moduli theory. With modern techniques of algebraic geometry it should be possible to build a bridge between differential equations and class field constructions in dimension > 1. On the analytic side we found hyperbolic lattices generated by some well-represented monodromy elements; on the other hand arithmetic lattices, e.g. of unitary type, played the main role in the fruitful Shimura theory. Feustel and me began with a study of single objects on the complex two-dimensional unit ball hoping for a successful synthesis. After first progress we called our objects "Picard modular groups", "Picard modular surfaces", in honor of the analytic work of Picard, at the end of the 19-th century, and the modular work of Shimura. These are brothers of the Hilbert modular surfaces, which were fascinating models for our purposes, reflecting also Hilbert's guideline. The research center for these objects was in Bonn with Hirzebruch as head of an international team. But beside of the hidden mathematical ditch there was a cold war wall through Germany. Our mathematics overjumped this wall in a remarkable manner. I learned at the Steklov-Institut of Mathematics of the Academy of Sciences of the USSR, that the best developments in mathematics in Germany happened during the 70-s without any doubt in Bonn. There were two political axioms in the GDR:
1.) To learn from Soviet Union, 2.) Don't follow calls from a NATO country.
Contradiction !

[1] This note is based on a lecture in the Arithmetic Geometry Seminar at Humboldt-University Berlin, January 2009. It was held in English because of the international audience.

Finally, we learned from both, from Soviet Union and from Bonn. After some years we were able to give also an impuls over the wall. I remember that Hirzebruch himself reviewed a paper of Feustel in the "Mathematical Reviews". At the end of Feustel's mathematical activities his foto appeared in "The Mathematical Intelligencer" at the begin of 1990, shortly after breaking the wall.

In order to localize Feustel's work inside of a longtime development, I start with a historical background. I could take Euler as starting point (see [Ho 86]). But I choosed two young mathematicians of legendary schools in Berlin and Paris as starting points. The first, H.A. Schwarz, comes from the Weierstraß school and the second, E. Picard, from Hermite's school. Next we describe the Proportionality Principle, which was actual during our common work. In the last section I establish the recent and finer Orbital Principle with application motivited by coding theory: I announce the first classification of the Picard modular surface of a (honest) natural congruence subgroup.

2 Historical Background

Complex dimension 1

We remember to Berlin's celebrated Weierstraß school. From there came Hermann Amandus Schwarz. After Gauß - with a breaking through case in complex function theory - he studied more generally *hypergeometric functions* as solutions of the ordinary *hypergeometric differential equations*

$$(1) \qquad z(1-z)f'' + (c-(a+b+1))f' - ab\cdot f = 0.$$

with function $f = f(z)$ of a complex variable z and constants a, b, c. For them the *Inverse Integer Condition* was introduced:

There exist positive integers l, m, n such that

$$(IIC\,1) \qquad 1-c = \tfrac{1}{l}\ ,\ c-a-b = \tfrac{1}{m}\ ,\ a-b = \tfrac{1}{n},$$

holds.

The Inverse Integer Relation $(IIC\,1)$ is interpreted as system of diophantine equations for integers l, m, n and rationals a, b, c with numerators restricted by $2lmn$ (by Cramer's Rule).

Assume that $(IIC\,1)$ is satisfied for a triple l, m, n. We fix it (together with the corresponding triple a, b, c). Moreover, we assume that

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1.$$

By the Gauß -Schwarz theory, one gets *hypergeometric integral* solutions of (1)

$$f(z) = \int u^{a-c}(u-1)^{c-b-1}(u-z)du,$$

taken over some special pathes. Among them there are two linearly independent ones. The quotient of both around a point of $\mathbb{C} \setminus \{0,1\}$ determines holomorphic function germs. Take one of them. It can be analytically extended (*monodromy*) to a multivalued function from $\mathbb{P}^1 \setminus \{0, 1\infty\}$ to the disc $\mathbb{D}: |\zeta| < 1$. The inverse function is a univalent holomorphic function $\varphi = \varphi(\zeta)$. This is an *automorphic function*. Precisely, it realizes analytically the quotient map

$$\varphi: \mathbb{D} \longrightarrow \Gamma \backslash \mathbb{D} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

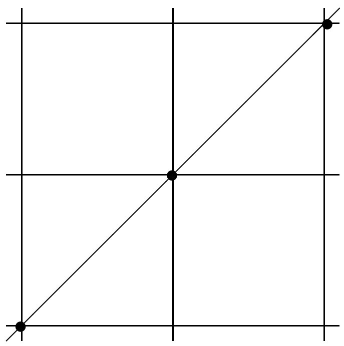
where $\Gamma = \Gamma_{l,m,n}$ is a well-determined discrete subgroup of $Aut_{hol}\mathbb{D} = \mathbb{PS}l_2(\mathbb{R})$. The fundamental domain of Γ is a hyperbolic triangle in $\mathbb{D}$ with angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$. The $\mathbb{D}$-lattices $\Gamma_{l,m,n}$ are called *Schwarz triangle groups.* [2]

Complex dimension 2

The work of the Weierstraß school in Berlin (also Kummer, Kronecker) was carefully studied by the members of the Hermite school in Paris. Émil Picard tried to lift the Schwarz theory to the second dimension with visible success. He presented first results in the 1880-s. Picard considered *two-dimensional hypergeometric integrals*

$$f(z_1, z_2) = \int u^{\lambda_1 - 1}(u-1)^{\lambda_2 - 1}(u - z_1)^{\lambda_3 - 1}(u - z_2)^{\lambda_4 - 1} du, \tag{2}$$

along special pathes avoiding $z_1, z_2 = 0, 1, \infty$ and $z_1 = z_2$. Values and integrations go around the 7 lines drawn in the following picture on the complex biproduct surface $\mathbb{P}^1 \times \mathbb{P}^1$:



$\Lambda = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{7 \text{ lines}\}$

Figure 1

They are solutions of an "Euler-Picard system of partial differential equations" (EP2) of two complex variables z_1, z_2. We refer to [Ho 86] for explicit equations and mention here only that they have rational functions of 2 variables as coefficients with poles at most on the 7 lines drawn in the above picture.
Picard introduced the following *Inverse Integral Condition 2* for the exponents in the integrand of (2):

[2]For more details we refer to [Y 97] and all the references given there.

$$(IIC\,2)\qquad \begin{cases} \lambda_i+\lambda_j & = \frac{1}{m_{ij}} \in \mathbb{N}^{-1}, \quad i\neq j \\ 2-\sum\limits_{j\neq i}\lambda_j & = \frac{1}{n_j} \ \in \mathbb{N}^{-1}, \quad i=1,..,4. \end{cases}$$

with convention $\infty = 0^{-1} \in \mathbb{N}^{-1} = \{\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, ...\}$. We remark that this has been also allowed earlier in the 1-dimensional case (Gauß, Schwarz, Riemann).

Theorem 2.1 *(stated by Picard). If and only if the 2-dimensional Inverse Integral Condition $(IIC\,2)$ is satisfied, then*

i) At each point on

$$\Lambda = (\mathbb{C}^* \setminus \{1\})^2 \setminus \{diagonal\} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{\textit{7 lines}\}$$

there is a basis of local solutions $U_0(z_1,z_2), U_1(z_1,z_2), U_2(z_1,z_2)$ of the Euler-Picard DE-system (EP2) consisting of hypergeometric integrals (2) (along different pathes).

ii) By monodromy around the 7 lines we can extend the quotients U_i/U_0, $i=1,2$, to locally holomorphic functions along pathes in $\mathbb{P}^1\times\mathbb{P}^1\setminus\{\textit{7 lines}\}$. Following different pathes, one gets multivalued functions.

iii) Choosing suitable coordinates, each projective map germ $(U_1 : U_2 : U_0)$ has values only in the complex ball $\mathbb{B}:\ |u_1|^2+|u|^2<|u_0|^2$.

iv) There is a discrete subgroup $\Gamma = \Gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ of the projective unitary group $\mathbb{PU}((2,1),\mathbb{C}) = Aut_{hol}\mathbb{B}$ uniformizing the multivalued map $U=(U_1:U_2:U_0)$. This means that for the (local finite analytic) quotient morphism $p:\mathbb{B}\longrightarrow\Gamma\backslash\mathbb{B}\supset\Lambda$ it holds that $\Gamma\backslash\mathbb{B}$ contains Λ and $p\circ U = id_\Lambda$.

The 19-th century was not ripe enough for giving a complete proof. The main gap was the statement, that the factorization in iv) leads back (up to a small set) to Λ. Hundred years later this gap has been closed by some work of Mostow and Deligne [DM 86] by means of the theory of Gauß-Manin connections. But already in 1896 LeVavassur solved completely in his thesis [LV 1896], advised by Picard, the Inverse Integer System $(IIC\,2)$.

Hirzebruch a.o. gave in [BHH 86] a nice geometric interpretation. After blowing up the 3 triple points of the 7-line curve in Figure 1 one gets the del Pezzo surface of degree 4. The inverse image of the curve is supported by 10 exceptional lines. The solutions of the Inverse Integer System are now transformed to ramification indices v_i of the coverings $\mathbb{B}\to\Gamma_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}\backslash\mathbb{B}$, at the 10 curves ($i=1...,10$). The finitely many solution tupels $(v_1,...v_{10})$ of a transformed Inverse Integer System allow some v_i to be ∞. The ∞-lines appear as compactification lines of the quotient surfaces. For the ball lattices only monodromy generators were known. At the end of the lecture we give an example in terms of arithmetic groups in Shimura theory.

3 The Proportional Principle

We consider pairs $(X', C'+T)$ with a smooth compact complex algebraic surface X' supporting a reduced divisor

$$C' + T = C'_1 + ... + C'_n + T_1 + ... + T_H$$

with *normal crossings*. This means (in our context) that all irreducible components C'_i and T_j of C' or T, respectively, are smooth, the intersections of two components are transversally and $C' + T$ has no other singularities than double points. We call the pair a *normal crossing model of a smooth orbital ball quotient surface*, say of $\Gamma\backslash\mathbb{B}$, if there exists a contraction $\hat{\rho}: X' \to \hat{X}$ onto the Baily-Borel compactified ball quotient surface $\hat{X} = \widehat{\Gamma\backslash\mathbb{B}} = \Gamma\backslash\mathbb{B} \cup \{K_1, ..., K_H\}$ such that

a) Two different curves on X' have different images on $\hat{X}$;

b) the curves T_j's are contracted to cusp points $K_j \in \hat{X}$;

c) The *finite part* $\hat{X} \setminus \{K_1, ..., K_H\} = X' \setminus$ *supp*(T') is smooth;

d) The image curve $\hat{C}$ of C' is the compactified branch divisor $\hat{C}$ of the quotient morphism $p: \mathbb{B} \to \Gamma\backslash\mathbb{B}$;

Consequences (see e.g. [Ho 98], ch. IV):

e) Each of the curves T_j is an elliptic curve or isomorphic to the complex projective line $\mathbb{P}^1$;

f) The image curve $\hat{C}$ of C' has at most (ordinary) triple points as singularities on the finite part;

g) the restriction ρ of $\hat{\rho}$ to the finite part is the simultaneous σ-process at all finite triple points of $\hat{C}$. Altogether, the exceptional divisor of $\hat{\rho}$ can be written as

$$E_{\hat{\rho}} = L_1 + ... + L_R + T_1 + ... + T_H,$$
$$R := \#\{\text{ finite triple points of } \hat{C}\}, \; H = \#\{\text{ cusp points of } \hat{X}\}$$

with exceptional curves of first kind $L_i \cong \mathbb{P}^1$ (selfintersection -1).

The attached Intersection Space

Consider the space $Div_{\mathbb{Q}}X'$ of Weil-Divisors on X' with coefficients in $\mathbb{Q}$ endowed with the symmetric intersection form $(.,.)$. We restrict the form to the n-dimensional subspace

$$\mathbb{Q}^n \circ \mathbf{C'} := \mathbb{Q} \cdot C'_1 + ... + \mathbb{Q} \cdot C'_n, \quad \mathbf{C'} := (C_1, ..., C_n).$$

The Gram matrix with respect to the basis $\mathbf{C}'$ is nothing else but the intersection matrix $G := ((C_i \cdot C_j))_{i,j=1..n}$. Sending the basis $\mathbf{C}'$ to the canonical basis we get an isomorphy of quadratic spaces

$$(\mathbb{Q}^n \circ \mathbf{C}', (.,.)) \cong (\mathbb{Q}^n, G)$$

from the $\mathbf{C}'$-*intersection space* onto its coordinate space. We decompose the intersection matrix G in the sum $D + S$, where D is the diagonal matrix with same diagonal as D, called *selfintersection matrix of* $\mathbf{C}'$. The pair $(\mathbb{Q}^n, D)$ is called *selfintersection space of* $\mathbf{C}'$. The complementary matrix $S := G - D$ is called the *honest intersection matrix* and $(\mathbb{Q}^n, S)$ the *honest intersection space* of $\mathbf{C}'$.

We deduce from $\mathbf{C}'$ some numerical vectors in $\mathbb{Q}^n$, namely

(i) the *selfintersecion vector* $\mathbf{s} = (s'_1, ..., s'_n) := ((C_1'^2), ..., (C_n'^2))$;

(ii) the *Euler vector* $\mathbf{e} := (e_1, ..., e_n)$, where e_i denotes the Euler number of C'_i.

(iii) the *cusp vector* $\mathbf{h} := (h_1, ..., h_n)$, where h_i is the number of cusps on C'_i, defined as $\#(C'_i \cap \mathit{supp}\, T)$;

(iv) the branch vector $\mathbf{v} := (v_1, ..., v_n)$ with branch (ramification) index $v_i \geq 2$ of $\hat{C}_i$ with respect to p (defined in d)).

We consider vector space $\mathbb{Q}^n$ also as $\mathbb{Q}$-algebra with componentwise multiplication $\circ$. The unit element is $\mathbf{1} := (1, ..., 1)$, and $\mathbf{v}$ has the inverse $\mathbf{v}^{-1} = (v_1^{-1}, ..., v_1^{-1})$.

Now we can formulate our first "Inverse Integer Conditions" for $\hat{C}$ being the branch divisor of a ball quotient surface. In [Ho 98] we called them "Relative Proportionality Conditions"

$$\text{(Prop 1)} \begin{cases} \mathbf{e} - (\mathbf{1} - \mathbf{v}^{-1}) \cdot S \; - \; \mathbf{h} = 2\mathbf{S} \circ \mathbf{v}; \\ \text{the selfintersection space } (\mathbb{Q}^n, D) \text{ is negative definit.} \end{cases}$$

Observe that (Prop 1) consists of n diophantine equations for all unknown values. In practice a part will be known. Then one has to solve the system for the remaining ones. More or less (Prop 1) is a heritage from dimension 1 with additional intersection numbers coming from surface embeddings. For our surfaces X' we denote the Euler number by $e(X')$ and the signature by $\tau(X')$. In a similar elementary manner we establish the diopantine equation

$$\text{(Prop 2)} \begin{cases} e(X') - 2H_0 - 2(\mathbf{1} - \mathbf{v}^{-1}) \cdot D \cdot^t \mathbf{v}^{-1} - \frac{1}{2}(\mathbf{1} - \mathbf{v}^{-1}) \cdot S \cdot^t (\mathbf{1} - \mathbf{v}^{-1}) \\ \qquad = 3\tau(X') - (\mathbf{v} - \mathbf{v}^{-1}) \cdot D \cdot^t \mathbf{v}^{-1} - (T^2) \quad > \quad 0, \end{cases}$$

where H_0 denotes the number of cusp curves on X' of genus 0 (cusp lines $\mathbb{P}^1$).

For the proofs one needs Riemann-Roch theory for surfaces with singularities. We refer to [Ho 98], from where we deduced (Prop 1) and (Prop 2) for smooth X', $\hat{X}\backslash\{\text{cusp points}\}$. Observe that (Prop 2) simplifies to $e(X') = 3\tau(X') - (T^2)$, if there is no branch divisor and no rational cusp line. This was first proved by Hirzebruch (case without cusp curves) and Mumford. That (Prop 2) is sufficient in these simplified cases for being a ball quotient were celebrated results by Yau and Miyaoka.

We want to classify the surface X' in a fine sense of Kodaira. We look for contracting morphism $\mu: X' \to X$ onto a well-known surface, say a minimal model of X'. For simplicity we assume that μ blows down M exceptional curve of first kind. Altogether we have a birational transformation from X to $\hat{X}$, precisely

$$X \xleftarrow{\mu} X' \xrightarrow{\hat{\rho}} \hat{X}$$

If we can calculate $e(X')$ and $\tau(X')$ from (Prop 2), then we get the important Euler and signature invariant of X by the relations

$$e(X') = e(X) + M\ ,\ \tau(X') = \tau(X) - M.$$

Conversely, we could start with a surface X, blow up M points to get X' and choose curves C', T on X' such that the properties a),b),c),e),f),g),(Prop 1) and (Prop 2) are satisfyed. Conjecture. In this case $\hat{X}$ is a ball quotient surface with (compactified) branch divisor $\hat{C}$. Problem). For/after affirmative answer of the conjecture for a special $(X', C' + T)$ one has to find the ball lattice Γ such that $\hat{X} = \Gamma\backslash\mathbb{B}$. If possible, find Γ in Shimura's theory, especially among Picard modular groups.

In the next section we present a way for solving this problem and give a new example.

4 The Orbital Principle

In [Ho 98] we introduced the category of orbital surfaces, see also [Ho 07]. The main objects look like $\mathbf{X}' := (X', \mathbf{vC}')$ with properties a),b),c) of the last section. Thereby $\mathbf{vC}' := v_1C_1' + ... + v_nC_n'$ is called an *orbital divisor*. Most important in orbital categories are *orbital coverings* accompanied with *orbital coverings*. These notions allow to define *orbital invariants* in a functorial manner, namely, e.g. numerical ones, as maps

$$\mathbf{c}:\ \{\text{objects}\} \longrightarrow \mathbb{R}$$

with the property

$$\mathbf{c}(\mathbf{Y}') = d \cdot \mathbf{c}(\mathbf{X}')$$

for all orbital coverings $\mathbf{Y}' \to \mathbf{X}'$ of degree d.

Theorem 4.1 *([Ho 98]). Both sides in (Prop 2) are orbital invariants on the category of orbital surfaces. The upper part is called* orbital Euler invariant *of* $\mathbf{X}'$ *and will be denoted by* $\mathbf{Eul}(\mathbf{X}')$. *The lower part is called* orbital signature invariant *of* $\mathbf{X}'$ *and will be denoted by* $\mathbf{Sig}(\mathbf{X}')$.

Now we consider the purely arithmetic category **Pic**. It consists, for simplicity of this talk, of all *orbital Picard modular surfaces*

$$\widehat{\mathbf{\Gamma}\backslash\mathbf{B}} = (\widehat{\Gamma\backslash\mathbb{B}}, \mathbf{v}\hat{\mathbf{C}}),$$

where $\mathbf{v}\hat{\mathbf{C}}$ denotes the (Bailey-Borel compactified) *orbital branch divisor* of the quotient map $\mathbb{B} \to \Gamma\backslash\mathbb{B}$, and Γ is a Picard modular group. The vector $\mathbf{v}$ collects the branch indices at the components $\hat{C}_i$ of the branch divisor $\hat{C}$ as before. We only allow as orbital finite coverings in **Pic**, say onto $\widehat{\mathbf{\Gamma}\backslash\mathbf{B}}$, those which are induced by sublattices Γ' of Γ. Then $\widehat{\mathbf{\Gamma}'\backslash\mathbf{B}} \to \widehat{\mathbf{\Gamma}\backslash\mathbf{B}}$ is simply supported by the finite factor morphism $\widehat{\Gamma'\backslash B} \to \widehat{\Gamma\backslash B}$. We restrict ourselves to the complete subcategory $\mathbf{Pic}^{\mathbb{N}}$ of *natural Picard modular congruence surfaces*. To describe the objects, we let

$$\Gamma_K = \mathbb{SU}((2,1), \mathcal{O}_K) = \{G \in \mathbb{G}l_3(\mathcal{O}_K);\ {}^t\bar{G}I_3G = I_3\},$$

with $I_3 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ be the *full Picard modular group* of the imaginary quadratic field (with coefficients in its ring $\mathcal{O}_K$ of integers). For each natural number $N > 0$ the *natural congruence subgroup* $\Gamma_K(N)$ is defined as kernel of the reduction map $\Gamma_K \to \mathbb{SU}((2,1), \mathcal{O}_K/N\mathcal{O}_K)$. Now, the objects of $\mathbf{Pic}^{\mathbb{N}}$ are all surfaces $\widehat{\Gamma_K(N)\backslash\mathbb{B}}$. On this orbital category we have an arithmetically defined orbital nvariant, namely

$$\text{(3)}\quad \lambda :\ \widehat{\Gamma_K(N)\backslash\mathbb{B}} \mapsto \delta_{K,N} \cdot N^8 \cdot \frac{L(3,\chi_K)}{\prod\limits_{p|N}(1-\frac{1}{p^2})^{-1} \cdot \prod\limits_{p|N}(1-\frac{\chi_K(p)}{p^3})^{-1}},$$

where $L(s,\chi_K)$ is the L-series of the Dirichlet character χ_K, the numerators are partial (Euler) products of Riemann's zeta function or of the just mentioned L-series, respectively. The first factor is elementary: $\delta_{K,N} = \begin{cases} \frac{1}{4}, & \text{if } 2|m \text{ and } 2|D_{K/\mathbb{Q}} \text{ (discriminant)} \\ 1, & \text{else.} \end{cases}$

The proof of (3) needs p-adic analysis, the globalizing strong approximation theorem, Tamagawa measure and Tamagawa number, see [Ho 98].

Also the numerator in (3) can be written in easily calculable terms. Namely, we remember to the functional equation for our L-series. It says that the the functions $L(1-s, \chi_K)$ and $L(s, \chi_K)$ coincide up to an elementary factor depending on K, see e.g. [I-R]. There one finds also the formula

$$L(1-m, \chi_K) = -\frac{1}{m} \cdot B_{m,\chi_K}, \quad m \in \mathbb{N}$$

which transfers together with the functional equation the L-value at $s = m = 3$ into the calculable higher Bernoulli number B_{3,χ_K}. Our highlight is the *Orbital Formula*

$$\text{(Orb 2)} \qquad \mathbf{Eul}(\mathbf{X}') = q_{N,\chi} \cdot B_{3,\chi} = 3 \cdot \mathbf{Sig}(\mathbf{X}'),$$

where $\chi = \chi_K$, $\mathbf{X}'$ is the orbital normal crossing model of the smooth orbital (assumed) quotient $\Gamma_K(N)\backslash\mathbb{B}$,

$$\begin{aligned}\mathbf{Eul}(\mathbf{X}') &= e(X') - 2H_0 - 2(\mathbf{1}-\mathbf{v}^{-1})\cdot D\cdot{}^t\mathbf{v}^{-1} - \tfrac{1}{2}(\mathbf{1}-\mathbf{v}^{-1})\cdot S\cdot{}^t(\mathbf{1}-\mathbf{v}^{-1})\\ \mathbf{Sig}(\mathbf{X}') &= 3\tau(X') - (\mathbf{v}-\mathbf{v}^{-1})\cdot D\cdot{}^t\,\mathbf{v}^{-1} - (T^2) \quad > \quad 0\end{aligned}$$

with the notations of (Prop 2) and $q_{N,\chi}$ is a rational number, which can be explicitely expressed by the above L-value relations. This is an elementary exercise for the reader.

Idea of Proof of (Orb 2). Each of the three numbers compared in (Orb 2) can be expressed as integral of an $\mathbb{SU}((2,1),\mathbb{C})$-invariant volume form over a fundamental domain (in $\mathbb{B}$) of the ball lattice. But all invariant volume forms are real multiples of one. By local considerations we recognized the factors we need for (Orb 2).

The double equation (Orb 2) is obviously a refinement of (Prop 2). From one diophantine equation we changed to two of them. For solving the system uniquely for a given Picard modular group, one needs a littlebit finite geometry, namely the reduction group $\Gamma_K/\Gamma_K(N)$ together with its representation on $(\mathcal{O}/N\mathcal{O})^3$. What we need is: (Prop 1), (Orb 2), finite geometry with Galois group $\Gamma_K/\Gamma_K(N)$ and Feustel's results. For instance, we can count the cusp number of $\Gamma_K(N)$ (= cusp point number of $\widehat{\Gamma_K(N)\backslash\mathbb{B}}$) , if we know the following

Theorem 4.2 *(Feustel) The cusp number of Γ_K coincides with the class number of the imaginary quadratic field K.*

Knowing this theorem we get the cusp number of $\Gamma_K(N)$ via the isotropy vectors in $\Gamma_K/\Gamma_K(N)$ with respect to the reduction $(2,1)$-metric. In a similar manner we can use Feustel's class number formulas for elliptic points and for branch curves. Knowing more and more values in our orbital diophantine equation system we get the solution. If we are happy enough - and we were successful in several cases - we get precise surface invariants with a classifying system of curves.

Recent Example

Let $K = \mathbb{Q}(i)$ be the field of Gauß numbers, $\mathcal{O}_K = \mathbb{Z}[i]$, $\Gamma = \mathbb{SU}((2,1),\mathbb{Z}[i])$, where $(2,1)$ stands for the hermitian metric on $\mathbb{C}^3$ representet by the diagonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. We will describe precisely the Baily-Borel compactified quotient surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$ of the natural congruence subgroup $\Gamma(2)$. We start with the (complex) projective plane $\mathbb{P}^2$. On the plane we consider a classically well-known configuration of nine projective lines. In old times it was used to construct *harmonic point quadruples* on a line, see e.g. [Ha 67]. We call it the *harmonic configuration* on $\mathbb{P}^2$. It is drawn in the following picture:

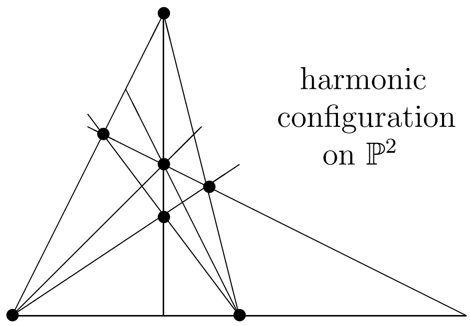


Figure 2

We mark the seven intersection points of more than two lines. After blowing them up, one gets the normal crossing model of the smooth orbital quotient $(\Gamma(2)\backslash\mathbb{B})'$. There are six lines supporting three of the marked points. After our blowing up, they get selfintersection index -2. After contracting these lines on $(\Gamma(2)\backslash\mathbb{B})'$ we finally receive

$\widehat{\Gamma(2)\backslash\mathbb{B}}$. This is a rational surface with precisely six singularities, namely, the cusp points.

The complete proof will be found in the thesis [P 09] of my last doctorand Maria Petkova in the framework of optimal codes by means of Shimura curves. By the way, we can now precisely describe a Picard modular group with the del Pezzo surface of degree 4 as quotient surface, mentioned at the end of section 3. It turns out that it is not a principal congruence subgroup of Γ; we found $< \Gamma(2), \sigma_1, \sigma_2 >$ instead, with $\sigma_1 = -i \cdot \left(\begin{smallmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $\sigma_2 = -i \cdot \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

The door is open now for refinements of the theory of modular forms of these groups (Shiga, Matsumoto), special values generating class fields (Riedel), explicit Shimura curves (Petkova) and uniformizations of special hypergeometric functions. I propose also for further classifications to accompany (Orb 2) by an L-value duplication (Orb 1) of the system (Prop 1) of diophantine equations in the same style as the double equation (Orb 2). For this purpose one has to study Feustel's paper [F 79].

References

[BHH 86] Barthel, G., Hirzebruch, F., Höfer, Th.: Geradenkonfigurationen und algebraische Flächen, Aspects of Mathematics D 4, Vieweg, Braunschweig-Wiesbaden, 1986.

[DM 86] Deligne, P., Mostow, G: Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math. I.H.E.S. **63** (1986), 5-90.

[F 79] Feustel, J.-M.: Volumina der Fundamentalgebiete der Spiegelscheiben Picardscher Modulgruppen, Preprint-Serie der Akademie der Wissenschaften der DDR, Report 13/79 (1979).

[Ha 67] Hartshorne, R.: Foundations of projective geometry, Harvard-LN, Benjamin, New York, 1967.

[Ho 86] Holzapfel, R.-P.: Geometry and Arithmetic around Euler partial differential equations, Dt. Verlag d. Wiss., Berlin / Reidel Publ. Comp., Dordrecht, 1986.

[Ho 98] Holzapfel, R.-P.: Ball and surface arithmetics, Aspekte der Mathematik, Vieweg, Braunschweig, 1998.

[Ho 07] Holzapfel, R.-P.: Relative proportionality on Picard- and Hilbert modular surfaces, in [HUY 07]:

[HUY 07] Holzapfel, R.-P., Uludag, A.M., Yoshida, M., (ed.): Arithmetic and Geometry Around Hypergeometric Functions, Lecture notes of a CIMPA Summer School 2005, Birkhäuser, Basel, Boston, Berlin 2007.

[I-R] Ireland,K., Rosen,M.: A classical introduction to modern number theory, Grad. Texts in Math. **84**, Springer, NY-Heidelberg-Berlin, 1982.

[LV 1896] Le Vavasseur, R.: Sur le système d'équation aux dérivées partielles simultanées auxquelles satisfait la séries hypergéometriques à deux variables, Journ. Fac. Toulouse **7** (1896), 1-205.

[P 09] Petkova, M.: Families of algebraic curves with application in the coding theory and crypthography, thesis, HU Berlin, to appear 2009.

[Y 97] Yoshida, Masaaki: Hypergeometric functions, my love, Aspects of Math. E 32, Birkhäuser, 1997.