English Translation by R.-P.Holzapfel from
Mitteilungen der DMV **26**, Heft 2-3. S.76-79 (Sept. 2018)

Abstract

In August 2018 Peter Scholze received the Fields-Medail for transforming Arithmetic Algebraic Geometry to Galois representations, and for the

development of new cohomology theories.

It seems to be impossible to present on some few pages a precise version or an complete impression the many pioneering results of P.Scholze. Even for a precise declaration of the above short Laudatio of the International Mathematical Union we

would need a much longer text. But perhaps it is possible to give a vague idea of notion of "Perfectoid Spaces" and to arrange an impression, why these spaces revolutionized a big part of the Arithmetic Algebraic Geometry. This is an attempt.

Contents

1 Number Theory and p-adic Geometry

Originally, Number Theory deals with integral number solutions of polynomial equations. For instance, the "Last Theorem" of Fermat states that, if $n > 2$, there is no solution triple of numbers $\{x, y, z \in \mathbb{Z}^* = \mathbb{Z} \setminus 0\}$ of the n-th Fermat equation $x^n + y^n = z^n$. Division by z shows the equivalence with the statement: The equation $x^n + y^n = 1$ has no solution pair $x, y \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$.

Real number solutions are easier to find (by drawing in the coordinate plane). The following curve shows the real solutions of $x^3 + y^3 = 1$. Especially, it is clear that there are infinitely many of such solutions, and it seems to be surprising that there exists no point on the curve with rational coordinates, except for $(0, 1)$ or $(1, 0)$.

The real numbers rise up from a distance function. Thats a metric on the rational number field $\mathbb{Q}$ mapping to any pair x,y their (archimedian) distance $|x-y|$, and the rationals are completed to $\mathbb{R}$ by "filling in infinitely many number gaps".

This procedure can be managed analogeously substituting the archimedean norm $|.|$ by a p-adian one $|.|_p$. Here p is any fixed prime number and $|x|_p$ is small, if the p-exponent of x is great. This p-exponent is equal to m in the decomposition $0 \neq x = \frac{a}{b} \cdot p^m$ with p-prime integers a, b. (For $x = 0$ one sets $m = \infty$). Now one sets $|x|_p := \beta^{-m}$, where $\beta > 1$ is a (fixed) real number. The completion of $\mathbb{Q}$ with respect to the above p-adic metric is denoted by $\mathbb{Q}_p$.

The p-adic metric extended to $\mathbb{Q}_p$ allows us to speak of convergent sequences and of rows with coefficients in $\mathbb{Q}_p$. In some sense it is here easier to recognize convergence because a series $\sum_{n=1} x_n$ with $x_n \in \mathbb{Q}_p$ converges in $\mathbb{Q}_p$ if and only if $(x_n)_n$ is a zero-sequence. This is a consequence of the fact that the stronger (non- archimedean) inequality $|x+y|_p \leq max|x|_p, |y|_p$ holds. Of course the geometry of p-adic numbers comes with several unusual effects. So two p-adic discs are either disjoint or one of them lies inside of the other.

Similar to (real analytic or complex) manifolds, which are geometric formations constructed by means of zeros of convergent power series in several variables (with real or complex) coefficients, one can also define

p-adic spaces (locally[1]) as zero sets of convergent power series with coefficients in $\mathbb{Q}_p$ (or in a field extension of $\mathbb{Q}_p$).

Whenever thereby in number theory an equation as $x^n + y^n = 1$ for $x, y \in \mathbb{Q}$ is studied, one can regard also solutions in $\mathbb{Q}_p$ and the corresponding geometry. Has an equation solutions in $\mathbb{Q}$, then it has naturally also solutions in $\mathbb{R}$ and in $\mathbb{Q}_p$ for all primes p. The inverse conclusion is generally not true, even if the solutions in $\mathbb{Q}_p$ (or in an extension field) are well-understood. Further arguments are necessary to understand solutions in $\mathbb{Q}$. Nevertheless is the investigation of p-adic spaces actually a central research object for the theory of numbers. And this area has been revolutionized by P.Scholze.

Before we come to the perfectoid spaces and their role, I will consider two further aspects: Geometry in characteristic p and the Galois theory of coverings.

2 Geometry in Characteristic p

The fields $\mathbb{R}$ and $\mathbb{C}$ with their ordinary absolute norms or the fields $\mathbb{Q}_p$ are only (but particularly important) special cases of fields, or more generally of commutative rings, endowed with a norm. Another example is the field

$$\mathbb{F}_p((t)) = \{f = \sum_{n>>-\infty} a_n t^n;\ a_n \in \mathbb{F}_p\}$$

[1] It was was the perception of John Tate during the 60-s of last century how to define "local" in this connection.

of Laurent series with finite principal part and coefficients in field $\mathbb{F}_p$ with p elements. Here the absolute value is given by $|f| = \beta^{-o(f)}$, where $o(f)$ is the zero order[2] of $f \in \mathbb{F}_p((t))$ and $\beta > 1$ is a fixed real number again. This absolute value (norm) is non-archimedean too, t.m. it satisfies inequality

For a big class of such rings with absolute values it is possible to define geometric objects. Here there are several theories. Scholze uses for this purpose the theory of adic spaces developed by R. Huber. It has the adventage to deal only with very weak finiteness presuppositions. It is important for the definition of perfectoid spaces being highly non-finite.

Both, the fields $\mathbb{F}_p$ and $\mathbb{F}_p((t))$, have characteristic p. Hence it holds that $(a+b)^p = a^p + b^p$ for all elements a, b of the field. Though geometry over fields with charact6eristic p for mathematicians only working with real or complex numbers seem to be esoterically, so several algebraic and arithmetic problems can be solved in a simpler manner, if the map $x \mapsto x^p$ is not only compatible with multiplication but also with addition. On this line there are for example a lot of deep arithmetical conjectures in the Langlands Program, which are open over $\mathbb{Q}$ but already proved for their analogies in characteristic p.

3 Coverings and Galois Theory

A classical theorem of topology classifies for sufficiently connected spaces X, e.g. for real or complex manifolds, the (unramified) coverings, t.m. spaces over X, which are locally of the form $F \times X$ for a non-void discrete space F. It states the existence of a bijection between the

coverings[3] of X and the subgroups of the fundamental group of X. Thereby the finite coverings correspond to the subgroups of finite index.

This statement has an algebraic analogy in algebraic geometry, where also

can be defined (algebraic) fundamental groups. Classical is the case of a field k, which is a point from the view of algebraic

geometry[4]. In this language "Puctured connected unramified coverings" are just given by finite separable k-extensions inside of a fixed algebraic closure $\overline{k}$ of k. And the main theorem of Galois Theory just states that the set of such extensions corresponds bijectively to the set of subgroups of finite index of the Galois group of k. Thus, it is a precise analogue of the above topological theorem. Especially, two fields have the same "Covering Theory", if they have the same Galois group.

4 A Perfectoid Point

Longtimes ago it was known that the fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ posses many formal similarities, though the former has characteristic 0 but the latter has character-

[2]These can also be negative (if f has a pole).

[3]More precisely: of isomorphy classes of punctured connected covers.

[4]Since there exist many different fields, there also are many different points in algebraic geometry.

istic p. So we can, via p-adic development, the elements of $\mathbb{Q}_p$ also consider as "Laurent series in p with coefficients between 0 and $p-1$:

$$\mathbb{Q}_p = \{ \sum_{n>>\infty} a_n p^n ; a_n \in \{0, 1, ..., p-1\}\},$$

and the p-adic valuation is again the "zero order of the series". Furthermore, a poroposition of Fontaine and Weinberger says that the Galois groups of $\mathbb{Q}_p$ and of $\mathbb{F}_p((t))$ approach each other, if one adjoints the p^n-th root of p respectively the t^n-th of t. The Galois groups of the "limits" $\mathbb{Q}_p(p^{1/p^\infty})$ respectively $\mathbb{F}_p((t^{1/p^\infty}))$ coincide. Therefore they have the same set of finite unramified coverings. This construction is a special case of perfectoid spaces and of their "tilts".

5 Perfectoid Spaces and the Tilt

Scholzes interpretation of the Fontaine/Winterberger result is the construction of a "tilting functor" $(.)^\flat$ corresponding to a so called perfectoid space in characzteristic 0 a perfectoid space in characteristic p. For our 0-dimensional special case it holds that

$$\mathbb{Q}_p(p^{1/p^\infty})^\flat = \mathbb{F}_p((t^{1/p^\infty})).$$

Similarly to the understanding of $\mathbb{Q}_p(p^{1/p^\infty})$ as limit of the finite extensions $\mathbb{Q}_p(p^{1/p^n})$ of $\mathbb{Q}_p(p)$ one gets from p-adic manifolds, taking from the coordinate functions higher and higher p^n-th roots, perfectoid spaces of characteristic 0. This can be imagined as a kind of "fractionalisation" of the p-adic manifolds. Such a perfectoid space construction X can be mapped to its tilt $X^\flat$. This is an adic space of characteristic p. For several purposes it is easier to understand than the p-adic manifold we startet with.

Moreover Scholze shows that the covering theory of a perfectoid space of characteristic 0 is the same as those of its tilt. It allows us in a very new but elegant manner to change from 0-characteristical spaces p-characteristical ones.

Indeed, short time back, Scholze refered to us a direct way for corresponding to any p-adic space a geometric object in characteristic p, a so called "diamond", not using the procedure of "fractionalisation". It allows, for instance, to construct the so called "local Shimura varieties" with methods in characteristic p. The general existence of such varieties could be only conjectured before Scholze construction has been done.

6 Examples of Applications

During the last years the theory of perfectoid spaces was extremely fruitful. I want to outline on this place only two applications. They should enlighten "applications to Galois representations" on the one hand and the "development of new cohomology theories" on the others from the above short laudations. This is only a little part of the world opened by Scholze's mathematical world.

The first application is Scholze's construction of Galois group

representations associated to the cohomology of locally symmetric spaces for the group $\mathbb{G}l_n$ over totally real fields or with complex multiplication. The existence of such Galois representations had been conjectured since 40 years by Ash, Grunewald and others. For cohomologies with rational values it has been proved recently in a work of Harris, Lan, Taylor and Thorne. It's noteworthy to point out that Scholze is able with his results to join the things with Galois representations with values in $\mathbb{Z}/p^n\mathbb{Z}$. While rational cohomology classes appear not so often, there exist lots of torsion classes. The most important tool again is the perfectoisation: in the case of Shimura varieties, in which cohomologies can be found those of locally symmetric spaces as by-product.

As second application should be mentioned the construction of a new cohomology theory for smooth compact p-adic spaces with good reduction. Scholze, Bhatt and Morrow presented this in a common paper. Cohomology groups are important algebraic

invariants in geometry. For example, the genus of a compact Riemann surface is nothing else but the rank of a certain cohomology group.

For p-adic spaces there exist several such cohomology theories. In their work Bhatt, Morrow and Scholze built a new one, such that the most important known cohomologies are specialisations of this new creation. The authors use their techniques for the comparision of torsion classes with respect to different cohomologies. This was a central unsolved problem in p-adic geometry, being open since the work of Fontaine, Messing, Faltings and Tsuji in the 90-s years, when they investigated successfully the torsionfree contributions.

This is only a small part of results of last years with which Scholze

renewed parts of fundaments of new arranged arithmetic algebraic geometry. His outcomes beam wide to other branches of mathematics as topology or theory of automorphic forms.

Remarkable on Scholze's results is the beauty and simplicity of his ideas. Indeed, the practical realization of them demand for technical abilities. But one has in his publications the feeling that he reveals inside of mathematics natural structures, which - sometimes after - in their clarity and elegance really should have been obvious. Thanks to Scholz the mathematics became again much more fascinating and exciting.