Arithmetic and Geometry around a Shimura Quartic

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Abstract

We consider and construct (finite and infinite) towers of Picard modular surfaces with trivial (t.m. rational) function fields but non-trivial discriminants, geometrically known as orbital (branch) divisors, of the involved coverings. It is convenient to regard them as special cases of Galois-Reflection towers, which will be defined in arbitrary dimensions. We prove that (finitely many) reflections generate the Picard modular groups defining such a tower. We use this knowledge for explicit algebraic-geometric classifications of the Baily-Borel compactifications of tower members and explicit description of the corresponding Picard modular groups by means of reflections. Finally, we turn to dimension 1 considering arithmetic subdiscs of the 2-ball and their algebraic image curves. On this way we get an explicit tower of Shimura curves embedded in the constructed surface tower. Via reductions mod p one gets important towers of coding theory, whose members can be explicitly determined step by step.

Look at the globe with drawn equator and two meridians, all orthogonally to each other. These three circles describe the norm-1 curve configuration on a special Picard modular surface visualizing an octahedral group action. The six intersection points are the cusp singularities of the surface. The notions will be explained now in a general setting of arbitrary dimension.

Let V be the space $\mathbb{C}^{n+1}$ endowed with hermitian metric $< .,. >$ of signature $(n, 1)$. Explicitly we will work with the diagonal representation. For $v \in V$ we call $n(v) =< v, v >$ the norm of v. The space of all vectors with negative (positive) norms is denoted by $V^-(V^+)$. The image $\mathbb{P}V^-$ of V^- in $\mathbb{P}V = \mathbb{P}^n$ is the hyperball denoted by $\mathbb{B}^n$. The unitary group $\mathbb{U}((n, 1), \mathbb{C})$ acts transitively on it. Now let K be an imaginary quadratic number field, $\mathcal{O}_K$ its ring of integers.

The arithmetic subgroup $\Gamma_K = \mathbb{U}((n, 1), \mathcal{O}_K)$ is called the *full Picard modular group.* Each subgroup Γ of finite index is a *Picard modular group.*

The ball quotients $\Gamma\backslash\mathbb{B}^n$ are quasiprojective. They have a minimal algebraic compactification $\widehat{\Gamma\backslash\mathbb{B}^n}$ constructed by Baily and Borel in [2]. The authors proved that these compactifications are normal projective complex varieties. The Picard modular groups of fixed K act also on the hermitian $\mathcal{O}_K$-lattice $\Lambda = (\mathcal{O}_K)^{n+1}$.

Let $\mathfrak{a} \in \Lambda$ be a primitive positive vector and $\mathfrak{a}^\perp$ its orthogonal complement in V. It is a hermitian subspace of V of signature $(n - 1, 1)$. The intersection $\mathbb{D}_\mathfrak{a} := \mathbb{P}\mathfrak{a}^\perp \cap \mathbb{B}^n$ is isomorphic to $\mathbb{B}^{n-1}$. We call it an *arithmetic subball* of $\mathbb{B}^n$.

Take all elements of Γ acting on $\mathbb{D}_\mathfrak{a}$: $\Gamma_\mathfrak{a} := \{\gamma \in \Gamma; \gamma(\mathbb{D}_\mathfrak{a}) = \mathbb{D}_\mathfrak{a}\}$ This is an arithmetic group. The image $p(\mathbb{D}_\mathfrak{a})$ along the quotient projection $p : \mathbb{B}^n \to \Gamma\backslash\mathbb{B}^n$ is an algebraic subvariety $H_\mathfrak{a}$ of $\Gamma\backslash\mathbb{B}^n$ of codimension 1. The algebraic subvarieties $H_\mathfrak{a}$ are called *arithmetic hypersurfaces* of the Picard modular variety

$\Gamma\backslash\mathbb{B}^n$. The same notion is used for the compactifications. The *norm* $n(H_\mathfrak{a})$ of $H_\mathfrak{a}$ is defined as $n(\mathfrak{a})$.

An element $\sigma \in \Gamma$ is called a Γ*-reflection*, iff it has an n-dimensional eigenspace $V_\mathfrak{a} \subset V$ of eigenvalue 1 and a positive eigenvector $\mathfrak{a} = \mathfrak{a}(\sigma)$ of other eigenvalue. The latter can be chosen primitive in Λ. The eigenspace $V_\mathfrak{a}$ is then nothing else but $\mathfrak{a}^\perp$, and σ acts identically on the arithmetic subball $\mathbb{D}_\mathfrak{a} = \mathbb{P}V_\mathfrak{a} \cap \mathbb{B}^n$ of $\mathbb{B}^n$. We call such $\mathbb{D}_\mathfrak{a}$ a Γ*-reflection subball* of $\mathbb{B}^n$. The hypersurface $H_\mathfrak{a}$ of the primitive eigenvector $\mathfrak{a} = \mathfrak{a}(\sigma)$ of a Γ-reflection σ is called a Γ*-reflection hypersurface*.

Let $\ldots \Gamma_{i+1} \subset \Gamma_i \subset \cdots \subset \Gamma_1 \subset \Gamma$, (1), be a (finite or infinite) normal series of subgroups of finite index of the Picard modular group Γ. We call it a Γ*-Reflection series*, if Γ_i is generated by Γ_{i+1} and finitely many reflections for each in (1) occurring pair $(i+1, i)$. The corresponding Galois tower of finite Galois coverings $\cdots \to \Gamma_{i+1}\backslash\mathbb{B}^n \to \Gamma_i\backslash\mathbb{B}^n \to \cdots \to \Gamma_1\backslash\mathbb{B}^n$, (2), with the normal factors Γ_i/Γ_{i+1} as Galois groups, is then called a *Galois-Reflection tower*. In this case each normal factor is generated by a coset of Γ-reflections.

Theorem: Let $\Gamma\backslash\mathbb{B}^n$ be simply-connected and smooth. Then Γ is generated by finitely many Γ-reflections.

Corollary: Let $\Gamma' \subset \Gamma_N \subset \cdots \subset \Gamma_1$, $(1')$, be a normal series of Picard modular subgroups of the ball lattice Γ, and $\Gamma'\backslash\mathbb{B}^n \to \Gamma_N\backslash\mathbb{B}^n \to \cdots \to \Gamma_1\backslash\mathbb{B}^n$, $(2')$, the corresponding Galois tower of Picard modular varieties. If the varieties $\Gamma_i\backslash\mathbb{B}^n$, $i = 1, ..., N$, are smooth and simply-connected, then $(1')$ is a Galois-Reflection series with Galois-Reflection Tower $(2')$.

Example: Uludag constructed in [9] the first (and only until now) infinite Galois-Reflection tower in dimension > 1. It consists (compactified) of orbital projective planes $\mathbb{P}^2$. The successive Galois coverings $\mathbb{P}^2 \to \mathbb{P}^2$ have $K4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein's Vierergruppe) as Galois groups. The first member is the orbital $\mathbb{P}^2$ with Apollonius (branch divisor) configuration, see Holzapfel [4], first appearance by Yoshida [6]. In [4] we proved that the congruence subgroup $\Gamma_1 = \Gamma(1-i)$ is the uniformizing ball lattice, where $\Gamma = SU((2,1), \mathbb{Z}[i])$.

We use Galois-Reflection towers step by step for explicit descriptions of the uniformizing ball lattices, if the orbital Picard modular surfaces are explicitly known and vice versa. The main goal is the first algebraic-geometric classification of the Picard modular surface of a natural congruence subgroup:

Proposition: A (singular) model of $\Gamma(2)\backslash\mathbb{B}^2$ is the space quartic $U^2T^2 - X^4 - Y^4 - T^4 + 2X^2Y^2 + 2X^2T^2 + 2Y^2T^2 = 0$ (in $\mathbb{P}^3$ with coordinates $(u : x : y : t)$). It has $\mathbb{P}^1 \times \mathbb{P}^1$ as smooth model. Blowing up suitable six points of $\mathbb{P}^1 \times \mathbb{P}^1$ we get the minimal desingularisation $\overline{(\Gamma(2)\backslash\mathbb{B}^2)}$ of $\widehat{\Gamma(2)\backslash\mathbb{B}^2}$ (resolution of cusp singularities).

For the proof we climb step by step through a Galois-Reflection diagram supported by the Uludag Tower:

$$\begin{array}{ccccc} & \widehat{\Gamma(2)\backslash\mathbb{B}} & \to & \mathbb{P}^1 \times \mathbb{P}^1 & \\ & | & & | & (T) \\ \cdots\ Uludag's\ Tower\ \cdots \mathbb{P}^2 \to & \mathbb{P}^2 & \to & \mathbb{P}^2 & \end{array}$$

where each arrow corresponds to a reflection. The geometric rectangle consists of ball quotients of the following (short) Γ-Reflection series:

$$\begin{array}{ccc} \Gamma(2) & \rightarrow & < \Gamma(2), \sigma, \sigma' > \\ \downarrow & & \downarrow \\ < \Gamma(2), \rho > & \rightarrow & \Gamma(1-i) =< \Gamma(2), \rho, \sigma, \sigma' > \end{array}$$

with explicitly known reflections ρ, σ, σ'.

First we know the second orbital $\mathbb{P}^2$ of the Uludag Tower, whose orbital (branch) divisor is drawn in [9], supported on a Ceva-configuration with 7 lines, which one can already find by Hirzebruch [2], p. 81. Its uniformizing ball lattice is $< \Gamma(2), \rho >$. Knowing the branch divisor of the left vertical double cover we know that $\widehat{\Gamma(2)\backslash\mathbb{B}^2}$ is the normalization of $\mathbb{P}^2$ along the function field extension $\mathbb{C}(x,y,w)/\mathbb{C}(x,y)$ with $w = \sqrt{(x^4 + y^4 - 2x^2.y^2 - 2x^2 - 2y^2 + 1)}$.

This gives the quartic space equation. To get $\mathbb{P}^1 \times \mathbb{P}^1$ as model one goes through the right side of the rectangle. In the thesis [5] of Matsumoto one can find the orbital $\mathbb{P}^1 \times \mathbb{P}^1$, which is easily recognized as double cover of the orbital Apollonius plane. The normalization of a birational transform of Matsumoto's $\mathbb{P}^1 \times \mathbb{P}^1$ finishes the proof of the theorem. Moreover, knowing all orbital divisors, one can see that $\overline{\Gamma(2)\backslash\mathbb{B}^2}$ is the $K4$-cover of orbital del Pezzo surface No. 20 in the Hirzebruch's table [2], p. 201, with orbital divisor supported by 10 projective lines, p. 196 [2].

In the case $n = 2$ the subballs are $K-$linear discs, which define algebraic curves on the ball quotient surface $\Gamma\backslash\mathbb{B}^2$. The quotient curve $C = \Gamma\backslash\mathbb{D} \subset \Gamma\backslash\mathbb{B}^2$, the projection of the $K-$linear disc $\mathbb{D}$, is an algebraic curve, which embedded model on the Picard modular surface $\Gamma\backslash\mathbb{B}^2$ is defined over $\overline{\mathbb{Q}}$ (the proof is based on an article of Shiga, Wolfart [7]). The particular consideration of this case is strongly motivated by results from coding theory. It was shown by T. Zink [8] that towers, i.e. sequences of finite covers, of Shimura curves defined over $\mathbb{F}_{q^2}$, are asymptotically optimal. They correspond to sequences of codes with good parameters.

In [3] N.Elkies defines a construction for towers $(X_n \to X_{n-1} \to \cdots \to X_0)$ of Shimura curves, which is based only on the first two curves X_0, X_1. In general it is quite difficult to compute equations for X_0 and X_1. The aim of our work is to find a way to obtain equations for X_0 and X_1, in the case where they are Shimura curves from $K-$ linear discs on Picard modular surfaces. For $K = \mathbb{Q}(i)$ and $\Gamma = SU((2,1), \mathcal{O}_K)(1-i)$ the curves of norm 1 and 2 have been completely described [4]. Studying systematically the curves of small norm, the next step is to try to compute an equation for a norm 3 Shimura curve. This is possible and we obtain a plane quadric defined by $Sh : 16xy + 4xz + 4yz - 3z^2 = 0$. We recognize the Shimura curve X_1 of Elkis' Tower on the quartic space model of $\Gamma(2)\backslash\mathbb{B}^2$. It has the plane *Shimura quartic* model $16x^2y^2 + 4x^2z^2 + 4y^2z^2 - 3z^4 = 0$, which is an elliptic curve with j-invariant 2048/3.

To obtain the globe mentioned at the beginning one has to extend a little bit the Galois-Reflection Diagram (T).

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