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The Neville–Aitken formula for rational interpolants with prescribed poles

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Using a polynomial description of rational interpolation with prescribed poles a simple purely algebraic proof of a Neville–Aitken recurrence formula for rational interpolants with prescribed poles is presented. It is used to compute the general Cauchy–Vandermonde determinant explicitly in terms of the nodes and poles involved.

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1. Preliminaries and notations

Let m, n be non-negative integers and let $(a_i) = (a_0, a_1, a_2, \dots)$ and $(b_j) = (b_1, b_2, \dots)$ be given sequences of (not necessarily distinct) complex numbers that are disjoint:

$$\{a_0, a_1, a_2, \dots\} \cap \{b_1, b_2, \dots\} = \emptyset. \tag{1}$$

Given a complex function f which is sufficiently often differentiable at the multiple points a_i the *rational interpolant* $r_{m,n}$ *of* f *of degree* m, n *with prescribed poles* $b_1, \dots, b_n$ *and nodes* $a_0, \dots, a_m$ counting multiplicities in both cases is the rational function

$$r_{m,n} = p_{m,n}/B_n, \tag{2}$$

where $p_{m,n}$ is a polynomial of degree m at most and

$$B_n(z) := (z - b_1) \cdot \dots \cdot (z - b_n)$$

such that

$$f - r_{m,n} \tag{3}$$

has zeros $a_0, \dots, a_m$ counting multiplicities.

Observe that $r_{m,n}$ is uniquely determined by these properties. Depending on data function f, $r_{m,n}$ has no other poles than $b_1, \ldots, b_n$ or ∞ counting multiplicities, where the multiplicity of ∞ is $\min\{0, deg\, p_{m,n} - n\}$ with $deg\, p$ denoting the exact degree of a polynomial p.

A Neville–Aitken type procedure for computing $r_{m,n}$ recursively is given in [1]. It is based upon the general Neville–Aitken algorithm [2].

In this note we will give a short direct proof of the rational Neville–Aitken recurrence relation starting from an alternative purely algebraic definition of $r_{m,n}$.

Define

$$A_m(z) := (z - a_0) \cdots (z - a_m). \tag{4}$$

THEOREM 1

Let ϕ be any polynomial interpolating f at the nodes $a_0, \ldots, a_m$ counting multiplicities. If $p_{m,n}$ is the polynomial of degree m at most left when $\phi \cdot B_n$ is divided by A_m, i.e.

$$\phi \cdot B_n \equiv p_{m,n} \ (\text{mod}\ A_m), \quad deg\, p_{m,n} \leqslant m, \tag{5}$$

then

$$r_{m,n} = p_{m,n}/B_n. \tag{6}$$

Proof

The proof is a slight modification of Walsh's classical existence and unicity proof for the rational interpolant with prescribed poles [6]. Clearly, $r_{m,n}$ is of the form required. Next we use that the polynomials A_m and B_n are relatively prime. By construction, there exists a polynomial Q such that

$$\left(\phi - \frac{p_{m,n}}{B_n}\right) \cdot B_n = A_m \cdot Q.$$

Therefore, $r_{m,n} = p_{m,n}/B_n$ agrees with ϕ and consequently also with f at $a_0, \ldots, a_m$ counting multiplicities. □

2. An algebraic proof of the Neville–Aitken recurrence formula for rational interpolants with prescribed poles

In [1] a Neville–Aitken algorithm computing

$$\left(r_{i,j} \mid i + j \leqslant m + n,\ i \leqslant m\right)$$

recursively is derived from the general Neville–Aitken algorithm [2] via explicit representations of Cauchy–Vandermonde determinants. In this section we give a simple direct proof of the rational Neville–Aitken recurrence formula which is purely algebraic.

Subsequently, knowing its weight factors, one can easily derive the explicit formula of the Cauchy–Vandermonde determinant. This seems to be simpler than running the opposite direction.

We suppose the data function f to be fixed. Corresponding to a polynomial h let f_h be the Hermite interpolation polynomial of f, where the nodes are the zeros of h counting multiplicities.

If q is another polynomial such that h and q are relatively prime by $p[h; q]$, we denote the remainder of the polynomial division of $q \cdot f_h$ by h:

$$p[h; q] \equiv q \cdot f_h \ (\mathrm{mod}\ h) \quad \text{and} \quad deg\, p[h; q] < deg\, h. \tag{7}$$

Finally, according to theorem 1,

$$r[h; q] := p[h; q]/q \tag{8}$$

is the unique rational function of degree m, n, $m := deg\, h - 1$, $n := deg\, q$, with prescribed poles the zeros of q that interpolates f at the zeros of h counting multiplicities.

THEOREM 2

Let $h, h_1, h_2, h_3, q, q_1, q_2, q_3$ be monic complex polynomials and let $\alpha_1 \neq \alpha_2$, β be complex numbers. Let $p_i := p[h_i; q_i]$ and $r_i := p_i/q_i$ for $i = 1, 2, 3$.

(a) Suppose that $h_i(z) = (z - \alpha_i) \cdot h(z)$ for $i = 1, 2$ and $h_3(z) = (z - \alpha_1)(z - \alpha_2) \cdot h(z)$ with $h_3(\beta) \neq 0$ and that $q_i = q$ for $i = 1, 2$ and $q_3(z) = (z - \beta)q(z)$. Then,

$$r_3(z) = \frac{r_1(z)(z - \alpha_2)(\beta - \alpha_1) - r_2(z)(z - \alpha_1)(\beta - \alpha_2)}{(\alpha_2 - \alpha_1)(z - \beta)}. \tag{9}$$

(b) Suppose that $h_i(z) = (z - \alpha_i) \cdot h(z)$ for $i = 1, 2$ and $h_3(z) = (z - \alpha_1)(z - \alpha_2) \cdot h(z)$ and that $q_i = q$ for $i = 1, 2, 3$. Then,

$$r_3(z) = \frac{r_2(z)(z - \alpha_1) - r_1(z)(z - \alpha_2)}{(\alpha_2 - \alpha_1)}. \tag{10}$$

Proof

(a) Let $\phi := f_{h_3}$. Since $\phi q_3 \equiv p_3 \ (\mathrm{mod}\ h_3)$ we also have

$$\phi q_3 \equiv p_3 \ (\mathrm{mod}(z - \alpha_i) \cdot h) \quad \text{for } i = 1, 2.$$

On the other hand, by definition

$$\phi q_i = \phi q \equiv p_i \ (\mathrm{mod}(z - \alpha_i) \cdot h) \quad \text{for } i = 1, 2.$$

This implies

$$p_3 \equiv \phi \cdot (z - \beta) \cdot q \equiv (z - \beta) \cdot p_i \ (\mathrm{mod}(z - \alpha_i) \cdot h) \quad \text{for } i = 1, 2.$$

Thus, there exist polynomials F_1, F_2 with

$$p_3(z)=(z-\alpha_i)h(z)F_i(z)+(z-\beta)p_i(z) \quad \text{for } i=1,2. \tag{11}$$

Since by the assumptions in (a) $deg\, p_3 \leqslant deg\, h+1$ and $deg\, p_i \leqslant deg\, h$ $(i=1,2)$, it follows that both $F_1(z)=: F_1$ and $F_2(z)=: F_2$ are constants. Consequently,

$$F_1=\frac{p_3(\beta)}{(\beta-\alpha_1)h(\beta)}, \quad F_2=\frac{p_3(\beta)}{(\beta-\alpha_2)h(\beta)}.$$

Observe that $F_1=F_2=0$ iff $p_3(\beta)=0$. In this case according to (11), $p_3(z)=(z-\beta)p_i(z)$ for $i=1,2$. As a consequence, $r_3=r_1=r_2$ and (9) holds. Otherwise

$$F_2=\frac{\beta-\alpha_1}{\beta-\alpha_2}F_1.$$

Multiplication of (11) for $i=1$ by $(z-\alpha_2)\cdot F_2$ and for $i=2$ by $(z-\alpha_1)\cdot F_1$, respectively, and subtraction yield

$$p_3(z)\left[\frac{z-\alpha_2}{\beta-\alpha_2}(\beta-\alpha_1)-(z-\alpha_1)\right]\cdot F_1$$
$$=(z-\beta)\left[p_1(z)\frac{z-\alpha_2}{\beta-\alpha_2}(\beta-\alpha_1)-p_2(z)(z-\alpha_1)\right]\cdot F_1,$$

from which (9) is easily derived.

(b) Also under the assumptions of (b) as in the proof of (a)

$$p_3 \equiv p_i \ (\text{mod}(z-\alpha_i)h) \quad \text{for } i=1,2$$

follows. Accordingly, there exist constants F_1, F_2 with

$$p_3(z)=(z-\alpha_i)\cdot h(z)\cdot F_i+p_i \quad \text{for } i=1,2. \tag{12}$$

Consequently, $F_1=F_2$ is the leading coefficient of p_3. A similar reasoning and calculation as used in part (a) applied to (12) results in (10). □

Remarks

(i) Letting $\beta\to\infty$ in (9) gives a second proof of (10).

(ii) Theorem 2 is identical with [1, theorem 9] although the notations are different. In [1] from this theorem an algorithm is derived computing the values $r_{i,j}(z)$ for $i+j\leqslant m+n$, $i\leqslant m$ with $O(l^2)$ arithmetical operations where $l=\max\{m+1, n\}$.

3. Computation of Cauchy–Vandermonde determinants

The rational interpolant (2) belongs to a particular Cauchy–Vandermonde space spanned by the functions basic for the partial fraction decomposition of $r_{m,n}$.

More generally, Cauchy–Vandermonde systems are constructed as follows. Let $\mathcal{B}=(b_1, b_2, b_3, \ldots)$ be a fixed sequence of points of the extended complex plane $\overline{\mathbb{C}}$ which will serve as "prescribed poles". Notice that repetition of points is allowed. By $\nu_k(x)$ we denote the multiplicity of x in $\mathcal{B}_{k-1} := (b_1, \ldots, b_{k-1})$. With $\mathcal{B}$ we associate a system $\mathcal{U}=(u_1, u_2, \ldots)$ of basic rational functions defined by

$$u_k(z) = \begin{cases} z^{\nu_k(b_k)} & \text{if } b_k = \infty, \\ \dfrac{1}{(z-b_k)^{\nu_k(b_k)+1}} & \text{if } b_k \in \mathbb{C}, \end{cases} \tag{13}$$

which will be called the *Cauchy–Vandermonde system* generated by $\mathcal{B}$. To $\mathcal{B}_k$ corresponds the basis $\mathcal{U}_k = (u_1, \ldots, u_k)$ of the k-dimensional *Cauchy–Vandermonde space span* $\mathcal{U}_k$.

COROLLARY 1

$\mathcal{U}$ is an extended complete Chebyshev system on $\mathbb{C}\setminus\{b_1, b_2, \ldots\}$.

Proof

Any element from *span* $\mathcal{U}_k$ is a rational function with prescribed poles $b_1, \ldots, b_k$, that means it is of the form (6). Thus, by theorem 1 any Hermite interpolation problem with *span* $\mathcal{U}_k$ and nodes from $\mathbb{C}\setminus\{b_1, b_2, \ldots\}$ has a unique solution. □

Let $\mathcal{A}=(a_1, a_2, \ldots)$ be a fixed sequence of complex numbers which will serve as *interpolation points* or *nodes* taking into account multiplicities. By $\mu_k(x)$ we denote the multiplicity of x in $\mathcal{A}_{k-1} = (a_1, \ldots, a_{k-1})$. Notice that

$$mult(\mathcal{A}_m) := \prod_{k=1}^{m} \mu_k(a_k)!$$

measures in some sense repetition of nodes in $\mathcal{A}_m$.

From corollary 1 it follows that any *Cauchy–Vandermonde determinant*

$$V \mid \mathcal{U}_m;\ \mathcal{A}_m \mid := V\begin{vmatrix} u_1, \ldots, u_m \\ a_1, \ldots, a_m \end{vmatrix} := det\left(D^{\mu_i(a_i)} u_j(a_i)\right)$$

is different from zero provided $\mathcal{A}_m \cap \mathcal{B}_m = \emptyset$.

How to compute $V \mid \mathcal{U}_m;\ \mathcal{A}_m \mid$ explicitly in terms of the poles and nodes involved? We will do this starting from theorem 2 and using a little "general interpolation theory". To simplify notations we adopt the convention that finite products of extended complex numbers β_j have to be understood according to

$$\prod_{j\in J}{}^{*} \beta_j := \prod_{j \in J} \beta_j^{*},$$

where

$$\beta_j^* = \begin{cases} 1 & \text{if } \beta_j = 0 \text{ or } \beta_j = \infty, \\ \beta_j & \text{iff } \beta_j \in \mathbb{C}\setminus\{0\}. \end{cases}$$

Moreover, to get a simple sign factor we assume that both systems $\mathscr{A}_m$ and $\mathscr{B}_m$ are *consistently ordered* according to

$$\mathscr{A}_m = (a_1, \dots, a_m) = \Big(\underbrace{\alpha_1, \dots, \alpha_1}_{m_1}, \alpha_2, \dots, \underbrace{\alpha_p, \dots, \alpha_p}_{m_p}\Big) \subset \mathbb{C},$$

$$\mathscr{B}_m = (b_1, \dots, b_m) = \Big(\underbrace{\beta_1, \dots, \beta_1}_{n_1}, \beta_2, \dots, \underbrace{\beta_q, \dots, \beta_q}_{n_q}\Big) \subset \overline{\mathbb{C}},$$

with $\alpha_1, \alpha_2, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_q$ pairwise distinct and $m_1 + \cdots + m_p = m$, $n_1 + \cdots + n_q = m$.

THEOREM 3

When $\mathscr{U}_m$ is generated by $\mathscr{B}_m$ according to (13) and when $\mathscr{A}_m$ and $\mathscr{B}_m$ are consistently ordered then

$$V \mid \mathscr{U}_m;\ \mathscr{A}_m \mid = mult(\mathscr{A}_m) \cdot \frac{\displaystyle\prod_{\substack{k,j=1 \\ k>j}}^{m}{}^{*} (a_k - a_j) \cdot \prod_{\substack{k,j=1 \\ k>j}}^{m}{}^{*} (b_k - b_j)}{\displaystyle\prod_{\substack{k,j=1 \\ k\geqslant j}}^{m}{}^{*} (a_k - b_j) \cdot \prod_{\substack{k,j=1 \\ k>j}}^{m}{}^{*} (b_k - a_j)}. \tag{14}$$

Proof

Let f be a fixed complex function which is defined and suffiently often differentiable at the multiple points of $\mathscr{A}_m$. Suppose $r_1 \in \mathscr{U}_{m-1}$ and $r_2 \in \mathscr{U}_{m-1}$ are the rational interpolants of f with respect to $\mathscr{A}_{m-1} = (a_1, \dots, a_{m-1})$ and $\mathscr{A}'_{m-1} = (a_2, \dots, a_m)$, respectively. Let $r \in \mathscr{U}_m$ interpolate f at $\mathscr{A}_m$ and set

$$h(z) := (z - a_2) \cdots (z - a_{m-1}),$$

$$h_1(z) := (z - a_1) \cdot h(z),$$

$$h_2(z) := (z - a_m) \cdot h(z),$$

$$q_1(z) := \prod_{j=1}^{m-1}{}^{*} (z - b_j) = q_2(z),$$

$$q_3(z) := \prod_{j=1}^{m}{}^{*} (z - b_j).$$

From theorem 2 with $\alpha_1 = a_1$, $\alpha_2 = a_m$ and $\beta = b_m$ in case (a) and $\beta = b_m = \infty$ in case (b) we deduce that always

$$\begin{aligned} r &= r_1 \cdot \gamma_2 + r_2 \cdot \gamma_1 \\ &= r_1 + \gamma_1 \cdot (r_2 - r_1), \end{aligned} \tag{15}$$

where

$$\gamma_1(z) = \frac{(z - a_1) \cdot (a_m - b_m)^*}{(a_m - a_1) \cdot (z - b_m)^*}$$

and

$$\gamma_1 + \gamma_2 = 1.$$

On the other hand, by Newton's interpolation formula [3]

$$r = r_1 + [a_1, \dots, a_m] f \cdot r_{m-1} u_m,$$

where $[a_1, \dots, a_m]f$ is the leading coefficient of r (that before u_m) and

$$r_{m-1} u_m(z) = V \begin{vmatrix} u_1, \dots, u_{m-1}, u_m \\ a_1, \dots, a_{m-1}, z \end{vmatrix} \Big/ V \begin{vmatrix} u_1, \dots, u_{m-1} \\ a_1, \dots, a_{m-1} \end{vmatrix}$$

is a Newton remainder. By comparison with (15)

$$\gamma_1 (r_2 - r_1) = [a_1, \dots, a_m] f \cdot r_{m-1} u_m. \tag{16}$$

We claim that

$$\gamma_1 \cdot (r_2 - r_1) = \frac{(z - a_1)(a_m - b_m)^*}{(a_m - a_1)(z - b_m)^*} \cdot \frac{(z - a_2) \cdots (z - a_{m-1})}{(z - b_1)^* \cdots (z - b_{m-1})^*} \cdot c, \tag{17}$$

where

$$c = [a_1, \dots, a_m] f \cdot \frac{a_m - a_1}{(a_m - b_m)^*} \cdot \frac{(b_m - b_1)^* \cdots (b_m - b_{m-1})^*}{(b_m - a_1)^* \cdots (b_m - a_{m-1})^*} \tag{18}$$

is a constant factor depending on f.

In fact,

$$r_2 - r_1 = p / q_1,$$

with p a polynomial of degree $m - 2$ depending on f with zeros $a_2, \dots, a_{m-1}$. This proves (17). It remains to compute c. To show (18) consider the partial fraction decomposition of

$$\frac{(z - a_1) \cdots (z - a_{m-1})}{(z - b_1)^* \cdots (z - b_{m-1})^* (z - b_m)^*} = \sum_{\mu=1}^{m} d_\mu \cdot u_\mu(z).$$

Here it is easily seen that

$$d_m = \frac{(b_m - a_1)^* \cdots (b_m - a_{m-1})^*}{(b_m - b_1)^* \cdots (b_m - b_{m-1})^*}.$$

Comparing the leading coefficient (that before u_m) in (16) and (17) yields

$$[a_1,\dots,a_m]f = c\cdot\frac{(a_m-b_m)^*}{a_m-a_1}\cdot d_m.$$

As a consequence we get (18).

Then, according to (16), (17) and (18)

$$r_{m-1}u_m(z)=\frac{1}{d_m}\cdot\frac{(z-a_1)\cdots(z-a_{m-1})}{(z-b_1)^*\cdots(z-b_{m-1})^*(z-b_m)^*}.$$

Therefore, as a function of z

$$V\begin{vmatrix}u_1,\cdots,u_{m-1},u_m\\ a_1,\cdots,a_{m-1},z\end{vmatrix}=V\begin{vmatrix}u_1,\cdots,u_{m-1}\\ a_1,\cdots,a_{m-1}\end{vmatrix}\frac{(b_m-b_1)^*\cdots(b_m-b_{m-1})^*}{(b_m-a_1)^*\cdots(b_m-a_{m-1})^*}\cdot\frac{(z-a_1)\cdots(z-a_{m-1})}{(z-b_1)^*\cdots(z-b_{m-1})^*(z-b_m)^*}$$

is a rational function which is known explicitly.

Since

$$V\begin{vmatrix}u_1,\dots,u_{m-1},u_m\\ a_1,\dots,a_{m-1},a_m\end{vmatrix}=\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu_m(a_m)}V\begin{vmatrix}u_1,\dots,u_{m-1},u_m\\ a_1,\dots,a_{m-1},z\end{vmatrix}_{z=a_m},$$

the derivative can be computed by Leibniz' rule. Observing

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\mu_m(a_m)}\frac{(z-a_1)\cdots(z-a_{m-1})}{(z-b_1)^*\cdots(z-b_{m-1})^*(z-b_m)^*}\Bigg|_{z=a_m}=\mu_m(a_m)!\frac{\prod_{j=1}^{m-1}{}^*(a_m-a_j)}{\prod_{j=1}^{m}{}^*(a_m-b_j)}$$

and putting all things together, yields the formula

$$V\begin{vmatrix}u_1,\dots,u_m\\ a_1,\dots,a_m\end{vmatrix}=V\begin{vmatrix}u_1,\dots,u_{m-1}\\ a_1,\dots,a_{m-1}\end{vmatrix}\cdot\mu_m(a_m)!\frac{\prod_{j=1}^{m-1}{}^*(b_m-b_j)\prod_{j=1}^{m-1}{}^*(a_m-a_j)}{\prod_{j=1}^{m-1}{}^*(b_m-a_j)\prod_{j=1}^{m}{}^*(a_m-b_j)}.\tag{19}$$

Since

$$V\begin{vmatrix}u_1\\ a_1\end{vmatrix}=\frac{\mu_1(a_1)!}{(a_1-b_1)^*}$$

an induction argument proves (14). □

Remarks

(i) We note that (14) can also be proved more directly as follows: A moment's reflection shows

$$V\begin{vmatrix} u_1, \ldots, u_{m-1}, u_m \\ a_1, \ldots, a_{m-1}, z \end{vmatrix} = e \cdot \frac{(z-a_1)\cdots(z-a_{m-1})}{(z-b_1)^* \cdots (z-b_m)^*}, \tag{20}$$

with a constant e. Hence, using the notations of the proof of theorem 3, by comparing coefficients of u_m on both sides

$$V\begin{vmatrix} u_1, \ldots, u_{m-1} \\ a_1, \ldots, a_{m-1} \end{vmatrix} = e \cdot d_m.$$

Since d_m is computed above, the constant e in (20) is known and gives a representation for the left hand side of (20) from which (19) follows as above.

(ii) More general Cauchy–Vandermonde determinants and alternative representations thereof are determined in [4].

(iii) For the particular case of multiple poles but simple knots the Cauchy–Vandermonde determinant has been computed in [5].

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