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Symmetric coupling of boundary elements and Raviart–Thomas-type mixed finite elements in elastostatics

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Dedicated to Professor G. C. Hsiao on occasion of his 60th birthday.

Summary. Both mixed finite element methods and boundary integral methods are important tools in computational mechanics according to a good stress approximation. Recently, even low order mixed methods of Raviart–Thomas-type became available for problems in elasticity. Since either methods are robust for critical Poisson ratios, it appears natural to couple the two methods as proposed in this paper. The symmetric coupling changes the elliptic part of the bilinear form only. Hence the convergence analysis of mixed finite element methods is applicable to the coupled problem as well. Specifically, we couple boundary elements with a family of mixed elements analyzed by Stenberg. The locking-free implementation is performed via Lagrange multipliers, numerical examples are included.

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1. Introduction

In the classical finite element approach, the displacements are the unknowns while the stresses are computed afterwards in lower accuracy. In many applications, the stresses rather than the displacements are of primary interest. In the boundary element method (BEM), displacements and stresses in the interior of the domain are approximated with the same order.

Regarding finite elements, the approximation of the stresses can be improved by mixed methods. Here we consider Raviart–Thomas-type finite elements due to Stenberg [11], which are an improvement of Arnold–Brezzi–Douglas's PEERS element (plane elasticity element with reduced symmetry) [1]. The unknowns are three independent fields, namely the stress tensor, which is not a priori assumed to be symmetric, the rotation, which acts as a Lagrange multiplier to enforce the symmetry of the stress tensor in a weak form, and the displacement vector.

The analysis is based on the theory of mixed methods but refined by using mesh-dependent norms (Pitkäranta and Stenberg [9], [10]), by stability proofs on patches of elements and by weakening of the symmetry condition.

Another advantage of these elements is the capability of modeling nearly incompressible elasticity; more precisely, the relative error is independent of Poisson's ratio ν: there is no locking.

In many applications, for example, if nonlinearities in a bounded domain Ω_{F} are present as well as homogeneous, isotropic linear elastic material in an unbounded, e.g., exterior, domain Ω_{B}, one might combine the finite element method (in Ω_{F}) with the boundary integral method (acting on $\partial\Omega_{\mathrm{B}}$ but treating the problem in Ω_{B}). The symmetric coupling with boundary elements was proposed and analyzed by Costabel in [6] where the displacements inside the FEM domain are sought in the Sobolev space $H^1(\Omega_{\mathrm{F}})$; traces on the interface are inserted into the boundary integral equations, while the equilibrium of tractions across the interface is satisfied in a weak sense only.

In contrast, in the mixed FEM under consideration here, the stresses σ are required to satisfy $\sigma \in L^2(\Omega_{\mathrm{F}})$ and $\operatorname{div}\sigma \in L^2(\Omega_{\mathrm{F}})$, while the displacements are sought in $L^2(\Omega_{\mathrm{F}})$ only. Hence, the discrete tractions across interelement sides are continuous and, consequently, our coupled scheme is designed to yield continuous tractions across the interface between FEM and BEM while continuity of the displacements across the interelement sides and the interface is satisfied in a weak sense.

The construction of finite element spaces with continuous interelement tractions may be cumbersome and hence is enforced by using Lagrange multipliers. Then all unknowns, except the Lagrange multipliers, are discontinuous across the interelement sides and can be eliminated on each element before assembling the global linear system. We extended this scheme to the coupled system.

A similar coupling is possible with other mixed finite element methods than those considered in this paper. The weakening of the symmetry of the stress tensor is a particular way to construct stable finite element spaces, but is by no means necessary for the coupling.

The paper is organized as follows: A model problem (as depicted in Fig. 1) is described in its strong form in Sect. 2 and rewritten in a weak form. We prove existence and uniqueness of solutions and a norm estimate by using Brezzi's theory of mixed problems. The discretization is described in Sect. 3 where we state convergence estimates under some hypotheses on the mixed finite element methods. Proofs are given in Sect. 4 while in Sect. 5 Stenberg's locking-free family of methods is discussed. We describe the implementation by using Lagrange multipliers in Sect. 6 and give some numerical examples in Sect. 7.

2. The coupled problem

Let Ω be a bounded polygonal (resp. polyhedral) domain in $\mathbb{R}^d$, $d = 2$ (resp. $d = 3$) with boundary $\partial\Omega = \Gamma_u \cup \Gamma_t$ with disjoint Γ_u and Γ_t, either having a positive surface measure.

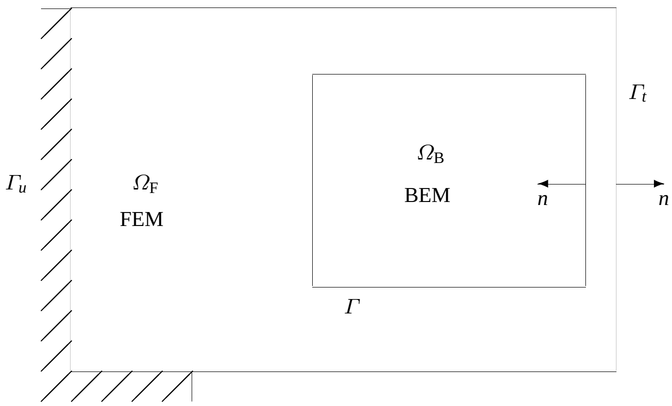

Fig. 1. Notation

Throughout this paper, we consider the linear elasticity problem

$$
\begin{array}{rcll}
\operatorname{div} \sigma & = & -f & \text{in } \Omega \\
\sigma & = & \mathbb{C}\varepsilon(u) & \text{in } \Omega \\
u & = & 0 & \text{on } \Gamma_u \\
\sigma n & = & 0 & \text{on } \Gamma_t .
\end{array} \tag{2.1}
$$

The displacement field is denoted by u, the related (linear Green) strain tensor is $\varepsilon(u) = \frac{1}{2}(\operatorname{grad} u + (\operatorname{grad} u)^{\mathrm{T}})$. The elasticity tensor $\mathbb{C}$ describes the stress–strain relationship; in the simplest case we have $\sigma = \mathbb{C}\varepsilon = \lambda \operatorname{tr} \varepsilon I + 2\mu\varepsilon$ where λ where μ are the Lamé coefficients and I denotes the identity matrix. (In the two-dimensional case, this is the constitutive equation of plane strain).

As depicted in Fig. 1, the domain Ω is partitioned into $\Omega = \Omega_{\mathrm{F}} \cup \Gamma \cup \Omega_{\mathrm{B}}$; the outward unit normal of Ω_{F} is denoted by n.

On Ω_{F} a mixed finite element method based on the Hellinger–Reissner principle is applied. This means the displacement $u : \Omega_{\mathrm{F}} \to \mathbb{R}^d$ and the stress $\sigma : \Omega_{\mathrm{F}} \to \mathbb{R}^{d\times d}$ are independent unknowns. Furthermore, the stress tensor σ is not a priori assumed to be symmetric. Symmetry will be enforced in a weak form by a Lagrange multiplier technique. To obtain a variational formulation, we choose test functions $\tau : \Omega_{\mathrm{F}} \to \mathbb{R}^{d\times d}$ satisfying $\tau n = 0$ on Γ_t, and gain from $(2.1)_b$

$$\int_{\Omega_{\mathrm{F}}} \tau : \mathbb{C}^{-1}\sigma \, d\Omega - \int_{\Omega_{\mathrm{F}}} \tau : \varepsilon(u) \, d\Omega = 0$$

so that integration by parts gives

$$\int_{\Omega_{\mathrm{F}}} \tau : \mathbb{C}^{-1}\sigma \, d\Omega + \int_{\Omega_{\mathrm{F}}} \operatorname{div} \tau \cdot u \, d\Omega + \int_{\Omega_{\mathrm{F}}} \tau : \gamma \, d\Omega = \int_{\Gamma} \tau n \cdot u \, ds$$

where the rotation $\gamma := \frac{1}{2}(\operatorname{grad} u - (\operatorname{grad} u)^{\mathrm{T}})$ is a new variable for the skew-symmetric part of the displacement gradient,

$$\gamma \in \mathscr{W} := \{\eta \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \eta + \eta^{\mathrm{T}} = 0\}.$$

The stresses and displacements are sought in

$$\begin{aligned} \mathscr{H} &:= \{\tau \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \operatorname{div}\tau \in L^2(\Omega_{\mathrm{F}})^d, \quad \tau n = 0 \text{ on } \Gamma_t\}, \\ \mathscr{L} &:= L^2(\Omega_{\mathrm{F}})^d. \end{aligned}$$

$\mathscr{H}$ is a Hilbert space when endowed with the norm

$$\|\tau\|_{\mathrm{div}} := (\|\tau\|^2_{0,\Omega_{\mathrm{F}}} + \|\operatorname{div}\tau\|^2_{0,\Omega_{\mathrm{F}}})^{1/2}.$$

In summary, the variational form of the problem in Ω_{F} reads: Given a body load f and a displacement φ on Γ, find $(u,\sigma,\gamma) \in \mathscr{L} \times \mathscr{H} \times \mathscr{W}$ satisfying

$$\text{(2.2)} \quad \begin{aligned} \int_{\Omega_{\mathrm{F}}} \tau : \mathbb{C}^{-1}\sigma\, d\Omega + \int_{\Omega_{\mathrm{F}}} \operatorname{div}\tau\cdot u\, d\Omega & \\ + \int_{\Omega_{\mathrm{F}}} \tau : \gamma\, d\Omega &= \int_{\Gamma} \tau n\cdot\varphi\, ds \qquad \forall \tau \in \mathscr{H} \\ \int_{\Omega_{\mathrm{F}}} \operatorname{div}\sigma\cdot v\, d\Omega &= -\int_{\Omega_{\mathrm{F}}} f\cdot v\, d\Omega \quad \forall v \in \mathscr{L} \\ \int_{\Omega_{\mathrm{F}}} \sigma : \eta\, d\Omega &= 0 \qquad \forall \eta \in \mathscr{W}. \end{aligned}$$

The identity $(2.2)_a$ is derived above, $(2.2)_b$ is a weak form of $(2.1)_a$, and $(2.2)_c$ is a weak form of the symmetry of σ.

We assume that the body load $f \in L^2(\Omega)^d$ vanishes on Ω_{B} for simplicity and that the Lamé coefficients are constant on Ω_{B}. Then, at any point $x \in \Omega_{\mathrm{B}}$, the displacement field can be represented by the Betti formula

$$u(x) = -\int_{\Gamma} G(x,y)\, T_y u(y)\, ds_y + \int_{\Gamma} \left(T_y G(x,y)\right)^{\mathrm{T}} u(y)\, ds_y\,.$$

Here $T_y u(y) = \mathbb{C}\varepsilon(u(y))\, n(y)$ is the traction corresponding to u at a point $y \in \Gamma$, and $T_y G(x,y)$ are the columnwise tractions of $G(x,y)$ at y. $G(x,y)$ is the fundamental solution and equals

$$\begin{aligned} &\frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)} \left\{\log\frac{1}{|x-y|} I + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(x-y)(x-y)^{\mathrm{T}}}{|x-y|^2}\right\} \quad \text{if } d = 2, \\ &\frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)} \left\{\frac{1}{|x-y|} I + \frac{\lambda+\mu}{\lambda+3\mu}\frac{(x-y)(x-y)^{\mathrm{T}}}{|x-y|^3}\right\} \quad \text{if } d = 3. \end{aligned}$$

Letting $x \to \Gamma$ we obtain with the classical jump relations the boundary integral equation

$$\text{(2.3)} \qquad \tfrac{1}{2} u = -Vt + Ku$$

with $t(y) = T_y u(y)$ and the integral operators

$$\begin{aligned} (Vt)(x) &= \int_{\Gamma} G(x,y)\, t(y)\, ds_y, \quad x \in \Gamma, \\ (Ku)(x) &= \int_{\Gamma} \left(T_y G(x,y)\right)^{\mathrm{T}} u(y)\, ds_y, \quad x \in \Gamma\,. \end{aligned}$$

Applying the traction operator T_x we get another boundary integral equation

(2.4) $$\tfrac{1}{2}t = -K't - Wu$$

where

$$\begin{aligned}(K't)(x) &= \int_\Gamma T_x G(x,y)\, t(y)\, ds_y, \quad x \in \Gamma, \\ (Wu)(x) &= -T_x \int_\Gamma \big(T_y G(x,y)\big)^{\mathrm{T}} u(y)\, ds_y, \quad x \in \Gamma.\end{aligned}$$

The symmetric coupling of (2.2) with the integral equations (2.3) and (2.4) is performed as follows: Introduce a new variable $\varphi := u|_\Gamma$ belonging to the trace space

$$H^{1/2} := H^{1/2}(\Gamma)^d := \{u|_\Gamma : u \in H^1(\Omega)^d\},$$

and use continuity of the tractions on Γ, i.e., $t = \sigma n$. Then, (2.3) reads

$$\varphi = -V(\sigma n) + (\tfrac{1}{2}I + K)\varphi$$

and this is inserted in the right-hand side of $(2.2)_a$ while (2.4) reads

$$W\varphi + (\tfrac{1}{2}I + K')(\sigma n) = 0$$

and this is added in a weak form to (2.2). (This coupling is in a sense 'dual' to the more classical approach, see e.g. [4, 6, 7], where a new variable is introduced for σn on the interface while for $u|_\Gamma$ continuity is used.) The resulting weak formulation is rewritten in a saddle point structure: Find $(\sigma, \varphi, u, \gamma) \in \mathscr{H} \times H^{1/2} \times \mathscr{L} \times \mathscr{W}$ such that

(2.5) $$\begin{aligned} a(\sigma,\varphi;\tau,\psi) + b(\tau;u,\gamma) &= 0 \\ b(\sigma;v,\eta) &= -\int_{\Omega_{\mathrm{F}}} f \cdot v\, d\Omega \end{aligned}$$

for all $(\tau, \psi, v, \eta) \in \mathscr{H} \times H^{1/2} \times \mathscr{L} \times \mathscr{W}$. Here,

$$\begin{aligned} a(\sigma,\varphi;\tau,\psi) &:= \int_{\Omega_{\mathrm{F}}} \tau : \mathbb{C}^{-1}\sigma\, d\Omega \\ &\quad + \langle \tau n\,,\, V(\sigma n)\rangle - \left\langle \tau n\,,\, (\tfrac{1}{2}I + K)\varphi \right\rangle \\ &\quad - \langle \psi\,,\, W\varphi\rangle - \left\langle \psi\,,\, (\tfrac{1}{2}I + K')(\sigma n)\right\rangle \\ b(\sigma;v,\eta) &:= \int_{\Omega_{\mathrm{F}}} \operatorname{div}\sigma \cdot v\, d\Omega + \int_{\Omega_{\mathrm{F}}} \sigma : \eta\, d\Omega. \end{aligned}$$

Throughout this paper, $\langle \varphi\,,\, \psi\rangle$ denotes the extension of the L^2-scalar product $\int_\Gamma \varphi \cdot \psi\, ds$ to the duality in $H^{-1/2} \times H^{1/2}$; $H^s := H^s(\Gamma)^d$.

Remark 2.1. It is known that the mappings $V : H^{-1/2} \to H^{1/2}$, $K : H^{1/2} \to H^{1/2}$, $K' : H^{-1/2} \to H^{-1/2}$, $W : H^{1/2} \to H^{-1/2}$ are well defined, linear and continuous (cf., e.g., [5]). V and W are symmetric; K' is dual to K and W is positive semi-definite and $\ker W = \ker \varepsilon|_\Gamma$, i.e., the kernel of W consists of the (linearized) rigid body motions. W is positive definite on $(H^{1/2}/\ker\varepsilon)^2$. For $d = 3$, V is positive definite. For $d = 2$, V is positive definite when restricted to $H_0^{-1/2}$ where

$$H_0^s := \{w \in H^s : \int_\Gamma w\, ds = 0\} \equiv H^s/\mathbb{R}^d.$$

We refer to [7] for proofs in case $d = 3$ and mention that the proofs work verbatim in case $d = 2$ (provided the radiation condition gives a sufficiently strong decay which is guaranteed owing to the restriction on $H_0^{-1/2}$).

Remark 2.2. We note that σn is defined in $H^{-1/2}$ via Green's formula even if $\sigma \in \mathscr{H}$ as follows. Given $v \in H^{1/2}$ extend it to some $v \in H^1(\Omega_\mathrm{F})^d$ with $v|_{\Gamma_u} = 0$. Then, let

$$\langle v\,,\, \sigma n\rangle := \int_{\Omega_\mathrm{F}} \sigma : \operatorname{grad} v\, d\Omega + \int_{\Omega_\mathrm{F}} v \cdot \operatorname{div} \sigma\, d\Omega.$$

The right-hand side is well defined and depends linearly and continuously on v and σ. Furthermore,

$$\|\sigma n\|_{-1/2,\Gamma} \le C\, \|\sigma\|_{\mathrm{div}}\,. \tag{2.6}$$

In view of the remarks, a is a symmetric and continuous bilinear form on $(\mathscr{H} \times H^{1/2})^2$, and b is continuous on $\mathscr{H} \times (\mathscr{L} \times \mathscr{W})$.

Theorem 2.1. *For every $f \in L^2(\Omega_\mathrm{F})^d$ the saddle point problem* (2.5) *has a unique solution satisfying*

$$\|\sigma\|_{0,\Omega_\mathrm{F}} + \|\operatorname{div}\sigma\|_{0,\Omega_\mathrm{F}} + \|\varphi\|_{1/2,\Gamma} + \|u\|_{0,\Omega_\mathrm{F}} + \|\gamma\|_{0,\Omega_\mathrm{F}} \le C\, \|f\|_{0,\Omega_\mathrm{F}} \tag{2.7}$$

with a positive constant C which is independent of f.

Proof. We apply the theory of saddle point problems, cf., e.g., [3, Sect. II.1]. It is sufficient to verify surjectivity of b and the inf–sup condition on a. As it is well known, the bilinear form b has the following surjectivity property: For all $(v, \eta) \in \mathscr{L} \times \mathscr{W}$ there is some $\tau \in \mathscr{H}$ satisfying

$$\operatorname{div}\tau = v \quad \text{and} \quad \operatorname{as}\tau := \tfrac{1}{2}(\tau - \tau^\mathrm{T}) = \eta. \tag{2.8}$$

Therefore, it remains to prove that, for a constant $\alpha > 0$,

$$\inf_{(\sigma,\varphi)} \sup_{(\tau,\psi)} \frac{a(\sigma,\varphi;\tau,\psi)}{(\,\|\sigma\|_{0,\Omega_\mathrm{F}} + \|\varphi\|_{1/2,\Gamma})(\,\|\tau\|_{0,\Omega_\mathrm{F}} + \|\psi\|_{1/2,\Gamma})} \ge \alpha \tag{2.9}$$

where in $\inf_{(\sigma,\varphi)}$ and $\sup_{(\tau,\psi)}$ the nonzero arguments run through $\ker B$,

$$\ker B := \{\tau \in \mathscr{H} : \operatorname{div}\tau = 0 \text{ and } \operatorname{as}\tau = 0\} \times H^{1/2}\,.$$

We will prove (2.9) in two steps partly arguing as in [4]: Given $(\sigma,\varphi)\in\ker B$ let $\varphi=\varphi_0+r_0$ where $\varphi_0\in H^{1/2}/\ker\varepsilon$ and $r_0\in\ker\varepsilon$, $\ker\varepsilon$ the (linearized) rigid body motions. Let $t_c\in\mathbb{R}^d$ be defined by

$$\int_\Gamma(\sigma n-t_c)\,ds=0.$$

Then, let $\sigma_c:=\mathbb{C}\varepsilon(u_c)$ where $u_c\in H^1(\Omega_F)^d$ is the solution of $\operatorname{div}\sigma_c=0$ on Ω_F while $\sigma_c n=t_c$ on Γ, $\sigma_c n=0$ on Γ_t and $u_c=0$ on Γ_u. Furthermore, let $\varphi_c\in\mathbb{R}^d$ satisfy

$$\left\langle V(\sigma n-t_c)+(\tfrac{1}{2}I+K)\varphi\,,\,t_c\right\rangle-\langle\sigma n\,,\,u_c\rangle=\langle t_c\,,\,\varphi_c\rangle\,. \tag{2.10}$$

(If $t_c=0$, every $\varphi_c\in\mathbb{R}^d$ solves (2.10).) Note that φ_c can be chosen such that $\|\varphi_c\|_{1/2,\Gamma}\le C(\|\sigma\|_{0,\Omega_F}+\|\varphi\|_{1/2,\Gamma})$. Thus,

$$\|\sigma-\sigma_c\|_{0,\Omega_F}+\|\varphi-\varphi_c\|_{1/2,\Gamma}\le C(\|\sigma\|_{0,\Omega_F}+\|\varphi\|_{1/2,\Gamma}) \tag{2.11}$$

where $C>0$ is independent of σ and φ. Integration by parts shows

$$\int_{\Omega_F}\sigma_c:\mathbb{C}^{-1}\sigma\,d\Omega=\langle\sigma n\,,\,u_c\rangle\,. \tag{2.12}$$

Then, using (2.10) and $K\varphi_c=\frac{1}{2}\varphi_c$,

$$\begin{aligned}a(\sigma,\varphi;\sigma-\sigma_c,-\varphi+\varphi_c)&=\int_{\Omega_F}\sigma:\mathbb{C}^{-1}\sigma\,d\Omega\\&\quad+\langle\sigma n-t_c\,,\,V(\sigma n-t_c)\rangle+\langle\varphi\,,\,W\varphi\rangle\,.\end{aligned}$$

Since W defines a positive definite bilinear form on $(H^{1/2}/\ker\varepsilon)^2$ and since V is positive definite on $(H_0^{-1/2})^2$ (recall $\sigma n-t_c\in H_0^{-1/2}$ by definition of t_c) we obtain

$$a(\sigma,\varphi;\sigma-\sigma_c,-\varphi+\varphi_c)\ge C_1(\|\sigma\|^2_{0,\Omega_F}+\|\varphi_0\|^2_{1/2,\Gamma}) \tag{2.13}$$

with $C_1>0$ depending only on $\mathbb{C}$ and W.

Now we show that for every rigid body motion r_0 there is a stress field τ_0 with $(\tau_0,0)\in\ker B$ and

$$a(0,r_0;\tau_0,0)=\langle r_0\,,\,r_0\rangle\,. \tag{2.14}$$

For example, let $\tau_0\in\mathscr{H}$ satisfy $\operatorname{div}\tau_0=0$ in Ω_F and $\tau_0 n=-r_0$ on Γ. Since $Kr_0=\frac{1}{2}r_0$, we conclude (2.14).

In the second step we assume for contradiction that (2.9) is false. Hence we may find a sequence (σ_j,φ_j) in $\ker B$ with $\|\sigma_j\|_{0,\Omega_F}+\|\varphi_j\|_{1/2,\Gamma}=1$ and

$$\sup_{(\tau,\psi)\in\ker B\setminus\{0\}}a(\sigma_j,\varphi_j;\tau,\psi)/(\|\tau\|_{0,\Omega_F}+\|\psi\|_{1/2,\Gamma})<1/j \tag{2.15}$$

for all j. Let $\varphi_j=\varphi_{0j}+r_j$ where $\varphi_{0j}\in H^{1/2}/\ker\varepsilon$ and $r_j\in\ker\varepsilon$.

According to (2.11), (2.13) and (2.15) we get

$$0 = \lim_{j\to\infty} (\|\sigma_j\|_{0,\Omega_{\mathrm{F}}} + \|\varphi_{0j}\|_{1/2,\Gamma})$$

and hence, since r_j is a bounded sequence in a finite dimensional space,

$$(\sigma_j, \varphi_j) \to (0, r_0) \quad \text{as } j \to \infty \tag{2.16}$$

for a subsequence (which is not relabeled) and a rigid body motion r_0. In particular, $\|r_0\|_{1/2,\Gamma} = 1$. Let τ_0 satisfy (2.14) with r_0 as in (2.16). Since a is continuous, (2.16) shows

$$0 = \lim_{j\to\infty} \frac{a(\sigma_j, \varphi_j; \tau_0, 0)}{\|\tau_0\|_{0,\Omega_{\mathrm{F}}}} = \frac{a(0, r_0; \tau_0, 0)}{\|\tau_0\|_{0,\Omega_{\mathrm{F}}}} > 0$$

owing to (2.14) and $r_0 \neq 0$. This contradiction verifies (2.9); the proof is finished. □

3. Approximation and convergence

For the finite element method we consider a regular family of triangulations $\mathscr{T}_h$ of $\overline{\Omega}_{\mathrm{F}}$. Two different triangles (resp. tetrahedrons) in $\mathscr{T}_h$ are either disjoint or have one common side or edge or vertex. Let the sides of finite elements in $\mathscr{T}_h$ on the interface Γ define a partition $\mathscr{E}_h$ of Γ and take $\mathscr{E}_h$ as boundary elements for simplicity. These partitions give rise to finite-dimensional subspaces $\mathscr{H}_h$, $H_h^{1/2}$, $\mathscr{L}_h$ and $\mathscr{W}_h$ which are assumed to satisfy (H1)–(H3); examples are considered in Sect. 5.

(H1) (Conformity and approximation property) There holds

$$\mathscr{H}_h \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h \subset \mathscr{H} \times H^{1/2} \times \mathscr{L} \times \mathscr{W}$$

and $\mathbb{R}^d \subset H_h^{1/2}$*. For all* $(\tau, \psi, v, \eta) \in \mathscr{H} \times H^{1/2} \times \mathscr{L} \times \mathscr{W}$*,*

$$\begin{aligned} 0 \quad = \quad & \lim_{h\to 0} \Big(\inf_{\tau_h \in \mathscr{H}_h} \|\tau - \tau_h\|_{\mathrm{div}} + \inf_{\psi_h \in H_h^{1/2}} \|\psi - \psi_h\|_{1/2,\Gamma} \\ & + \inf_{v_h \in \mathscr{L}_h} \|v - v_h\|_{0,\Omega_{\mathrm{F}}} + \inf_{\eta_h \in \mathscr{W}_h} \|\eta - \eta_h\|_{0,\Omega_{\mathrm{F}}} \Big). \end{aligned}$$

(H2) (Equilibrium condition) For each $\tau_h \in \mathscr{H}_h$*, the condition*

$$b(\tau_h; v_h, \eta_h) = 0 \quad \forall (v_h, \eta_h) \in \mathscr{L}_h \times \mathscr{W}_h$$

implies $\operatorname{div} \tau_h = 0$.

Remark 3.1. Note that a piecewise polynomial function τ_h satisfies $\operatorname{div} \tau_h \in L^2(\Omega_{\mathrm{F}})^d$ if and only if the tractions $\tau_h n$ are continuous across interelement sides (which is thus implied by $\mathscr{H}_h \subset \mathscr{H}$).

(H3) (inf–sup condition) The bilinear form b satisfies the inf–sup condition in some mesh-dependent norm. That is there exist a norm $\|\cdot\|_{\mathscr{H},h}$ *on some space* $\mathscr{H}_{(h)} \subset \mathscr{H}$ *with* $\mathscr{H}_h \subset \mathscr{H}_{(h)}$ *and*

$$\|\tau_h\|_{0,\Omega_{\mathrm{F}}} \leq \|\tau_h\|_{\mathscr{H},h} \leq C\,\|\tau_h\|_{\mathrm{div}} \tag{3.1}$$

for all $\tau_h \in \mathscr{H}_h$ *and a norm* $\|(\cdot,\cdot)\|_h$ *in* $\mathscr{L}_{(h)} \times \mathscr{W}_h$ *with* $\mathscr{L}_h \subset \mathscr{L}_{(h)} \subset \mathscr{L}$ *such that a and b are uniformly continuous (i.e., with h-independent bounds), and there is a positive constant* β *such that for all* $(v_h, \eta_h) \in \mathscr{L}_h \times \mathscr{W}_h$

$$\sup_{\tau_h \in \mathscr{H}_{0h}\setminus\{0\}} \frac{b(\tau_h; v_h, \eta_h)}{\|\tau_h\|_{\mathscr{H},h}} \geq \beta\,\|(v_h,\eta_h)\|_h . \tag{3.2}$$

where $\mathscr{H}_{0h} := \{\tau_h \in \mathscr{H}_h : \tau_h n|_\Gamma = 0\}$. *Both C and* β *are assumed to be independent of h.*

Remark 3.2. To ensure the continuity of a, we need the estimate

$$\|\tau n\|_{-1/2,\Gamma} \leq C\,\|\tau\|_{\mathscr{H},h} \quad \forall \tau \in \mathscr{H}_{(h)} \tag{3.3}$$

Note that (3.3) does not affect (3.2) owing to $\mathscr{H}_{0h}$.

Example 3.1. *Let* $\mathscr{T}_h$ *be a finite element triangulation of* Ω_{F} *and let* $\mathscr{S}_h$ *denote the set of sides in the interior of* Ω_{F} *and* $\overline{\mathscr{S}_h}$ *the set of all finite element sides. The value of the jump of* v *across an interelement side S is denoted by* $[v]$, *while the diameter of S is* h_S. *We follow Stenberg [10, 11]: The first mesh-dependent norm*

$$\|\tau\|^2_{\mathscr{H},h} := \|\tau\|^2_{0,\Omega_{\mathrm{F}}} + \sum_{S\in\overline{\mathscr{S}_h}} h_S \int_S |\tau n|^2\, ds$$

is defined on the 'intermediate' space

$$\mathscr{H}_{(h)} := \{\tau \in \mathscr{H} : \tau n \in L^2(S)^d \quad \forall S \in \overline{\mathscr{S}_h}\},$$

and the second

$$\begin{aligned}\|(v,\eta)\|^2_h \quad := \quad & \sum_{T\in\mathscr{T}_h} (\,\|\varepsilon(v)\|^2_{0,T} + \|\eta - \omega(v)\|^2_{0,T}) \\ & + \sum_{S\in\mathscr{S}_h} h_S^{-1} \int_S |[v]|^2\, ds + \sum_{S\subset\Gamma_u} h_S^{-1} \int_S |v|^2\, ds\end{aligned}$$

on $\mathscr{L}_{(h)} \times \mathscr{W}_h$ *with* $\omega(v) := \frac{1}{2}(\operatorname{grad} v - (\operatorname{grad} v)^{\mathrm{T}})$ *and*

$$\mathscr{L}_{(h)} := \{v \in L^2(\Omega)^d : v|_T \in H^1(T)^d \quad \forall T \in \mathscr{T}_h\}.$$

On $\mathscr{H}_h$, *the norms* $\|\cdot\|_{\mathscr{H},h}$ *and* $\|\cdot\|_{0,\Omega_{\mathrm{F}}}$ *are uniformly equivalent, i.e., for all* $\tau_h \in \mathscr{H}_h$

$$\|\tau_h\|_{0,\Omega_{\mathrm{F}}} \leq \|\tau_h\|_{\mathscr{H},h} \leq C\,\|\tau_h\|_{0,\Omega_{\mathrm{F}}}. \tag{3.4}$$

In these norms, the bilinear forms a and b are continuous, i.e., there exists $C > 0$ such that for all $(\sigma, \varphi), (\tau, \psi) \in \mathscr{H}_{(h)} \times H^{1/2}$

$$a(\sigma, \varphi; \tau, \psi) \le C(\|\sigma\|_{\mathscr{H},h} + \|\varphi\|_{1/2,\Gamma})(\|\tau\|_{\mathscr{H},h} + \|\psi\|_{1/2,\Gamma}) \tag{3.5}$$

and for all $(\tau, v, \eta) \in \mathscr{H}_{(h)} \times \mathscr{L}_{(h)} \times \mathscr{W}$

$$b(\tau; v, \eta) \le C \|\tau\|_{\mathscr{H},h} \|(v, \eta)\|_h . \tag{3.6}$$

To show continuity of a, it remains to verify (3.3)*. Recall*

$$\|\tau n\|_{-1/2,\Gamma} = \sup_{v \in H^{1/2}(\Gamma)^d} \frac{\langle \tau n\, ,\, v\rangle}{\|v\|_{1/2,\Gamma}} .$$

Each $v \in H^{1/2}(\Gamma)^d$ can be extended to $v \in H^1(\Omega_{\mathrm{F}})^d$ with $v|_{\Gamma_u} = 0$ and $\|v\|_{1,\Omega_{\mathrm{F}}} \le C\|v\|_{1/2,\Gamma}$. Integration by parts yields

$$\langle \tau n\, ,\, v\rangle = \int_{\Omega_{\mathrm{F}}} \tau : \operatorname{grad} v \, d\Omega + b(\tau; v, 0).$$

Further, $\|(v, 0)\|_h \le \|v\|_{1,\Omega_{\mathrm{F}}}$ since the jumps $[v]$ on the interelement sides vanish. Thus, using the continuity of b in the mesh-dependent norms, we obtain

$$\langle \tau n\, ,\, v\rangle \le C \|\tau\|_{\mathscr{H},h} \|v\|_{1,\Omega_{\mathrm{F}}} .$$

This proves (3.3)*.*

Remark 3.3. We assume throughout this paper that the solution to (2.5) satisfies $\sigma \in \mathscr{H}_{(h)}$ and $u \in \mathscr{L}_{(h)}$ for all $f \in L^2(\Omega_{\mathrm{F}})^d$, which is a regularity condition. In the situation of Example 3.1, the condition $u \in H^s(\Omega_{\mathrm{F}})^d$ with $s > 3/2$ is sufficient.

The *discretized saddle point problem* consists in finding $(\sigma_h, \varphi_h, u_h, \gamma_h) \in \mathscr{H}_h \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h$ such that for all $(\tau_h, \psi_h, v_h, \eta_h) \in \mathscr{H}_h \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h$

$$\begin{aligned} a(\sigma_h, \varphi_h; \tau_h, \psi_h) + b(\tau_h; u_h, \gamma_h) &= 0 \\ b(\sigma_h; v_h, \eta_h) &= -\int_{\Omega_{\mathrm{F}}} f \cdot v_h \, d\Omega \end{aligned} \tag{3.7}$$

Theorem 3.1. *Assuming (H1)–(H3), the discrete problem* (3.7) *has exactly one solution. There exists some h-independent constant $C > 0$ such that*

$$\begin{aligned} &\|\sigma - \sigma_h\|_{\mathscr{H},h} + \|\varphi - \varphi_h\|_{1/2,\Gamma} \\ &\quad \le C\Big(\inf_{\tau_h \in \mathscr{H}_h} \|\sigma - \tau_h\|_{\mathscr{H},h} + \inf_{\psi_h \in H_h^{1/2}} \|\varphi - \psi_h\|_{1/2,\Gamma} + \inf_{\eta_h \in \mathscr{W}_h} \|\gamma - \eta_h\|_{0,\Omega_{\mathrm{F}}}\Big). \end{aligned}$$

To derive L^2-estimates for the displacements, we need a regularity–approximation estimate and a mapping $P_h : \mathscr{L} \to \mathscr{L}_h \times \mathscr{W}_h$.

(H4) (Regularity–approximation property) In the mesh-dependent norms we have the following estimate where $\varrho(h)$ is an h-dependent constant such that

$$\inf_{(\tau_h,\psi_h,v_h,\eta_h)\in\mathscr{H}_h\times H_h^{1/2}\times\mathscr{L}_h\times\mathscr{W}_h} \|(\sigma-\tau_h,\varphi-\psi_h,u-v_h,\gamma-\eta_h)\|_h \leq \varrho(h)\,\|f\|_{0,\Omega_F} \tag{3.8}$$

for all $f\in L^2(\Omega_F)^d$ and $(\sigma,\varphi,u,\gamma)$ solving (2.5) *(cf. Theorem 2.1). The mesh-dependent norm $\|\cdot\|_h$ in* (3.8) *is defined by*

$$\|(\tau,\psi,v,\eta)\|_h := \|\tau\|_{\mathscr{H},h} + \|\psi\|_{1/2,\Gamma} + \|(v,\eta)\|_h . \tag{3.9}$$

Lemma 3.1. *There is a linear mapping $P_h:\mathscr{L}\to\mathscr{L}_h\times\mathscr{W}_h$ which satisfies*

$$b(\tau_h;v,0)=b(\tau_h;P_h(v))$$

for all $\tau_h\in\mathscr{H}_h$ and all $v\in\mathscr{L}$.

Modifications of this mapping P_h are frequently used in the literature for a proof of the inf–sup condition; we conversely may construct it from (H3). In particular cases, P_h is known explicitly (see Sect. 5).

Theorem 3.2. *Assuming (H1)–(H4), there exists some h-independent constant $C>0$ such that, with $\varrho(h)$ as in (H4),*

$$\|P_h(u)-(u_h,\gamma_h-\gamma)\|_{0,\Omega_F} \leq C\cdot\varrho(h)\cdot\Big(\inf_{\tau_h\in\mathscr{H}_h}\|\sigma-\tau_h\|_{\mathscr{H},h} + \inf_{\psi_h\in H_h^{1/2}}\|\varphi-\psi_h\|_{1/2,\Gamma} + \inf_{\eta_h\in\mathscr{W}_h}\|\gamma-\eta_h\|_{0,\Omega_F}\Big).$$

Remark 3.4. To obtain an estimate for $\|u-u_h\|_{0,\Omega_F}$, let $(\tilde{u},\tilde{\gamma}):=P_h(u)$ and observe that

$$\|u-u_h\|_{0,\Omega_F} \leq \|u-\tilde{u}\|_{0,\Omega_F} + \|P_h(u)-(u_h,\gamma_h-\gamma)\|_{0,\Omega_F}. \tag{3.10}$$

The last term is estimated by Theorem 3.2, and, according to the specific P_h, an a priori estimate of $\|u-\tilde{u}\|_{0,\Omega_F}$ is available.

Proofs are given in Sect. 4 while examples are studied in Sect. 5.

4. Proofs

The proofs of Theorem 3.1 and 3.2 use several lemmas where

$$\begin{aligned} Z &:= \{\tau\in\mathscr{H} : \operatorname{div}\tau=0 \text{ and } \tau=\tau^{\mathrm{T}}\}\\ Z_h &:= \{\tau_h\in\mathscr{H}_h : b(\tau_h;v_h,\eta_h)=0 \quad \forall (v_h,\eta_h)\in\mathscr{L}_h\times\mathscr{W}_h\}\\ \ker B_h &:= Z_h\times H_h^{1/2}. \end{aligned}$$

Throughout this section, $C>0$ denotes a generic h-independent constant.

Lemma 4.1. [11] *For all* $\tau_h \in Z_h$

$$\int_{\Omega_{\mathrm{F}}} \tau_h : \mathbb{C}^{-1}\tau_h \, d\Omega \geq C \, \|\tau_h\|^2_{0,\Omega_{\mathrm{F}}}.$$

Lemma 4.2. *For all* $\overline{\sigma} \in Z$ *there exists* $\overline{\sigma}_h \in Z_h$ *satisfying*

$$\|\overline{\sigma} - \overline{\sigma}_h\|_{\mathrm{div}} \leq C \inf_{\tau_h \in \mathscr{H}_h} \|\overline{\sigma} - \tau_h\|_{0,\Omega_{\mathrm{F}}}.$$

Proof. The result follows as in [11]; for related results we refer to [3, Proposition II.2.5] and [2, Remark III.4.6]. We give a proof for completeness and to stress that (H1)–(H3) are sufficient (even for different situations on the boundary). Let $\tilde{\sigma}$ be the best approximant to $\overline{\sigma} \in Z$ in $\mathscr{H}_h$ with respect to the $L^2(\Omega_{\mathrm{F}})$-norm. By Lemma 4.1 and (H3), the mixed finite element problem

$$\begin{aligned} \int_{\Omega_{\mathrm{F}}} \overline{\sigma}_h : \mathbb{C}^{-1}\tau_h \, d\Omega + b(\tau_h; \overline{u}_h, \overline{\gamma}_h) &= L_1(\tau_h) \\ b(\overline{\sigma}_h; v_h, \eta_h) &= L_2(v_h, \eta_h) \end{aligned}$$

(with linear forms L_1, L_2) satisfies ellipticity and inf–sup conditions in mesh-dependent norms so that we have a unique solution $(\overline{\sigma}_h, \overline{u}_h, \overline{\gamma}_h) \in \mathscr{H}_h \times \mathscr{L}_h \times \mathscr{W}_h$ satisfying

$$\int_{\Omega_{\mathrm{F}}} (\overline{\sigma}_h - \overline{\sigma}) : \mathbb{C}^{-1}\tau_h \, d\Omega + b(\tau_h; \overline{u}_h, \overline{\gamma}_h) = 0 \tag{4.1}$$

$$b(\overline{\sigma}_h - \tilde{\sigma}; v_h, \eta_h) = 0 \tag{4.2}$$

for all $(\tau_h, v_h, \eta_h) \in \mathscr{H}_h \times \mathscr{L}_h \times \mathscr{W}_h$. Note that $\overline{\sigma}_h - \tilde{\sigma} \in Z_h$ according to (4.2). Hence, Lemma 4.1 yields

$$\begin{aligned} C \, \|\overline{\sigma}_h - \tilde{\sigma}\|^2_{0,\Omega_{\mathrm{F}}} &\leq \int_{\Omega_{\mathrm{F}}} (\overline{\sigma}_h - \tilde{\sigma}) : \mathbb{C}^{-1}(\overline{\sigma}_h - \tilde{\sigma}) \, d\Omega \\ &= \int_{\Omega_{\mathrm{F}}} (\overline{\sigma} - \tilde{\sigma}) : \mathbb{C}^{-1}(\overline{\sigma}_h - \tilde{\sigma}) \, d\Omega \end{aligned}$$

owing to (4.1). Thus, by Cauchy's inequality,

$$\|\overline{\sigma}_h - \tilde{\sigma}\|_{0,\Omega_{\mathrm{F}}} \leq C \, \|\overline{\sigma} - \tilde{\sigma}\|_{0,\Omega_{\mathrm{F}}}.$$

Then, the triangle inequality and (H2) finish the proof of the lemma. □

Lemma 4.3. *There is a constant* $\alpha > 0$ *such that*

$$\inf_{(\sigma_h,\varphi_h)} \sup_{(\tau_h,\psi_h)} \frac{a(\sigma_h, \varphi_h; \tau_h, \psi_h)}{(\|\sigma_h\|_{\mathscr{H},h} + \|\varphi_h\|_{1/2,\Gamma})(\|\tau_h\|_{\mathscr{H},h} + \|\psi_h\|_{1/2,\Gamma})} \geq \alpha \tag{4.3}$$

where the nonzero arguments in $\inf_{(\sigma_h,\varphi_h)}$ *and* $\sup_{(\tau_h,\psi_h)}$ *belong to* $\ker B_h$.

Proof. According to the equilibrium condition (H2) it is sufficient to consider the norm $\|\cdot\|_{0,\Omega_F}$ instead of $\|\cdot\|_{\mathscr{H},h}$ (because, by (3.1), $\|\tau_h\|_{0,\Omega_F} \le \|\tau_h\|_{\mathscr{H},h} \le C\,\|\tau_h\|_{\text{div}} = C\,\|\tau_h\|_{0,\Omega_F}$ for all $\tau_h \in Z_h$). We proceed as in the proof of Theorem 2.1. Assuming that (4.3) is false we find a sequence of meshes, discrete spaces and discrete functions (σ_j, φ_j) in $\ker B_{h_j}$ with $\|\sigma_j\|_{0,\Omega_F} + \|\varphi_j\|_{1/2,\Gamma} = 1$ and

$$\sup_{(\tau,\psi)\in\ker B_{h_j}\setminus\{0\}} a(\sigma_j, \varphi_j; \tau, \psi)/(\|\tau\|_{0,\Omega_F} + \|\psi\|_{1/2,\Gamma}) < 1/j \tag{4.4}$$

for all j. Let $\varphi_j = \varphi_{0j} + r_j$ where $\varphi_{0j} \in H^{1/2}/\ker\varepsilon$ and $r_j \in \ker\varepsilon$.

Since (σ_j, φ_j) are bounded in $\mathscr{H} \times H^{1/2}$ we may extract a weakly convergent subsequence (not relabeled); so assume,

$$(\sigma_j, \varphi_{0j}) \rightharpoonup (\sigma, \varphi_0) \quad \text{(weakly) in } \mathscr{H} \times H^{1/2}/\ker\varepsilon$$

for some $(\sigma, \varphi_0) \in \mathscr{H} \times H^{1/2}/\ker\varepsilon$. By (H2) we have $\operatorname{div}\sigma_j = 0$ and so $\operatorname{div}\sigma = 0$. Define $t_j \in \mathbb{R}^d$ by

$$\int_\Gamma (\sigma_j n - t_j)\, ds = 0.$$

The weak convergence of (σ_j) causes strong convergence of t_j in $\mathbb{R}^d$ so that

$$\lim_{j\to\infty} t_j =: \bar{t} \in \mathbb{R}^d. \tag{4.5}$$

Let $\bar\sigma \in Z$ satisfy $\bar\sigma n = \bar t$ on Γ. Assume $\bar t \ne 0$ first. Then, for any $t_j \ne 0$, we choose $\bar\sigma_j \in Z_{h_j}$ as in Lemma 4.2. If $\bar t = 0$ we choose $\bar\sigma_j = 0 = \bar\sigma$. Note $\bar\sigma_j \to \bar\sigma$ strongly in $\mathscr{H}$ as $j \to \infty$ because of Lemma 4.2, (H1) and (4.5).

By $\mathbb{R}^d \subset H_h^{1/2}$ (cf., (H1)) we have $-\varphi_j + c_j \in H_h^{1/2}$ for each $c_j \in \mathbb{R}^d$. We define $c_j \in \mathbb{R}^d$ with minimal Euclid norm in $\mathbb{R}^d$ such that

$$\begin{aligned} \langle c_j\,,\, t_j\rangle \;=\;& \langle t_j\,,\, V(2\sigma_j n - t_j)\rangle \\ & - \left\langle \bar\sigma_j n\,,\, V(\sigma_j n) - (\tfrac{1}{2}I + K)\varphi_j \right\rangle - \int_{\Omega_F} \bar\sigma_j : \mathbb{C}^{-1}\sigma_j\, d\Omega. \end{aligned} \tag{4.6}$$

Note that $(\bar\sigma_j, c_j)$ are bounded in $\mathscr{H} \times H^{1/2}$. Using (4.6) we compute

$$\begin{aligned} a(\sigma_j, \varphi_j; \sigma_j - \bar\sigma_j, -\varphi_j + c_j) &= \int_{\Omega_F} \sigma_j : \mathbb{C}^{-1}\sigma_j\, d\Omega \\ &\quad + \langle \sigma_j n - t_j\,,\, V(\sigma_j n - t_j)\rangle + \langle \varphi_j\,,\, W\varphi_j\rangle \\ &\ge C\left(\|\sigma_j\|^2_{0,\Omega_F} + \|\varphi_{0j}\|^2_{1/2,\Gamma}\right) \end{aligned}$$

where $C > 0$ depends on W and the constant in Lemma 4.1 only. Since (4.4) and $\|\sigma_j - \bar\sigma_j\|_{0,\Omega_F} + \|-\varphi_j + c_j\|_{1/2,\Gamma} \le C$ the above estimate proves

$$(\sigma_j, \varphi_j) \to (0, r_0) \quad \text{(strongly) in } \mathscr{H} \times H^{1/2},$$

$(\sigma, \varphi_0) = 0$; $r_0 \in \ker\varepsilon$. Thus, $\|r_0\|_{1/2,\Gamma} = 1$.

Let $\hat{\sigma} \in Z$ satisfy $\hat{\sigma} n = -r_0$ on Γ. By Lemma 4.2 we find a discrete stress field $\hat{\sigma}_j \in Z_{h_j}$ which, by (H1), converges towards $\hat{\sigma}$ in $(\mathscr{H}, \|\cdot\|_{\mathrm{div}})$. Letting $(\tau, \psi) := (\hat{\sigma}_j, 0) \in \ker B_{h_j} \setminus \{0\}$ in (4.4),

$$a(\sigma_j, \varphi_j; \hat{\sigma}_j, 0) < \frac{1}{j} \|\hat{\sigma}_j\|_{0,\Omega_{\mathrm{F}}}.$$

By strong convergence, for $j \to \infty$, $a(0, r_0; \hat{\sigma}, 0) \leq 0$. But, by construction of $\hat{\sigma}$ and because $\|r_0\|_{1/2,\Gamma} = 1$, $a(0, r_0; \hat{\sigma}, 0) = \langle r_0 \, , \, r_0 \rangle > 0$. This contradiction proves the lemma. □

Proof of Lemma 3.1. Let Z_h^0 denote the set of functionals Φ on $\mathscr{H}_h$ with $\Phi(\tau_h) = 0$ for all $\tau_h \in Z_h$. By (H3), the mapping

$$\begin{cases} \mathscr{L}_h \times \mathscr{W}_h & \to & Z_h^0 \\ (v_h, \eta_h) & \mapsto & b(\cdot; v_h, \eta_h) \end{cases}$$

is an isomorphism [2, 3] (assuming $\mathscr{L}_h \times \mathscr{W}_h$ be endowed with the mesh-dependent norm). For all $v \in \mathscr{L}$, (H2) implies $b(\cdot; v, 0)|_{\mathscr{H}_h} \in Z_h^0$. Hence, for each $v \in \mathscr{L}$, there exists $P_h(v) := (v_h, \eta_h) \in \mathscr{L}_h \times \mathscr{W}_h$ satisfying

$$b(\cdot; v_h, \eta_h)|_{\mathscr{H}_h} = b(\cdot; v, 0)|_{\mathscr{H}_h}. \quad \square$$

Proof of Theorem 3.1. We follow [11, Proof of Theorem 3.1] and let $(\tilde{\sigma}, \tilde{\varphi}, \tilde{\gamma})$ be the best approximant to $(\sigma, \varphi, \gamma)$ in $\mathscr{H}_h \times H_h^{1/2} \times \mathscr{W}_h$. Let $\mathscr{H}_h \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h$ be endowed with the norm (3.9). Because of (H3) and Lemma 4.3 we get the inf–sup condition for the discrete spaces. With the theory of saddle point problems there exists $(\tau_h, \psi_h, v_h, \eta_h)$ in $\mathscr{H}_h \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h$ with

$$\begin{aligned}
&\|(\tau_h, \psi_h, v_h, \eta_h)\|_h \leq C \quad \text{and} \\
&\|\tilde{\sigma} - \sigma_h\|_{\mathscr{H},h} + \|\tilde{\varphi} - \varphi_h\|_{1/2,\Gamma} + \|P_h(u) - (u_h, \gamma_h - \tilde{\gamma})\|_h \\
&\quad \leq \quad a(\tilde{\sigma} - \sigma_h, \tilde{\varphi} - \varphi_h; \tau_h, \psi_h) \\
&\qquad\quad + b(\tau_h; P_h(u) - (u_h, \gamma_h - \tilde{\gamma})) + b(\tilde{\sigma} - \sigma_h; v_h, \eta_h) \\
&\quad = \quad a(\tilde{\sigma} - \sigma, \tilde{\varphi} - \varphi; \tau_h, \psi_h) \\
&\qquad\quad + b(\tilde{\sigma} - \sigma; v_h, \eta_h) - b(\tau_h; 0, \gamma - \tilde{\gamma})
\end{aligned}$$

where we used the discrete equations and $b(\tau_h; P_h(u) - (u, 0)) = 0$ according to Lemma 3.1. Since a and b are bounded (in the discrete norms)

$$\begin{aligned}
&\|\tilde{\sigma} - \sigma_h\|_{\mathscr{H},h} + \|\tilde{\varphi} - \varphi_h\|_{1/2,\Gamma} + \|P_h(u) - (u_h, \gamma_h - \tilde{\gamma})\|_h \\
&\quad \leq \quad C \left(\|\tilde{\sigma} - \sigma\|_{\mathscr{H},h} + \|\tilde{\varphi} - \varphi\|_{1/2,\Gamma} + \|\tilde{\gamma} - \gamma\|_{0,\Omega_{\mathrm{F}}} \right).
\end{aligned}$$

This and the triangle inequality prove

$$\begin{aligned}
(4.7) \quad &\|\sigma - \sigma_h\|_{\mathscr{H},h} + \|\varphi - \varphi_h\|_{1/2,\Gamma} + \|P_h(u) - (u_h, \gamma_h - \gamma)\|_h \\
\leq \quad & C \left(\inf_{\tau_h \in \mathscr{H}_h} \|\sigma - \tau_h\|_{\mathscr{H},h} + \inf_{\psi_h \in H_h^{1/2}} \|\varphi - \psi_h\|_{1/2,\Gamma} + \inf_{\eta_h \in \mathscr{W}_h} \|\gamma - \eta_h\|_{0,\Omega_{\mathrm{F}}} \right).
\end{aligned}$$

The proof of Theorem 3.1 is finished. □

Proof of Theorem 3.2. We follow [11], assume (H4) and continue in the notations of the proof of Theorem 3.1. With Theorem 2.1 we find (Π, Ψ, z, μ) in $\mathscr{H} \times H^{1/2} \times \mathscr{L} \times \mathscr{W}$ satisfying, for all (τ, ψ, v, η) in $\mathscr{H} \times H^{1/2} \times \mathscr{L} \times \mathscr{W}$,

$$\begin{aligned} a(\Pi, \Psi; \tau, \psi) + b(\tau; z, \mu) &= 0 \\ b(\Pi; v, \eta) &= \int_{\Omega_{\mathrm{F}}} \Big(P_h(u) - (u_h, \gamma_h - \gamma)\Big) \cdot (v, \eta)\, d\Omega. \end{aligned}$$

Taking $(\tau, \psi, v, \eta) = (\sigma - \sigma_h, \varphi - \varphi_h, P_h(u) - (u_h, \gamma_h - \gamma))$ we obtain

$$\begin{aligned} &\|P_h(u) - (u_h, \gamma_h - \gamma)\|^2_{0,\Omega_{\mathrm{F}}} = b(\Pi; P_h(u) - (u_h, \gamma_h - \gamma)) \\ &= \ a(\Pi, \Psi; \sigma - \sigma_h, \varphi - \varphi_h) \\ &\quad + b(\sigma - \sigma_h; z, \mu) + b(\Pi; P_h(u) - (u_h, \gamma_h - \gamma)) \\ &= \ a(\sigma - \sigma_h, \varphi - \varphi_h; \Pi - \tilde{\Pi}, \Psi - \tilde{\Psi}) \\ &\quad + b(\sigma - \sigma_h; z - \tilde{z}, \mu - \tilde{\mu}) + b(\Pi - \tilde{\Pi}; P_h(u) - (u_h, \gamma_h - \gamma)) \end{aligned}$$

using Galerkin equations and the definition of P_h in Lemma 3.1 for the best approximants $(\tilde{\Pi}, \tilde{\Psi}, \tilde{z}, \tilde{\mu})$ in $\mathscr{H}_h \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h$ to (Π, Ψ, z, μ). Considering norms and using (4.7) for $\sigma - \sigma_h$, $\varphi - \varphi_h$ and $P_h(u) - (u_h, \gamma_h - \gamma)$ we gain

$$\begin{aligned} \|P_h(u) - (u_h, \gamma_h - \gamma)\|^2_{0,\Omega_{\mathrm{F}}} \ &\leq \ C \cdot \|(\Pi - \tilde{\Pi}, \Psi - \tilde{\Psi}, z - \tilde{z}, \mu - \tilde{\mu})\|_h \\ &\cdot \left(\|\tilde{\sigma} - \sigma\|_{\mathscr{H},h} + \|\tilde{\varphi} - \varphi\|_{1/2,\Gamma} + \|\tilde{\gamma} - \gamma\|_{0,\Omega_{\mathrm{F}}} \right). \end{aligned}$$

By (H4), we have an a priori estimate of the form

$$\|(\Pi - \tilde{\Pi}, \Psi - \tilde{\Psi}, z - \tilde{z}, \mu - \tilde{\mu})\|_h \leq \varrho(h)\, \|P_h(u) - (u_h, \gamma_h - \gamma)\|_{0,\Omega_{\mathrm{F}}}.$$

Using this in the former estimate and dividing by $\|P_h(u) - (u_h, \gamma_h - \gamma)\|_{0,\Omega_{\mathrm{F}}}$ we conclude the claimed estimate. □

5. A family of elements

In the sequel we describe a family of finite element spaces due to Stenberg [11]. For each tetrahedron (resp. triangle) $T \in \mathscr{T}_h$ we define a bubble function b_T by

$$b_T(x) = \prod_{i=0}^{d} \lambda_i(x),$$

where $\lambda_0, \ldots, \lambda_d$ are the barycentric coordinates in T. By $P_k(T)$ we denote the space of polynomials of degree $\leq k$ on T. For $d = 3$ we define

$$\begin{aligned} B_l(T) \ := \ \{&(\tau_{ij}) : (\tau_{i1}, \ldots, \tau_{i3}) = \operatorname{curl}(b_T w_{i1}, \ldots, b_T w_{i3}), \\ &w_{ij} \in P_l(T),\ i, j = 1, 2, 3\} \end{aligned}$$

with $\operatorname{curl} z = \nabla \times z$, whereas for $d = 2$, with $\operatorname{curl} z = (\partial_2 z, -\partial_1 z)$,

$$B_l(T) := \{(\tau_{ij}) : (\tau_{i1}, \tau_{i2}) = \operatorname{curl}(b_T w_i),\ w_i \in P_l(T),\ i = 1, 2\}.$$

First we define the finite element spaces for polynomial degree $k \geq 2$,

$$\begin{aligned}
\mathscr{H}_h &:= \{\tau_h \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \operatorname{div}\tau_h \in L^2(\Omega_{\mathrm{F}})^d, \tau_h n = 0 \text{ on } \Gamma_t, \\
&\qquad \tau_h|_T \in P_k(T)^{d\times d} + B_{k-1}(T) \quad \forall T \in \mathscr{T}_h\}, \\
\mathscr{L}_h &:= \{v_h \in L^2(\Omega_{\mathrm{F}})^d : v_h|_T \in P_{k-1}(T)^d \quad \forall T \in \mathscr{T}_h\}, \\
\mathscr{W}_h &:= \{\eta_h \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \eta_h + \eta_h^{\mathrm{T}} = 0, \\
&\qquad \eta_h|_T \in P_k(T)^{d\times d} \quad \forall T \in \mathscr{T}_h\}.
\end{aligned}$$

For the lowest order method we need the space of (linearized) rigid body motions on T,

$$R(T) := \begin{cases} \{a + b \times x : a, b \in \mathbb{R}^3\}, & d = 3, \\ \{(a, b) + c(-x_2, x_1) : a, b, c \in \mathbb{R}\}, & d = 2, \end{cases}$$

and the skew-symmetric tensors

$$Q(T) := \{b_T \varrho : \varrho \in P_0(T)^{d\times d},\ \varrho + \varrho^{\mathrm{T}} = 0\}.$$

Then, the finite element spaces for $k = 1$ are

$$\begin{aligned}
\mathscr{H}_h &:= \{\tau_h \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \operatorname{div}\tau_h \in L^2(\Omega_{\mathrm{F}})^d, \tau_h n = 0 \text{ on } \Gamma_t, \\
&\qquad \tau_h|_T \in P_1(T)^{d\times d} \oplus Q(T) \oplus B_0(T) \quad \forall T \in \mathscr{T}_h\}, \\
\mathscr{L}_h &:= \{v_h \in L^2(\Omega_{\mathrm{F}})^d : v_h|_T \in R(T) \quad \forall T \in \mathscr{T}_h\}, \\
\mathscr{W}_h &:= \{\eta_h \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \eta_h + \eta_h^{\mathrm{T}} = 0, \\
&\qquad \eta_h|_T \in P_1(T)^{d\times d} \quad \forall T \in \mathscr{T}_h\}.
\end{aligned}$$

For the discretized displacements in the boundary element method we take piecewise polynomials of degree $\kappa \geq 1$,

$$H_h^{1/2} := \{\psi_h \in C(\Gamma)^d : \psi_h|_S \in P_\kappa(S)^d\ \forall S \in \mathscr{E}_h\}.$$

The degree κ can be chosen independently of k.

As shown in [11], the hypotheses of Theorem 3.1 are satisfied, using the mesh-dependent norms of Example 3.1 above. Besides the well-known approximation properties of piecewise polynomials in $L^2(\Omega)^d$ and $H^{1/2}$ we have [10, Lemma 3.1]

$$\inf_{\tau_h \in \mathscr{H}_h} \|\sigma - \tau_h\|_{\mathscr{H},h} \leq Ch^s \|\sigma\|_{s,\Omega_{\mathrm{F}}}$$

with $0 \leq s \leq k + 1$ provided the exact solution is sufficiently regular. Therefore we can expect convergence rates

$$\begin{aligned}
&\|\sigma - \sigma_h\|_{0,\Omega_{\mathrm{F}}} + \|\varphi - \varphi_h\|_{1/2,\Gamma} \\
&\quad \leq Ch^s \left(\|\sigma\|_{s,\Omega_{\mathrm{F}}} + \|\varphi\|_{s+1/2,\Gamma} + \|\gamma\|_{s,\Omega_{\mathrm{F}}} \right)
\end{aligned} \tag{5.1}$$

with $0 \le s \le \min\{k+1, \kappa + 1/2\}$.

Next, we investigate the consequences of Theorem 3.2. For the higher order methods, i.e., $k \ge 2$, we have $\operatorname{div} \tau_h \in \mathscr{L}_h$ for all $\tau_h \in \mathscr{H}_h$ and thus the mapping P_h (defined in Lemma 3.1) is the L^2-projection from $\mathscr{L}$ onto $\mathscr{L}_h$. Also for the lowest order method, $k = 1$, we have $\|u - P_h u\|_{0,\Omega_{\mathrm{F}}} \le Ch^r \|u\|_{r,\Omega_{\mathrm{F}}}$ with $r \le k$. Thus, from (3.10) we get

$$\|u - u_h\|_{0,\Omega_{\mathrm{F}}} \le Ch^r \|u\|_{r,\Omega_{\mathrm{F}}} + C\varrho(h)h^s \left(\|\sigma\|_{s,\Omega_{\mathrm{F}}} + \|\varphi\|_{s+1/2,\Gamma} + \|\gamma\|_{s,\Omega_{\mathrm{F}}}\right)$$

with $r \le k$ and s as in (5.1). To comment on $\varrho(h)$ from (H4), let us remark that for some $q \in [0,1]$ we have some regularity property

$$\|u\|_{q+1,\Omega_{\mathrm{F}}} + \|u\|_{q+1/2,\Gamma} + \|\sigma\|_{q,\Omega_{\mathrm{F}}} \le C \|f\|_{0,\Omega_{\mathrm{F}}}.$$

From the approximation property

$$\inf_{(v_h,\eta_h)\in\mathscr{L}_h\times\mathscr{W}_h} \|(u - v_h, \gamma - \eta_h)\|_h \le Ch^q \|u\|_{q+1,\Omega_{\mathrm{F}}}, \quad q \le k-1,$$

we infer $\varrho(h) = Ch^q$ if $k \ge 2$ and $\varrho(h) = C$ if $k = 1$. Altogether,

$$\|u - u_h\|_{0,\Omega_{\mathrm{F}}} \le C(u) \cdot h^{\min\{r,s+q\}}. \tag{5.2}$$

6. Implementation using Lagrange multipliers

In the implementation of Stenberg's elements considered in Sect. 5, we a priori assume for convenience that the tractions are not continuous across interelement boundaries and across the FEM–BEM interface. The continuity will be enforced by a Lagrange multiplier such that the solution of the discretized equations (3.7) is obtained. We drop the conditions $\operatorname{div} \tau_h \in L^2(\Omega_{\mathrm{F}})^d$ and $\tau_h n = 0$ on Γ_t, i.e. for $k \ge 2$ we seek the stresses in

$$\hat{\mathscr{H}}_h := \{\tau_h \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \tau_h|_T \in P_k(T)^{d\times d} + B_{k-1}(T) \quad \forall T \in \mathscr{T}_h\},$$

whereas for $k = 1$

$$\hat{\mathscr{H}}_h := \{\tau_h \in L^2(\Omega_{\mathrm{F}})^{d\times d} : \tau_h|_T \in P_1(T)^{d\times d} \oplus Q(T) \oplus B_0(T) \quad \forall T \in \mathscr{T}_h\}.$$

Further, we approximate the tractions t_h on the interface by

$$H_h^{-1/2} := \{\chi_h \in L^2(\Gamma)^d : \chi_h|_S \in P_k(S)^d \ \forall S \in \mathscr{E}_h\}.$$

It is easily verified that $\tau_h n = 0$ on ∂T for all $\tau_h \in B_l(T)$, $l \ge 0$. Thus the Lagrange multiplier must be of polynomial degree k. So, let us introduce the space $\mathscr{M}_h$ of functions μ_h defined on the union of all finite element sides S such that

$$\mu_h|_S \in P_k(S)^d \quad \text{and} \quad \mu_h|_S = 0 \text{ if } S \subset \Gamma_u. \tag{6.1}$$

As a slight generalization, we assume that on Γ_t a traction g is given. The final form of our method is as follows: Find

$$(\sigma_h, t_h, \varphi_h, u_h, \gamma_h, \lambda_h) \in \hat{\mathscr{H}}_h \times H_h^{-1/2} \times H_h^{1/2} \times \mathscr{L}_h \times \mathscr{W}_h \times \mathscr{M}_h \quad \text{with}$$

$$\begin{aligned}
&\int_{\Omega_F} \tau_h : \mathbb{C}^{-1}\sigma_h \, d\Omega + \int_{\Omega_F} \tau_h : \gamma_h \, d\Omega \\
&\qquad + \sum_{T \in \mathscr{T}_h} \left\{ \int_T \operatorname{div} \tau_h \cdot u_h \, d\Omega - \int_{\partial T} \tau_h n \cdot \lambda_h \, ds \right\} = 0 \qquad \forall \tau_h \in \hat{\mathscr{H}}_h \\
&\qquad \langle \chi_h \, , \, \lambda_h \rangle + \langle \chi_h \, , \, Vt_h \rangle - \langle \chi_h \, , \, (\tfrac{1}{2} I + K)\varphi_h \rangle = 0 \qquad \forall \chi_h \in H_h^{-1/2} \\
&\qquad - \langle \psi_h \, , \, W\varphi_h \rangle - \langle \psi_h \, , \, (\tfrac{1}{2} I + K')t_h \rangle = 0 \qquad \forall \psi_h \in H_h^{1/2} \\
&\qquad \sum_{T \in \mathscr{T}_h} \int_T \operatorname{div} \sigma_h \cdot v_h \, d\Omega = - \int_{\Omega_F} f \cdot v_h \, d\Omega \qquad \forall v_h \in \mathscr{L}_h \\
&\qquad \int_{\Omega_F} \sigma_h : \eta_h \, d\Omega = 0 \qquad \forall \eta_h \in \mathscr{W}_h \\
&\qquad - \sum_{T \in \mathscr{T}_h} \int_{\partial T} \sigma_h n \cdot \mu_h \, ds + \langle t_h \, , \, \mu_h \rangle = - \int_{\Gamma_t} g \cdot \mu_h \, ds \qquad \forall \mu_h \in \mathscr{M}_h .
\end{aligned}$$

The last equation ensures the continuity of the tractions. Since σ_h, u_h and γ_h are discontinuous across interelement sides, these unknowns can be eliminated on each element before assembling the global system.

7. Numerical experiments

We investigate two model problems of two-dimensional plane strain. To be able to compute convergence rates, we take in the first example the exact solution for a cylindrical cavity of radius unity in an unbounded domain under uniform tension of magnitude one in x_1-direction, that is,

$$\begin{aligned}
u_r &= \frac{1+\nu}{2E}\left[\frac{1}{r} + \left(\frac{4(1-\nu)}{r} - \frac{1}{r^3}\right)\cos 2\theta\right] \\
u_\theta &= -\frac{1+\nu}{2E}\left(\frac{2-4\nu}{r} + \frac{1}{r^3}\right)\sin 2\theta
\end{aligned}$$

in polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. Here, Young's modulus is $E = \mu(3\lambda + 2\mu)/(\lambda + \mu) > 0$, and Poisson's ratio is $\nu = \lambda/(2(\lambda + \mu))$, $0 \le \nu < 1/2$.

We apply the finite element method in the square $\Omega_F = \{(x_1, x_2) : 1 < x_1 < 2,\ 0 < x_2 < 1\}$, while the boundary elements live on the boundary of the square $\Omega_B = \{(x_1, x_2) : 1 < x_1 < 2,\ -1 < x_2 < 0\}$. (This is slightly more general than the situation in Fig. 1.) Displacements are prescribed on $\Gamma_{Bu} = \{(x_1, -1) : 1 < x_1 < 2\}$, whereas on the rest of the boundary tractions are given. The material parameters are $E = 1$ and $\nu = 0.2$.

We use the lowest order method of Sect. 5 with polynomial degrees $k = 1$ and $\kappa = 1$. The meshes are uniform (as in Example 2, Fig. 3). Each refinement step is performed by halving all sides and all boundary elements. In Table 1, N denotes the number of finite elements in Ω_{F}, CR is the convergence rate and RE is the relative error in the L^2-norm, i.e.,

$$\frac{\|\sigma - \sigma_h\|_{0,\Omega_{\mathrm{F}}}}{\|\sigma\|_{0,\Omega_{\mathrm{F}}}} \text{ for } \sigma_h, \quad \frac{\|u - u_h\|_{0,\Omega_{\mathrm{F}}}}{\|u\|_{0,\Omega_{\mathrm{F}}}} \text{ for } u_h, \quad \frac{\|\varphi - \varphi_h\|_{0,\tilde{\Gamma}}}{\|\varphi\|_{0,\tilde{\Gamma}}} \text{ for } \varphi_h,$$

respectively, where $\tilde{\Gamma} := \partial\Omega_{\mathrm{B}} \setminus \Gamma_{\mathrm{B}u}$.

Table 1. Example 1, errors for $\nu = 0.2$

	σ_h		u_h		φ_h	
N	RE	CR	RE	CR	RE	CR
8	.1184		.07130		.02043	
32	.03951	1.58	.03333	1.10	.004352	2.23
128	.01153	1.78	.01651	1.01	.001049	2.05
512	.003135	1.88	.008228	1.01	.0002587	2.02
2048	.0008170	1.94	.004110	1.00	.00006447	2.00

Table 2. Example 1, errors for $\nu = 0.4998$

	σ_h		u_h		φ_h	
N	RE	CR	RE	CR	RE	CR
8	.1190		.07480		.02205	
32	.03969	1.58	.03714	1.01	.005533	1.99
128	.01159	1.78	.01832	1.02	.001294	2.10
512	.003144	1.88	.009122	1.01	.0003087	2.07
2048	.0008188	1.94	.004556	1.00	.00007019	2.14

Since the solution is smooth, the convergence rates depend on k and κ only. The computed convergence rates tend to the optimal values that can be expected from the approximation properties. For σ_h and φ_h the results are better than the value $3/2$ predicted by (5.1). The convergence rates for u_h agree with (5.2). On less uniform meshes, the results are similar.

For $\nu = 0.4998$, i.e. almost incompressible material, the relative error is nearly unchanged as seen in Table 2. This is in contrast to the poor behavior of standard (displacement) finite element methods as confirmed by numerical experiments in [8] where the above lowest order finite element method is compared with other standard and mixed approaches.

Inside the BEM domain, the stresses converge with higher order as Table 3 shows for the sampling points $P_1 = (0.3, -0.2)$ and $P_2 = (0.3, -0.7)$. This is a characteristic feature of boundary element methods.

In the second example, we take a solution with a singularity typically arising at a re-entrant corner. Using again polar coordinates (r, θ), $-\pi < \theta \leq \pi$, we

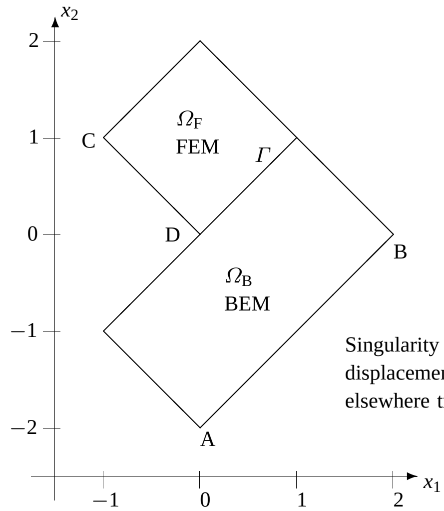

Fig. 2. Example 2

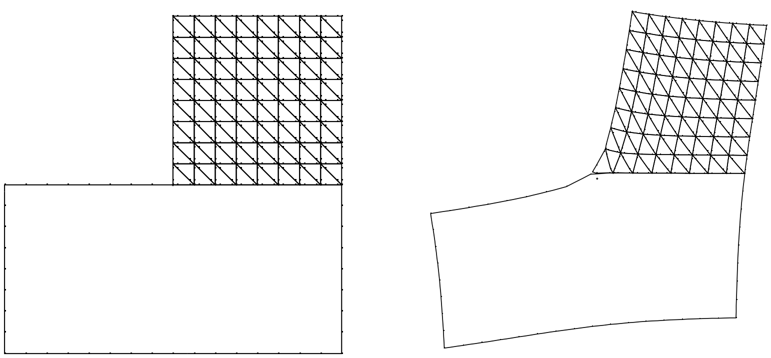

Fig. 3. Mesh and deformations for Example 2 (128 finite elements, 48 boundary elements)

impose the boundary conditions $\sigma n = 0$ for $\theta = \pm\omega$ where ω is half of the interior angle at the corner. According to [12], for plane strain,

$$\begin{aligned}
u_r &= \frac{1}{2\mu} r^{\alpha} \{-(\alpha+1)C_1 \cos((\alpha+1)\theta) + (C_3 - (\alpha+1))C_2 \cos((\alpha-1)\theta)\} \\
u_\theta &= \frac{1}{2\mu} r^{\alpha} \{(\alpha+1)C_1 \sin((\alpha+1)\theta) + (C_3 + \alpha - 1)C_2 \sin((\alpha-1)\theta)\} \\
\sigma_r &= r^{\alpha-1}\{-\alpha(\alpha+1)C_1 \cos((\alpha+1)\theta) + \alpha(3-\alpha)C_2 \cos((\alpha-1)\theta)\} \\
\sigma_\theta &= r^{\alpha-1}\alpha(\alpha+1)\{C_1 \cos((\alpha+1)\theta) + C_2 \cos((\alpha-1)\theta)\} \\
\sigma_{r\theta} &= r^{\alpha-1}\alpha\{(\alpha+1)C_1 \sin((\alpha+1)\theta) + (\alpha-1))C_2 \sin((\alpha-1)\theta)\}
\end{aligned}$$

where α solves

(7.1) $$\alpha \sin 2\omega + \sin(2\omega\alpha) = 0.$$

The constant C_1 is arbitrary, $C_2 = -C_1 \cos((\alpha+1)\omega)/\cos((\alpha-1)\omega)$ and $C_3 = 2(\lambda+2\mu)/(\lambda+\mu)$, λ and μ denoting the Lamé coefficients. In our example (see

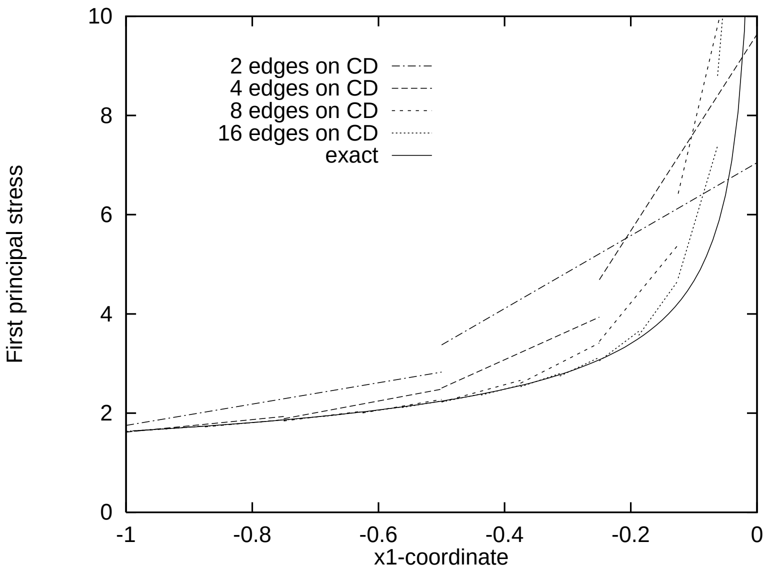

Fig. 4. Example 2, first principal stress on the side CD

Table 3. Example 1, stresses inside the BEM domain for $\nu = 0.2$

	$\sigma_h(P_1)$		$\sigma_h(P_2)$	
N	*RE*	*CR*	*RE*	*CR*
8	.6069E–1		.4031E–1	
32	.7739E–3	6.29	.3168E–2	3.67
128	.3444E–3	1.17	.1984E–3	4.00
512	.3856E–4	3.16	.4027E–4	2.30
2048	.4301E–5	3.16	.1104E–4	1.87

Fig. 2) we have $\omega = 3\pi/4$. Since we are interested in the most singular part of u, we take the smallest positive solution of (7.1), i.e. $\alpha = 0.544483736782463929\ldots$.

In the computations we choose $C_1 = 1$ and λ and μ corresponding to $E = 100$, $\nu = 0.3$. Further, we add a rigid body movement in x_1-direction such that $u = 0$ at the point $(2, 0)$. Again we employ the lowest order method ($k = 1$, $\kappa = 1$) on uniform meshes.

Since now $u \in H^{1+\alpha-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, the estimates (5.1) and (5.2) predict the convergence rate α. The numerical results for σ_h are in good agreement with this value (see Table 4).

In general, if a singularity is present, the convergence rates for u_h and φ_h cannot be expected to tend to a higher value than for σ_h. This is confirmed by numerical experiments with C_2 different from the above value; then, however, 'artificial' boundary conditions for σn on the wedge $\theta = \pm\omega$ are applied.

Computed deformations are shown in Fig. 3. In the FEM domain, as an approximation to u the Lagrange multipliers λ_h of Sect. 6 are used, with averaged

Table 4. Errors for Example 2

	σ_h		u_h		φ_h	
N	*RE*	*CR*	*RE*	*CR*	*RE*	*CR*
8	.2090		.1084		.05763	
32	.1413	0.5658	.04848	1.16	.02771	1.06
128	.09684	0.5456	.02344	1.05	.01341	1.05
512	.06640	0.5449	.01155	1.02	.006496	1.05
2048	.04552	0.5447	.005725	1.01	.003149	1.04
8192	.03121	0.5446	.002845	1.01	.001527	1.04

values at the vertices of the triangles. Figure 4 shows the first principal stress along the side CD (indicated in Fig. 2). The values clearly tend towards the exact solution when the mesh is refined.

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