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A POSTERIORI ERROR ESTIMATE FOR THE MIXED FINITE ELEMENT METHOD

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Abstract. A computable error bound for mixed finite element methods is established in the model case of the Poisson–problem to control the error in the H(div,Ω) $\times L^2(\Omega)$–norm. The reliable and efficient a posteriori error estimate applies, e.g., to Raviart–Thomas, Brezzi-Douglas-Marini, and Brezzi-Douglas-Fortin-Marini elements.

1. Mixed method for the Poisson problem

Mixed finite element methods are well-established in the numerical treatment of partial differential equations as regards a priori error estimates to guarantee convergence [BF]. In practical applications, a posteriori error control is at least of the same importance to guarantee a reliable approximation. Moreover, a posteriori error estimators indicate adaptive mesh-refinement criteria [EEHJ, V1] for an efficient computation.

In this paper we establish an efficient and reliable error estimator for the model example in the mixed finite element methods: Given $f \in L^2(\Omega)$, *the Poisson problem* consists in finding a function $u \in H^1_0(\Omega)$ that satisfies

$$\operatorname{div}(A\nabla u) + f = 0 \qquad \text{in } \Omega. \tag{1.1}$$

Here, $A \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$ is symmetric and uniformly elliptic, Ω is a convex bounded domain in the plane with polygonal boundary Γ. The Lebesgue and Sobolev spaces $L^2(\Omega)$ and $H^1_0(\Omega)$ are defined as usual (e.g., as in [H, LM]). We assume below that (1.1) is H^2–regular which, according to Ω being convex, means certain regularity on A (A the unit matrix as for the Laplace equation is clearly sufficient).

The mixed formulation is given by splitting (1.1) into two equations where $u \in H^1_0(\Omega)$ and $p \in L^2(\Omega)^2$ are unknown and have to satisfy

$$\operatorname{div} p + f = 0 \quad \text{and} \quad p = A\nabla u \qquad \text{in } \Omega. \tag{1.2}$$

It is well-known that (1.2) has a solution $(p, u) \in \mathrm{H}(\mathrm{div},\Omega) \times L^2(\Omega)$, where, as usual, $\mathrm{H}(\mathrm{div},\Omega) := \{q \in L^2(\Omega)^2 : \operatorname{div} q \in L^2(\Omega)\}$ is endowed with the norm given by

$$\|q\|^2_{\mathrm{H}(\mathrm{div},\Omega)} := \int_\Omega (|q|^2 + |\operatorname{div} q|^2)\, dx \qquad (q \in \mathrm{H}(\mathrm{div},\Omega)).$$

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The numerical approximation to (u,p) consists in prescribing finite dimensional subspaces L_h and M_h of $L^2(\Omega)$ and H(div,Ω), respectively, and computing $(p_h,u_h)\in M_h\times L_h$ that satisfies for all $(q_h,v_h)\in M_h\times L_h$

$$\begin{aligned}\int_\Omega (A^{-1}\cdot p_h)\cdot q_h\,dx+\int_\Omega u_h\cdot \operatorname{div} q_h\,dx &= 0,\\ \int_\Omega v_h\cdot \operatorname{div} p_h\,dx &= -\int_\Omega v_h\cdot f\,dx.\end{aligned}\tag{1.3}$$

It is well-known that the discrete problem (1.3) has a unique solution if a discrete inf–sup–condition holds for the discrete spaces M_h and L_h [BF] so we are interested in controlling the error

$$\epsilon := p-p_h\in \mathrm{H}(\mathrm{div},\Omega)\quad\text{and}\quad e:=u-u_h\in L^2(\Omega).\tag{1.4}$$

Moreover, if the discrete inf–sup–condition holds uniformly in h we have a constant $c_1>0$ such that

$$\|(\epsilon,e)\|_{\mathrm{H}(\mathrm{div},\Omega)\times L^2(\Omega)}\le c_1\cdot \inf_{(q_h,v_h)\in M_h\times L_h}\|(p-q_h,u-v_h)\|_{\mathrm{H}(\mathrm{div},\Omega)\times L^2(\Omega)},\tag{1.5}$$

i.e., the error is bounded from above and below by a constant times the best–approximation error. We refer to [BF] for the setting, examples, proofs, and more details. The Raviart–Thomas, Brezzi-Douglas-Marini, and Brezzi-Douglas-Fortin-Marini elements are also described in §3.1.

2. A posteriori error estimator

In the mixed finite element method, we consider a regular triangulation $\mathcal{T}_h$ of Ω satisfying the angle condition (cf. §4 for explanations) and define, for each $T\in\mathcal{T}_h$, h_T as the diameter of T, and, for any edge E of T, let $J(p_h\cdot t)$ denote the jump of $p_h\cdot t$ across E with t being the tangential unit vector along E; h_E denotes the length of E. Then, define

$$\begin{aligned}\eta_T^2 &:= \|f+\operatorname{div} p_h\|^2_{L^2(T)}+h_T^2\cdot\|\operatorname{curl}(A^{-1}p_h)\|^2_{L^2(T)}\\ &\quad +h_T^2\cdot\min_{v_h\in L_h}\|A^{-1}p_h-\nabla_h v_h\|^2_{L^2(T)}+\|h_E^{1/2}J(A^{-1}p_h\cdot t)\|^2_{L^2(\partial T)}\end{aligned}$$

for any $T\in\mathcal{T}_h$ and consider the sum of all element contributions

$$\eta_h := (\sum_{T\in\mathcal{T}_h}\eta_T^2)^{1/2}.$$

It is the aim of this paper to establish the following a posteriori error estimate.

Theorem 2.1. *For the Raviart–Thomas, the Brezzi-Douglas-Marini, or the Brezzi-Douglas-Fortin-Marini elements there is a positive constant c_1 which only depends on A, Ω, and on the shape of the elements and their polynomial degree k, such that*

$$\|(\epsilon,e)\|_{H(div,\Omega)\times L^2(\Omega)}\le c_2\cdot\eta_h.\tag{2.1}$$

Moreover, the reverse inequality holds as well provided that on each $T\in\mathcal{T}_h$, $A^{-1}p_h|_T\in\mathcal{P}_\ell$ and $\nabla_h u_h|_T\in\mathcal{P}_\ell$; $\mathcal{P}_k$ denotes the set of polynomials in two variables of total degree at most k. (Again, A the unit matrix as for the Laplace equation is clearly sufficient.)

Theorem 2.2. *For the Raviart–Thomas, the Brezzi-Douglas-Marini, or the Brezzi-Douglas-Fortin-Marini elements there is a positive constant c_3 which only depends* *on A, Ω, and on the shape of the elements and the polynomial degrees k and ℓ, such that*

$$c_3 \cdot \eta_h \leq \|(\epsilon, e)\|_{H(div,\Omega)\times L^2(\Omega)}. \tag{2.2}$$

The proofs of Theorems 2.1 and 2.2 will be given in §§4–6 under sharper but more technical assumptions while we first precede with some remarks in §3.

3. Remarks

Some supplements are in order to comment on the results displayed in Theorems 2.1 and 2.2.

3.1. **Examples for mixed finite elements.** The examples mentioned in Theorems 2.1 and 2.2 are briefly described for triangles $T \in \mathcal{T}_h$ by some $D_k(T) \subset \mathcal{C}(T)$ and $M_k(T) \subset \mathcal{C}(T)$ given in the following table where $k \geq 0$ and RT indicates entries for the Raviart–Thomas elements, BDM for the Brezzi-Douglas-Marini elements, and BDFM for the Brezzi-Douglas-Fortin-Marini elements.

Element	$M_k(T)$	$D_k(T)$
RT	$\mathcal{P}_k^2 + x \cdot \mathcal{P}_k$	$\mathcal{P}_k$
BDM	$\mathcal{P}_{k+1}^2$	$\mathcal{P}_k$
BDFM	$\{q \in \mathcal{P}_{k+1}^2 : (q \cdot n)\|_{\partial T} \in \mathcal{R}_k(\partial T)\}$	$\mathcal{P}_k$

Here, $\mathcal{P}_k$ denotes polynomials of total degree at most k and $\mathcal{R}_k(\partial T)$ denotes (not necessarily continuous) functions on ∂T which equal a polynomial of degree at most k on each edge of T. With the above sets $D_k(T)$ and $M_k(T)$ we define

$$\begin{aligned} L_h &:= \{v_h \in L^2(\Omega) : \forall T \in \mathcal{T}_h \; v_h|_T \in D_k(T)\}, \\ M_h &:= \{p_h \in \mathrm{H(div,}\Omega) : \forall T \in \mathcal{T}_h \; v_h|_T \in M_k(T)\}. \end{aligned}$$

For more information, in particular about other elements in $\mathbb{R}^n$ and about practical implementations using multipliers, we refer to [BF].

3.2. **Estimates in a weighted norm.** The results in §§4–6 give the following estimate with a different scaling in the equilibrium residual. Indeed, with $h : \Omega \to (0, \infty)$ defined by $h|_T = h_T$ on $T \in \mathcal{T}_h$ and by $h|_E = h_E$ on $E \in \mathcal{E}_h$ there holds

$$c_4 \cdot \eta_{h,\kappa}(p_h, u_h) \leq \| A^{-1/2}\epsilon \|_{L^2(\Omega)} + \| h^\kappa \operatorname{div} \epsilon \|_{L^2(\Omega)} + \| e \|_{L^2(\Omega)} \leq c_5 \cdot \eta_{h,\kappa}(p_h, u_h)$$

where $0 \leq \kappa \leq 1$ and ($\mathcal{E}_h$ denotes the set of edges in $\mathcal{T}_h$ and $\Gamma_h := \bigcup \mathcal{E}_h$)

$$\begin{aligned} \eta_{h,\kappa}(p_h, u_h) &:= \|h^\kappa \cdot (f + \operatorname{div} p_h)\|_{L^2(\Omega)} + \|h \cdot \operatorname{curl}(A^{-1}p_h)\|_{L^2(\Omega)} \\ &\quad + \min_{v_h \in L_h} \|h \cdot (A^{-1}p_h - \nabla_h v_h)\|_{L^2(\Omega)} + \|h^{1/2} \cdot J(A^{-1}p_h \cdot t)\|_{L^2(\Gamma_h)}. \end{aligned}$$

3.3. **Estimates for the stress variables.** The results in §§4–6 give the following estimate for the stress variable $p - p_h$, where $0 \leq \kappa \leq 1$,

$$\begin{aligned} c_6 \cdot \eta_{h,\kappa}(p_h) &\leq \| A^{-1/2}\epsilon \|_{L^2(\Omega)} + \| h^\kappa \operatorname{div} \epsilon \|_{L^2(\Omega)} \leq c_7 \cdot \eta_{h,\kappa}(p_h), \\ \eta_{h,\kappa}(p_h) &:= \|h^\kappa \cdot (f + \operatorname{div} p_h)\|_{L^2(\Omega)} + \|h \cdot \operatorname{curl}(A^{-1}p_h)\|_{L^2(\Omega)} \\ &\quad + \|h^{1/2} \cdot J(A^{-1}p_h \cdot t)\|_{L^2(\Gamma_h)}. \end{aligned}$$

We emphasize that this estimate holds also if (1.1) is not H^2-regular, so Ω may be an arbitrary bounded Lipschitz domain and $A_{ij} \in L^\infty(\Omega)$ is sufficient.

3.4. **On the term $A^{-1}p_h - \nabla_h v_h$.** In the definition of η_T, we may replace

$$h_T^2 \cdot \min_{v_h \in L_h} \|A^{-1}p_h - \nabla_h v_h\|^2_{L^2(T)}$$

by its upper bound

$$h_T^2 \cdot \|A^{-1}p_h - \nabla_h u_h\|^2_{L^2(T)}$$

without losing reliability and efficiency. Indeed, we conclude from Lemma 6.3 in §6 that Theorem 2.2 remains valid for this modified (less sharp but possibly simpler) estimator.

3.5. **Other estimates for the displacements.** The preceding estimates for the stress variables and standard arguments in the theory of mixed finite element methods give a posteriori bounds for $\Pi_{L_h} e$ and $u - u_h^*$ where u_h^* is the improved displacement field taking Lagrange multipliers in a practical implementation into account. Following the lines in [BF, p.186] we can verify that

$$\| \Pi_{L_h} u - u_h \|_{L^2(\Omega)} \le c_8 \cdot (\| h \cdot A^{-1/2} \epsilon \|_{L^2(\Omega)} + \| h \cdot \operatorname{div} \epsilon \|_{L^2(\Omega)})$$

which proves the a posteriori error estimate, $h_{max} := \| h \|_{L^\infty(\Omega)}$,

$$\| \Pi_{L_h} u - u_h \|_{L^2(\Omega)} \le c_9 \cdot h_{max} \cdot \eta_{h,1}(p_h).$$

Furthermore, let $\mathcal{L}_k^{1,NC} := \{v_h \in L^2(\Omega) : \forall T \in \mathcal{T}_h \ \forall \psi \in \mathcal{R}_k(\partial T) \ v_h|_T \in \mathcal{P}_k \wedge \int_{\partial T} J(u_h) \cdot \psi \, ds = 0\}$, let $u_h^* \in \mathcal{L}_k^{1,NC}$ denote the improved discrete displacement field defined in [BF, p.187] and let $\tilde{u}_h$ denote the $L^2(\Omega)$-best approximation to u in $\mathcal{L}_k^{1,NC}$. Then, as shown in [BF, Eq. (3.13)],

$$\| \tilde{u}_h - u_h^* \|_{L^2(\Omega)} \le c_{10} \cdot (\| h \cdot \epsilon \|_{L^2(\Omega)} + \| \Pi_{L_h} e \|_{L^2(\Omega)})$$

which results in the a posteriori error estimate

$$\| \tilde{u}_h - u_h^* \|_{L^2(\Omega)} \le c_{11} \cdot h_{max} \cdot \eta_{h,1}(p_h).$$

3.6. **Comments on the estimator by Braess and Verfürth.** Braess and Verfürth established a posteriori error estimates for mixed methods in [BV] involving integration by parts in $\int_\Omega u_h \cdot \operatorname{div} q \, dx$ (which appears, e.g., in (1.3)). Since u_h jumps across interelement boundaries those jumps count in their error indicator. Braess and Verfürth designed an error estimator working in mesh-dependent norms which is reliable and efficient in those norms but, somehow, is not efficient in the natural norm of H(div,Ω) $\times L^2(\Omega)$, seemingly because the displacement variable is overestimated in their mesh-dependent norm. In this paper, we outlaw any such integration by parts (with one well-chosen exception, cf. (5.12) below) and so jumps of displacements cannot arise at all. Instead, we emphasize a Helmholtz decomposition and are led to the estimator η_h which is reliable and efficient in the natural norm and avoids the saturation assumption that is important in [BV].

3.7. **Comments on A.** To estimate e in the proof of Theorem 2.1, we need that (1.1) is H^2-regular (see §4.2 below for details). Since Ω is convex, the condition $A \in C^{1,0}(\overline{\Omega})$ is sufficient for that (see, e.g., [G, Thm 3.2.1.2] for a proof). Moreover, even some discontinuities are allowed, because we only need that the restriction $u|_T$ of a solution u to an element T belongs to $H^2(T)$ (cf., (4.2) below). The following example proves that there exist problems (1.1) which are not H^2-regular but satisfy this assumption.

Example 1. Let $A(x) = \rho(x) \cdot I_{2\times 2}$ ($I_{2\times 2}$ the 2×2-unit matrix), and $\rho(x) > 0$ is piecewise constant (with polygonal lines of discontinuities), the possible singularities of such transmission problems are understood and some of them lead to H^2-regular problems (see, e.g., [N, Sec. 2.4] and the references quoted therein). For example, consider a square $\Omega := (0,1)^2$ and halve it along a diagonal $D := \overline{T_1} \cap \overline{T_2}$ into two (open) congruent triangles T_1 and T_2. Let $\rho(x) = \rho_j$ for $x \in T_j$, $j = 1, 2$, for two positive constants $\rho_1 \neq \rho_2$.

Then any $u \in H^1_0(\Omega)$ with $\operatorname{div}(A\nabla u) \in L^2(\Omega)$ satisfies $u|_{T_j} \in H^2(T_j)$ for $j = 1, 2$, but (1.1) is not H^2-regular.

Proof. The natural interface conditions along D show that $u \notin H^2(\Omega)$ (provided the normal derivatives (and hence their jump accross D) are non-zero which is generically the case). A careful study of the corner singularities tells us that $u|_{T_j} \in H^2(T_j)$ for $j = 1, 2$. (See, e.g., [N, Example 2.4] for a proof of that — there, it suffices to check that $D_2^D(\lambda) = 0$ and $\lambda > 0$ is possible only for $2 \geq \lambda$; cf., [N, page 102] for details and notation.) □

In Theorem 2.2 we stated the condition that $A^{-1}p_h$ is a polynomial on each element (but may be discontinuous on interelement boundaries). In the examples of §3.1, $p_h|_T$ is a polynomial so that A^{-1} is required to be a polynomial too. The analysis in §6 shows that this restriction can be weakened. Actually, A^{-1} has to be approximated by some polynomial A_T^{-1} for which we precede as in the proof given below while some additional approximation error $\| A^{-1} - A_T^{-1} \|_{L^\infty(\Omega)}$ arises in the bounds.

3.8. **Adaptive algorithms.** As in many contributions to self-adapting mesh-refinements (see, e.g., [EEHJ, V1, V2, V3] and the references quoted therein), based on an error estimator η_h we get an algorithm for efficient mesh-design: For each mesh $\mathcal{T}_{h_L}$ with a Galerkin solution (p_{h_L}, u_{h_L}) and local error estimators η_T, we refine $T \in \mathcal{T}_{h_L}$ (e.g., by halving its largest side) if (for example)

$$\eta_T \geq 0.5 \cdot \max_{T' \in \mathcal{T}_{h_L}} \eta_{T'}.$$

Then, further refinements to avoid hanging nodes lead to a new mesh $\mathcal{T}_{h_{L+1}}$ from which we start again.

4. Preliminaries

Theorem 2.1 holds under the following weaker assumptions on $\mathcal{T}_h$, A, on $L_h \subset L^2(\Omega)$ and $M_h \subset \mathrm{H}(\mathrm{div},\Omega)$. We emphasize that the Raviart–Thomas, Brezzi-Douglas-Marini, and Brezzi-Douglas-Fortin-Marini elements satisfy all the assumptions in this section.

4.1. **Assumptions on Ω.** The bounded Lipschitz domain Ω is assumed to be convex with a polygonal boundary. Depending only on Ω we have a constant $c_{12} > 0$ such that, for all $v \in H^1(\Omega)$ with integral mean v_0, Poincaré's inequality reads

$$\| v - v_0 \|_{2,\Omega} \leq c_{12} \cdot \| \nabla v \|_{2,\Omega}. \tag{4.1}$$

4.2. **Assumptions on A.** We assume that $A \in L^\infty(\Omega;\mathbb{R}^{2\times 2}_{sym})$ is uniformly elliptic, i.e., $A(x)$ is a symmetric and positive definite 2×2-matrix, with eigenvalues $\lambda_j(x) \in \mathbb{R}$ satisfying $0 < c_A \le \lambda_1(x), \lambda_2(x) \le C_A$ for almost all $x \in \Omega$. Then, by the Lax-Milgram lemma, the operator

$$-\operatorname{div}(A\nabla\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is invertible and the norm of the inverse is bounded by a constant $c_{13} > 0$ depending on c_A and c_{12}. Moreover, since Ω is convex, $A \in \mathcal{C}^{1,0}(\overline{\Omega})$ implies that

$$-\operatorname{div}(A\nabla\cdot) : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$$

is invertible [G] and there is a constant $c_{14} > 0$ such that

(4.2)
$$\| v \|_{2,2,\bigcup \mathcal{T}_h} \le c_{14} \cdot \| \operatorname{div}(A\nabla v) \|_{2,\Omega} \qquad (v \in H_0^1(\Omega) \text{ such that } \operatorname{div}(A\nabla v) \in L^2(\Omega)).$$

We emphasize that we only need an estimate on $\| v \|_{2,2,T}$ for each $T \in \mathcal{T}_h$, i.e., the assumption on A could be weakened in the sense that only (4.2) is required (cf., Example 1 where A is piecewise constant and satisfies (4.2) but (1.1) is not H^2-regular).

Finally, we need that A is elementwise smooth assuming that there exists a constant $c_{15} > 0$ such that

$$\max_{i,j,k=1,2} \| (\nabla_h A)_{ijk} \|_{\infty,\bigcup \mathcal{T}_h} \le c_{15}. \tag{4.3}$$

4.3. **Assumptions on $\mathcal{T}_h$.** The triangulation $\mathcal{T}_h$ is assumed to be regular in the sense of [C] and satisfies the angle condition which means that there is a constant $c_{16} > 0$ such that for all $T \in \mathcal{T}_h$

$$c_{16}^{-1} \cdot h_T^2 \le |T| \le c_{16} \cdot h_T^2 \tag{4.4}$$

where $|T|$ is the area of T. We define $S^0(\mathcal{T}_h) \subset L^2(\Omega)$ as the piecewise constant and $S^1(\mathcal{T}_h) \subset H^1(\Omega)$ or $S_0^1(\mathcal{T}_h) \subset H_0^1(\Omega)$ as continuous and piecewise affine functions; piecewise is understood with respect to $\mathcal{T}_h$. We consider Clement's interpolation operator [Cl] $r_h : H^1(\Omega) \to S^1(\mathcal{T}_h)$ which satisfies

$$\begin{aligned} \| v - r_h v \|_{2,T} &\le c_{17} \cdot h_T \cdot \| v \|_{1,2,\omega_T} \\ \| v - r_h v \|_{2,E} &\le c_{18} \cdot h_E^{1/2} \cdot \| v \|_{1,2,\omega_E} \end{aligned} \qquad (v \in H_0^1(\Omega)) \tag{4.5}$$

for each $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$, $\mathcal{E}_h$ being the set of element sides in $\mathcal{T}_h$. Here and below, $\| \cdot \|_{p,\omega}$ denotes the norm in $L^p(\omega)$ for $\omega \subset \Omega$ as well as for some edge $\omega = E$ while $\| \cdot \|_{m,p,\omega}$ and $| \cdot |_{m,p,\omega}$ denote norm and semi-norm in $W^{m,p}(\omega)$, respectively; in particular, we will occasionally write $\| \cdot \|_2$ instead of $\| \cdot \|_{2,\Omega}$ and $H^m(\omega)$ instead of $W^{m,2}(\omega)$. With $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we associate neighbourhoods ω_T and ω_E

$$\omega_T := \bigcup \{T' \in \mathcal{T}_h : \overline{T} \cap \overline{T'} \ne \emptyset\} \quad \text{and} \quad \omega_E := \bigcup \{T \in \mathcal{T}_h : E \subset \overline{T}\}.$$

Then, the positive constants c_{17} and c_{18} only depend on c_{16}. Moreover, let c_{19} be the maximal number of elements in ω_T which is h–independently bounded by the angle condition (depending on c_{16}). For all $E \in \mathcal{E}_h$ we fix one direction of a unit normal on E pointing in the outside of Ω in case that $E \subset \Gamma$. With $\Gamma_h := \bigcup \mathcal{E}_h$ we define $J : H^1(\bigcup \mathcal{T}_h) \to L^2(\Gamma_h)$, for $E \subset \Gamma_h$ and $v \in H^1(\bigcup \mathcal{T}_h)$ by

$$J(v)|_E := (v|_{T_+})|_E - (v|_{T_-})|_E \quad \text{if} \quad E = \overline{T_+} \cap \overline{T_-} \qquad (E \in \mathcal{E}_h; T_+, T_- \in \mathcal{T}_h)$$

and n_E points from T_+ into its neighbour element T_-; while

$$J(v)|_E := (v|_T)|_E \quad \text{if} \quad E = \overline{T} \cap \Gamma \qquad (E \in \mathcal{E}_h; T \in \mathcal{T}_h).$$

We define $W^{m,p}(\bigcup \mathcal{T}_h) := \{v \in L^p(\Omega) : \ \forall T \in \mathcal{T}_h \, v|_T \in W^{m,p}(T)\}$ and consider local versions of the differential operators div, ∇, curl (understood in the distributional sense, i.e., in $\mathcal{D}'(\Omega)$), namely, $\operatorname{div}_h, \operatorname{curl}_h : W^{1,2}(\bigcup \mathcal{T}_h)^2 \to L^2(\Omega)$ and $\nabla_h : W^{1,2}(\bigcup \mathcal{T}_h) \to L^2(\Omega)$ defined such that, e.g.,

$$\operatorname{div}_h v|_T := \operatorname{div}(v|_T) \quad \text{in } \mathcal{D}'(T) \qquad (T \in \mathcal{T}_h).$$

If there is no risk of confusion the local meshsize h is defined on both Ω and $\Gamma_h := \bigcup \mathcal{E}_h$ by $h|_T := h_T$ for $T \in \mathcal{T}_h$ and $h|_E := h_E$ for $E \in \mathcal{E}_h$, respectively.

4.4. **Assumptions on L_h.** We assume that $L_h \subset H^1(\bigcup \mathcal{T}_h)$ such that the $L^2(\Omega)$–orthogonal projection $\Pi_{L_h} : L^2(\Omega) \to L_h$ satisfies

$$\| v - \Pi_{L_h} v \|_{2,\Omega} \le c_4 \cdot \| h \cdot \nabla_h v \|_{2,\Omega} \qquad (v \in H^1(\bigcup \mathcal{T}_h)). \tag{4.6}$$

For example, if $S^0(\mathcal{T}_h) \subset L_h$, the Poincaré inequality (4.6) is satisfied with a positive constant c_{20} which only depends on the shape of the elements.

Furthermore, for the lower bound (Theorem 2.2) we assume $(\nabla_h u_h)|_T \in \mathcal{P}_\ell^2$ for all $T \in \mathcal{T}_h$.

4.5. **Assumptions on M_h.** We assume that

$$S^0(\mathcal{T}_h)^2 \cap \mathrm{H}(\mathrm{div},\Omega) \subset M_h \subset H^1(\bigcup \mathcal{T}_h) \cap \mathrm{H}(\mathrm{div},\Omega) \quad \text{and} \quad \operatorname{div} M_h = L_h. \tag{4.7}$$

Furthermore, in Theorem 2.2, we assume $(A^{-1}p_h)|_T \in \mathcal{P}_\ell^2$ for all $T \in \mathcal{T}_h$.

4.6. **Assumptions on an interpolation operator Π_h.** We assume that there exists an operator $\Pi_h : W \to M_h$ where $W = \mathrm{H}(\mathrm{div},\Omega) \cap L^s(\Omega)^2$ for some $s > 2$ as, e.g., in [BF, §III.3], such that the following diagram commutes

$$\begin{array}{ccc} W & \xrightarrow{\operatorname{div}} & L^2(\Omega) \\ \Pi_h \downarrow & & \downarrow \Pi_{L_h} \\ M_h & \xrightarrow{\operatorname{div}} & L_h \end{array} \tag{4.8}$$

where Π_{L_h} is the $L^2(\Omega)$–orthogonal projection. Let Id denote identity and let $\perp$ denote $L^2(\Omega)$–orthogonality. Then, the *commuting diagram property* in (4.8) reads

$$\operatorname{div}(\mathrm{Id} - \Pi_h) W \perp L_h. \tag{4.9}$$

Further, we assume that the interpolant satisfies a local error estimate (note that $H^1(\bigcup \mathcal{T}_h) \cap \mathrm{H}(\mathrm{div},\Omega) \subset W$)

$$\| h^{-1} \cdot (\mathrm{Id} - \Pi_h) q \|_2 \le c_{21} \cdot | q |_{1,2,\bigcup \mathcal{T}_h} \qquad (q \in H^1(\bigcup \mathcal{T}_h) \cap \mathrm{H}(\mathrm{div},\Omega)). \tag{4.10}$$

Finally, we assume that Π_h approximates the normal components on element edges such that we have, for any $E \in \mathcal{E}_h$, for any $v_h \in L_h$, and for all $q \in W$,

$$\int_T v_h \cdot (\mathrm{Id} - \Pi_h) q \cdot n_E \, dx = 0. \tag{4.11}$$

We refer to [BF] for proofs, further explanations and explicit definitions of Π_h in the examples under consideration.

5. Proof of Theorem 2.1

Theorem 2.1 is a direct consequence of the following two lemmas and the fact that $-\operatorname{div}\epsilon = f + \operatorname{div} p_h$. We recall that the local meshsize h is defined on $\Omega \setminus \Gamma_h$ by $h|_T := h_T$ for $T \in \mathcal{T}_h$ and on $\Gamma_h := \bigcup \mathcal{E}_h$ by $h|_E := h_E$ for $E \in \mathcal{E}_h$.

Lemma 5.1. *For* $c_{22} := \max\{\sqrt{8}c_{12} \cdot c_{18} \cdot C_A, \sqrt{2}c_{12} \cdot c_{17} \cdot c_{19} \cdot C_A, c_{20} \cdot c_A^{-1}\}$ *we have*

$$\begin{aligned}\| A^{-1/2}\epsilon \|_{2,\Omega} \quad &\leq \quad c_{22} \cdot \big(\| h \cdot \operatorname{curl}_h(A^{-1}p_h) \|_{2,\Omega}^2 \\ &\qquad + \| h \cdot (f + \operatorname{div} p_h) \|_{2,\Omega}^2 + \| h^{1/2} \cdot J(A^{-1/2}p_h \cdot t) \|_{2,\Gamma_h}^2\big)^{1/2}.\end{aligned}$$

Proof. We consider a Helmholtz decomposition of $A^{-1}p_h$ fixing $\alpha \in H_0^1(\Omega)$ with

$$\operatorname{div}(A\nabla\alpha) = \operatorname{div} p_h \quad \text{in } \mathcal{D}'(\Omega). \tag{5.1}$$

Then, there is some $\beta \in H^1(\Omega)$ satisfying $\int_\Omega \beta\, dx = 0$, $\operatorname{Curl}\beta \perp \nabla H_0^1(\Omega)$, and

$$p_h = A\nabla\alpha + \operatorname{Curl}\beta \tag{5.2}$$

($\perp$ denotes $L^2(\Omega)$-orthogonality). From (1.2) and (5.2) we obtain

$$\epsilon = A\nabla z - \operatorname{Curl}\beta \quad \text{with} \quad z := u - \alpha \in H_0^1(\Omega) \tag{5.3}$$

and hence the error decomposition

$$\int_\Omega (A^{-1}\epsilon)\cdot\epsilon\, dx = \int_\Omega (A\nabla z)\cdot\nabla z\, dx + \int_\Omega (A^{-1}\operatorname{Curl}\beta)\cdot\operatorname{Curl}\beta\, dx. \tag{5.4}$$

To estimate the first contribution of the right-hand side in (5.4) we integrate by parts and utilize $\operatorname{div}\epsilon \perp L_h$ (which follows from (1.2) and (1.3)). With (4.6), this leads to

$$\begin{aligned}\int_\Omega (A\nabla z)\cdot\nabla z\, dx &= \int_\Omega \nabla z\cdot\epsilon\, dx = -\int_\Omega z\cdot\operatorname{div}\epsilon\, dx \\ &= -\int_\Omega (z - \Pi_{L_h} z)\cdot\operatorname{div}\epsilon\, dx \leq c_{20}\cdot c_A^{-1}\cdot\| h\cdot\operatorname{div}\epsilon\|_2\cdot\| A^{1/2}\nabla z\|_2.\end{aligned} \tag{5.5}$$

To estimate the second contribution to the right-hand side of (5.4) we define $\beta_h := r_h\beta \in S^1(\mathcal{T}_h) \subset H^1(\Omega)$ utilizing Clement's operator r_h. Note that $\operatorname{Curl}_h\beta_h = \operatorname{Curl}\beta_h \in S^0(\mathcal{T}_h)^2 \subset L^\infty(\Omega)^2$ and $\operatorname{Curl}\beta_h \perp \nabla H_0^1(\Omega)$, whence $\operatorname{div}\operatorname{Curl}\beta_h = 0$, and

$$\operatorname{Curl}\beta_h \in \mathrm{H}(\mathrm{div},\Omega) \cap L^\infty(\Omega)^2 \cap M_h$$

according to (4.7). Therefore, (5.3) and (1.2)–(1.3) show

$$\begin{aligned}\int_\Omega (A^{-1}\operatorname{Curl}\beta)\cdot\operatorname{Curl}\beta_h\, dx &= -\int_\Omega (A^{-1}\epsilon)\cdot\operatorname{Curl}\beta_h\, dx \\ &= \int_\Omega e\cdot\operatorname{div}\operatorname{Curl}\beta_h\, dx = 0.\end{aligned} \tag{5.6}$$

From the integration by parts formula

$$\int_\omega (\phi\cdot\frac{\partial\psi}{\partial x_j} + \psi\cdot\frac{\partial\phi}{\partial x_j})\, dx = \int_{\partial\omega} \psi\cdot\phi\cdot n_j\, ds$$

(say, for $\phi, \psi \in H^1(\omega)$, $j = 1, 2$, $n = (n_1, n_2)^T \in \mathbb{R}^2$ the exterior unit-normal to the Lipschitz boundary $\partial\omega$) we gain

$$\int_\omega (\psi\cdot\operatorname{Curl}\phi + \phi\cdot\operatorname{curl}\psi)\, dx = \int_{\partial\omega} \phi\cdot(\psi\cdot t)\, ds \qquad (\phi, \psi_1\psi_2 \in H^1(\omega)) \tag{5.7}$$

where t is tangential on $\partial\omega$: $t_1 = -n_2$, $t_2 = n_1$, and where we differentiate between curls involved as

$$\operatorname{curl}\psi = \frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi_1}{\partial x_2} \quad \text{and} \quad \operatorname{Curl}\phi = (-\frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_1})^T.$$

Utilizing (5.6), (5.2), and (5.7) we infer (recall $\Gamma_h = \bigcup \mathcal{E}_h$)

$$\begin{aligned}\int_\Omega (A^{-1}\operatorname{Curl}\beta)\cdot\operatorname{Curl}\beta\,dx &= \int_\Omega A^{-1}p_h\cdot\operatorname{Curl}(\beta-\beta_h)\,dx \\ &= -\int_\Omega(\beta-\beta_h)\cdot\operatorname{curl}_h(A^{-1}p_h)\,dx + \int_{\Gamma_h} J(A^{-1}p_h\cdot t)\cdot(\beta-\beta_h)\,ds.\end{aligned}$$

According to (4.5), and since the number of elements in ω_T is bounded by c_{19},

$$\begin{aligned}\int_\Omega(\beta-\beta_h)\cdot\operatorname{curl}_h(A^{-1}p_h)\,dx &\leq c_{17}\cdot c_{19}\cdot\|\,h\cdot\operatorname{curl}_h(A^{-1}p_h)\,\|_2\cdot\|\,\beta\,\|_{1,2,\Omega}, \\ \int_{\Gamma_h} J(A^{-1}p_h\cdot t)\cdot(\beta-\beta_h)\,ds &\leq 2c_{18}\cdot\|\,h^{1/2}\cdot J(A^{-1}p_h\cdot t)\,\|_{2,\Gamma_h}\cdot\|\,\beta\,\|_{1,2,\Omega}.\end{aligned}$$

With Poincaré's inequality (4.1) and ellipticity of A we deduce

$$\|\,\beta\,\|_{1,2,\Omega} \leq c_{12}\cdot\|\,\nabla\beta\,\|_{2,\Omega} = c_{12}\cdot\|\,\operatorname{Curl}\beta\,\|_{2,\Omega} \leq c_{12}\cdot C_A\cdot\|\,A^{-1/2}\operatorname{Curl}\beta\,\|_{2,\Omega}.$$

The above estimates verify

$$\begin{aligned}\int_\Omega(A^{-1}\operatorname{Curl}\beta)\cdot\operatorname{Curl}\beta\,dx &\leq c_{23}\cdot\|\,A^{-1/2}\operatorname{Curl}\beta\,\|_{2,\Omega} \\ &\cdot\big(\|\,h^{1/2}\cdot J(A^{-1}p_h\cdot t)\,\|_{2,\Gamma_h} + \|\,h\cdot\operatorname{curl}_h(A^{-1}p_h)\,\|_2\big)\end{aligned}$$

where $c_{23} := c_{12}\cdot C_A\cdot\max\{2c_{18}, c_{17}\cdot c_{19}\}$. Together with (5.4) and (5.5) this establishes

$$\begin{aligned}\|\,A^{-1/2}\epsilon\,\|_2^2 \leq &\big(c_{20}^2\cdot c_A^{-2}\cdot\|\,h\cdot\operatorname{div}\epsilon\,\|_2^2 + 2c_{23}^2\cdot\|\,h^{1/2}\cdot J(A^{-1}p_h\cdot t)\,\|_{2,\Gamma_h}^2 \\ &+2c_{23}^2\cdot\|\,h\cdot\operatorname{curl}_h(A^{-1}p_h)\,\|_2^2\big)^{1/2}\cdot\|\,A^{-1/2}\epsilon\,\|_2\end{aligned}$$

and concludes the proof. □

Lemma 5.2. *For $c_{24} := (c_{20}^2\cdot c_A^{-2} + c_{14}^2\cdot c_{21}^2\cdot(C_A^2 + 4c_{15}^2))^{1/2}$ we have*

$$\|\,e\,\|_{2,\Omega} \leq c_{24}\cdot\big(\|\,h\cdot(f+\operatorname{div}p_h)\,\|_{2,\Omega}^2 + \min_{v_h\in L_h}\|\,h\cdot(A^{-1}p_h - \nabla_h v_h)\,\|_{2,\Omega}^2\big)^{1/2}.$$

Proof. There exists exactly one $\eta\in H_0^1(\Omega)$ with $\operatorname{div}(A\nabla\eta) = e$. According to (4.2), we have $\eta\in H_0^1(\Omega)\cap H^2(\bigcup\mathcal{T}_h)$ and

$$\max\{c_{14}^{-1}\cdot\|\,\eta\,\|_{2,2,\bigcup\mathcal{T}_h}, c_{13}^{-1}\cdot\|\,\eta\,\|_{1,2,\Omega}, c_A\cdot|\,\eta\,|_{1,2,\Omega}\} \leq \|\,e\,\|_{2,\Omega}. \tag{5.8}$$

By construction of η, integration by parts and with (1.2), (1.3) and (4.9) we infer, for any $v_h \in L_h$,

$$
\begin{aligned}
\| e \|_2^2 &= \int_\Omega (u - u_h) \cdot \operatorname{div}(A\nabla\eta)\, dx \\
&= -\int_\Omega p \cdot \nabla\eta\, dx - \int_\Omega u_h \cdot \operatorname{div} \Pi_h(A\nabla\eta)\, dx \\
&= -\int_\Omega \epsilon \cdot \nabla\eta\, dx - \int_\Omega (A^{-1}p_h) \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx \\
&= \int_\Omega \eta \cdot \operatorname{div}\epsilon\, dx + \int_\Omega (\nabla_h v_h - A^{-1}p_h) \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx \qquad (5.9)\\
&\quad - \int_\Omega \nabla_h v_h \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx.
\end{aligned}
$$

Letting $\eta_h := \Pi_{L_h}\eta$ we get from (1.2), (1.3), and (4.6) that

$$\int_\Omega \eta \cdot \operatorname{div}\epsilon\, dx = \int_\Omega (\eta - \eta_h) \cdot \operatorname{div}\epsilon\, dx \le c_{20} \cdot \| \nabla_h \eta \|_{2,\Omega} \cdot \| h \cdot \operatorname{div}\epsilon \|_{2,\Omega}. \qquad (5.10)$$

The second term on the right-hand side of (5.9) is

$$
\begin{aligned}
&\int_\Omega (\nabla_h v_h - A^{-1}p_h) \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx \\
&\qquad \le \quad \| h \cdot (\nabla_h v_h - A^{-1}p_h) \|_2 \cdot \| h^{-1} \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta) \|_2.
\end{aligned}
$$

According to (4.10) and letting $c_{25} := c_{21} \cdot (C_A^2 + 4c_{15}^2)^{1/2}$ we obtain

$$\| h^{-1} \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta) \|_{2,\Omega} \le c_{21} \cdot | A\nabla\eta |_{1,2,\bigcup \mathcal{T}_h} \le c_{25} \cdot \| \eta \|_{2,2,\bigcup \mathcal{T}_h}$$

and conclude

$$
\begin{aligned}
&\int_\Omega (\nabla_h v_h - A^{-1}p_h) \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx \\
&\qquad \le c_{25} \cdot \| h \cdot (\nabla_h v_h - A^{-1}p_h) \|_2 \cdot \| \eta \|_{2,2,\bigcup \mathcal{T}_h}. \qquad (5.11)
\end{aligned}
$$

The last term in (5.9) vanishes because the integral on Γ_h in the integration by parts is zero by (4.11) and so

$$\int_\Omega \nabla_h v_h \cdot (\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx = \int_\Omega v_h \cdot \operatorname{div}(\operatorname{Id} - \Pi_h)(A\nabla\eta)\, dx = 0 \qquad (5.12)$$

because of (4.9). Putting (5.9)—(5.12) together with (5.8) we have

$$\| e \|_2^2 \le \| e \|_2 \cdot (c_{20} \cdot c_A^{-1} \cdot \| h \cdot \operatorname{div}\epsilon \|_2 + c_{14} \cdot c_{25} \cdot \| h \cdot (\nabla_h v_h - A^{-1}p_h) \|_2)$$

and conclude the proof with Cauchy's inequality. □

6. Proof of Theorem 2.2

As indicated by the additional hypothesis $A^{-1}p_h|_T \in \mathcal{P}_\ell$ and $\nabla_h u_h|_T \in \mathcal{P}_\ell$, the lower bound is proved by inverse inequalities — a technique already elaborated in [V1, V2, V3]. The setting is simple: various weighted norms on polynomials on the reference element are equivalent and that by transforming backwards and forwards the equivalence constants of the current element only depend further on the change of the shape (i.e. on c_{16}) and the scaling (i.e. on h_T) during these transformations.

The proof of Theorem 2.2 is divided into Lemmas 6.1–6.3 where the positive constants $c_{26}, \dots, c_{36}$ arising below only depend on the shape of the elements, their maximal polynomial degree, and on ℓ.

Lemma 6.1. *For each $T \in \mathcal{T}_h$ and with β as defined in* (5.2),

$$h_T \cdot \| \operatorname{curl}(A^{-1} p_h) \|_{2,T} \le c_{26} \cdot \| A^{-1/2} \operatorname{Curl} \beta \|_{2,T}. \tag{6.1}$$

Proof. Fixing $\psi_T \in \mathcal{P}_3$ with $0 \le \psi_T \le 1 = \max \psi$ and zero boundary values on T we learn (e.g., from [V1, Lemma 1.3], or [V2, Lemma 4.1], or [V3, Lemma 5.1])

$$c_{27} \cdot \| \operatorname{curl}(A^{-1} p_h) \|_{2,T}^2 \le \| \psi_T^{1/2} \cdot \operatorname{curl}(A^{-1} p_h) \|_{2,T}^2. \tag{6.2}$$

Integration by parts, $\operatorname{curl}_h(A^{-1} p_h) = -\operatorname{curl}_h(A^{-1}\epsilon)$, and (5.3) verify

$$\begin{aligned} \| \psi_T^{1/2} \cdot \operatorname{curl}(A^{-1} p_h) \|_{2,T}^2 &= \int_T (A^{-1}\epsilon) \cdot \operatorname{Curl}(\psi_T \cdot \operatorname{curl}(A^{-1} p_h))\, dx \\ &= -\int_T (A^{-1} \operatorname{Curl} \beta) \cdot \operatorname{Curl}(\psi_T \cdot \operatorname{curl}(A^{-1} p_h))\, dx. \end{aligned} \tag{6.3}$$

Since $\psi_T \cdot \operatorname{curl}(A^{-1} p_h) \in \mathcal{P}_{\ell+2}$ with zero boundary values on T we have

$$| \psi_T \cdot \operatorname{curl}(A^{-1} p_h) |_{1,2,T} \le c_{28} \cdot h_T^{-1} \cdot \| \psi_T \cdot \operatorname{curl}(A^{-1} p_h) \|_{2,T} \tag{6.4}$$

(as, e.g., in [V2, Lemma 4.1] or [V3, Lemma 5.1]). Finally, Cauchy's inequality, (5.4), and (6.2)–(6.4) prove the lemma. □

Lemma 6.2. *For each $E \in \mathcal{E}_h$,*

$$\| h^{1/2} \cdot J(A^{-1} p_h \cdot t) \|_{L^2(E)} \le c_{29} \cdot \| A^{-1/2} \operatorname{Curl} \beta \|_{2,\omega_E}. \tag{6.5}$$

Proof. Let ψ_E denote that continuous function satisfying $0 \le \psi_E \le 1 = \max \psi_E$ on ω_E and $\psi_E|_T \in \mathcal{P}_2$ for each $T \in \mathcal{T}_h$ with $T \subset \omega_E$. Put $\sigma := J(A^{-1} p_h \cdot t)$ which is a polynomial of degree $\le k$ along E. As defined by backward and forward transformation and by continuous extension on the reference element in [V2, V3], there exists an extension operator $P : \mathcal{C}(E) \to \mathcal{C}(\omega_E)$ satisfying $P\sigma|_E = \sigma$ and

$$c_{30} \cdot h_E^{1/2} \cdot \| \sigma \|_{2,E} \le \| \psi_E^{1/2} \cdot P\sigma \|_{2,\omega_E} \le c_{31} \cdot h_E^{1/2} \cdot \| \sigma \|_{2,E}. \tag{6.6}$$

Similar to (6.2) (again established in [V1, V2, V3]) we gain

$$c_{32} \cdot \| \sigma \|_{2,E}^2 \le \| \psi_E^{1/2} \cdot \sigma \|_{2,E}^2 = -\int_E (\psi_E \cdot P\sigma) \cdot J(A^{-1}\epsilon \cdot t)\, ds. \tag{6.7}$$

An application of (5.7) to each element $T \subset \omega_E$ and of (5.3) result in

$$\begin{aligned} &-\int_E (\psi_E \cdot P\sigma) \cdot J(A^{-1}\epsilon \cdot t)\, ds \\ &= -\int_{\omega_E} (A^{-1}\epsilon) \cdot \operatorname{Curl}(\psi_E \cdot P\sigma)\, dx - \int_{\omega_E} (\psi_E \cdot P\sigma) \cdot \operatorname{curl}(A^{-1}\epsilon)\, dx \\ &= \int_{\omega_E} (A^{-1} \operatorname{Curl} \beta) \cdot \operatorname{Curl}(\psi_E \cdot P\sigma)\, dx + \int_{\omega_E} (\psi_E \cdot P\sigma) \cdot \operatorname{curl}(A^{-1} p_h)\, dx \\ &\le \| A^{-1} \operatorname{Curl} \beta \|_{2,\omega_E} \cdot | \psi_E \cdot P\sigma |_{1,2,\omega_E} + \| \psi_E \cdot P\sigma \|_{2,\omega_E} \cdot \| \operatorname{curl}(A^{-1} p_h) \|_{2,\omega_E}. \end{aligned}$$

Using (6.1) and (6.6) we infer

$$\begin{aligned} -\int_E (\psi_E \cdot P\sigma) \cdot J(A^{-1}\epsilon \cdot t)\, ds \le{}& \| A^{-1} \operatorname{Curl} \beta \|_{2,\omega_E} \cdot | \psi_E \cdot P\sigma |_{1,2,\omega_E} \\ &+ c_{26} \cdot c_{31} \cdot h_E^{-1/2} \cdot \| \sigma \|_{2,E} \cdot \| A^{-1/2} \operatorname{Curl} \beta \|_{2,\omega_E}. \end{aligned} \tag{6.8}$$

Since $\psi_E \cdot P\sigma$ is a certain extension of a polynomial it follows as an inverse inequality

$$| \psi_E \cdot P\sigma |_{1,2,\omega_E} \le c_{33} \cdot h_E^{-1} \cdot \| \psi_E \cdot P\sigma \|_{2,\omega_E}$$

(see, e.g., [V2, p.76] or [V3, Eq.(5.6)]). With (6.6), this proves

$$| \psi_E \cdot P\sigma |_{1,2,\omega_E} \leq c_{31} \cdot c_{33} \cdot h_E^{-1/2} \cdot \| \sigma \|_{2,E}. \tag{6.9}$$

After this, the lemma follows from (6.7)–(6.9). □

Lemma 6.3. *For each $T \in \mathcal{T}_h$,*

$$h_T \cdot \| A^{-1} p_h - \nabla_h u_h \|_{2,T} \leq c_{34} \cdot \big(\| e \|_{2,T} + h_T \cdot \| A^{-1/2} \epsilon \|_{2,T} \big). \tag{6.10}$$

Proof. As in (6.2), there holds

$$c_{35} \cdot \| A^{-1} p_h - \nabla u_h \|_{2,T}^2 \leq \| \psi_T^{1/2} \cdot (A^{-1} p_h - \nabla u_h) \|_{2,T}^2 \tag{6.11}$$

and integration by parts gives

(6.12)

$$\begin{aligned} &\| \psi_T^{1/2} \cdot (A^{-1} p_h - \nabla u_h) \|_{2,T}^2 \\ &= - \int_T \psi_T \cdot A^{-1} \epsilon \cdot (A^{-1} p_h - \nabla u_h) \, dx - \int_T e \cdot \operatorname{div}(\psi_T \cdot (A^{-1} p_h - \nabla u_h)) \, dx \\ &\leq \| A^{-1} \epsilon \|_{2,T} \cdot \| \psi_T \cdot (A^{-1} p_h - \nabla u_h) \|_{2,T} + \| e \|_{2,T} \cdot | \psi_T \cdot (A^{-1} p_h - \nabla u_h) |_{1,2,T}. \end{aligned}$$

As in (6.4), we deduce

$$| \psi_T \cdot (A^{-1} p_h - \nabla u_h) |_{1,2,T} \leq c_{36} \cdot h_T^{-1} \cdot \| \psi_T \cdot (A^{-1} p_h - \nabla u_h) \|_{2,T} \tag{6.13}$$

and, finally, conclude the lemma from (6.11)–(6.13). □

The lower bound in Theorem 2.2 is a direct consequence of Lemmas 6.1–6.3 and the (global) error decomposition (5.4).

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