Numer. Math. (1998) 81: 187–209

Numerische Mathematik

A posteriori error estimates for mixed FEM in elasticity

Carsten Carstensen[1], **Georg Dolzmann**[2]

[1] Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany; e-mail: cc@numerik.uni-kiel.de
[2] Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany; e-mail: georg@mis.mpg.de

Received July 17, 1997

Summary. A residue based reliable and efficient error estimator is established for finite element solutions of mixed boundary value problems in linear, planar elasticity. The proof of the reliability of the estimator is based on Helmholtz type decompositions of the error in the stress variable and a duality argument for the error in the displacements. The efficiency follows from inverse estimates. The constants in both estimates are independent of the Lamé constant λ, and so locking phenomena for $\lambda \to \infty$ are properly indicated. The analysis justifies a new adaptive algorithm for automatic mesh–refinement.

Mathematics Subject Classification (1991): 65N30, 65N15, 73C35

1. Introduction

The fundamental problem in linear elasticity is usually modelled as follows [Ci2, Va]: Let $\Omega \subset \mathbb{R}^d$ be the reference configuration of the elastic body under consideration with boundary $\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N}$, $\Gamma_{\rm D}$ not empty and connected, $\Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset$. Given a volume force $f : \Omega \to \mathbb{R}^d$, a displacement $u_{\rm D} : \Gamma_{\rm D} \to \mathbb{R}^d$ and a traction $g : \Gamma_{\rm N} \to \mathbb{R}^d$, find a displacement $u : \Omega \to \mathbb{R}^d$ and a stress tensor $\sigma : \Omega \to \mathbb{M}^{d\times d}_{\rm sym} := \{\tau \in \mathbb{M}^{d\times d} : \tau = \tau^{\rm T}\}$ satisfying

$$-\operatorname{div}\sigma = f, \quad \sigma = \mathbb{C}\mathbb{E}(u) \text{ in } \Omega, \tag{1.1}$$

$$u = u_{\rm D} \text{ on } \Gamma_{\rm D}, \qquad \sigma n = g \text{ on } \Gamma_{\rm N}, \tag{1.2}$$

Correspondence to: C. Carstensen

where the fourth order elasticity tensor $\mathbb{C}$ is bounded, positive definite, and satisfies the symmetry conditions $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} = \mathbb{C}_{klij}$. We write $\mathbb{E}(v) = \frac{1}{2}(\nabla v + (\nabla v)^{\mathrm{T}})$ for the infinitesimal strain tensor. In the following we restrict ourselves to the model of plane strain, i.e.

$$\mathbb{C}\mathbb{E}(u) = \lambda \operatorname{tr}(\mathbb{E}(u))\mathrm{Id} + 2\mu\mathbb{E}(u), \tag{1.3}$$

where λ and μ are the Lamé constants, $\operatorname{tr}(A) = A_{11} + \ldots + A_{dd}$ is the trace of the matrix A and Id is the $d \times d$ identity matrix. (Using ideas from [AF] it is easy to see that our estimates hold also for more general tensors $\mathbb{C}$.) It is a consequence of Korn's inequality and the Lax-Milgram lemma that problem (1.1)–(1.2) has a unique solution $(\sigma, u) \in L^2(\Omega; \mathbb{M}^{d\times d}_{\mathrm{sym}}) \times W^{1,2}(\Omega; \mathbb{R}^d)$ which satisfies the a priori estimate $\|u\|_{1,2;\Omega} + \|\sigma\|_{2;\Omega} \leq c_1\|f\|_{2;\Omega}$. In addition, the error estimate for the displacement requires the following regularity assumption

$$\|u\|_{2,2;\Omega} + \|\sigma\|_{1,2;\Omega} \leq c_2(\|f\|_{2;\Omega} + \|u_{\mathrm{D}}\|H^{3/2}(\Gamma_{\mathrm{D}}) + \|g\|H^{1/2}(\Gamma_{\mathrm{N}})). \tag{1.4}$$

A realistic hypothesis for (1.4) to hold is $0 < \operatorname{dist}(\Gamma_{\mathrm{D}}; \Gamma_{\mathrm{N}})$, i.e., the boundary condition does not change at some boundary point. Furthermore, the constant c_2 is supposed to be independent of λ (see Theorem 2.1 in [ADG] and Lemma A.1 in [Vo] for the cases $\Gamma_{\mathrm{N}} = \emptyset$ and $\Gamma_{\mathrm{D}} = \emptyset$, respectively; the general statement does not seem to be available in the literature).

Mixed methods are a powerful tool for the numerical solution of the system (1.1)–(1.2). They provide at the same time an approximation of the displacement and the stress tensor. A priori estimates have been established for a wide choice of different methods which satisfy the Babuška-Brezzi condition. A subtle choice of the discrete spaces avoids the common phenomenon of locking (i.e., the estimates are independent of the parameter λ in (1.3)). A difficulty in the design of stable numerical schemes is linked to the symmetry of the stress tensor σ and therefore FRAEIJS DE VEUBEKE [FdV] and following his ideas BREZZI-DOUGLAS-MARINI [BDM], ARNOLD-BREZZI-DOUGLAS [ABD] and STENBERG [St] weakened the symmetry condition and reformulated the elasticity problem: Find $u : \Omega \to \mathbb{R}^d$, $\sigma : \Omega \to \mathbb{M}^{d\times d}$ and $\gamma : \Omega \to \mathbb{M}^{d\times d}_{\mathrm{skew}} := \{\eta \in \mathbb{M}^{d\times d} : \eta + \eta^{\mathrm{T}} = 0\}$, such that

$$\sigma = \mathbb{C}(\nabla u - \gamma), \quad \sigma = \sigma^{\mathrm{T}}, \quad -\operatorname{div}\sigma = f \text{ in } \Omega, \tag{1.5}$$

$$u = u_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}, \qquad \sigma n = g \text{ on } \Gamma_{\mathrm{N}}. \tag{1.6}$$

In the following we will assume $u_{\mathrm{D}} = 0$. In the corresponding variational formulation one seeks $(\sigma, u, \gamma) \in \Sigma_g \times \mathcal{U} \times \mathcal{W}$ such that

$$a(\sigma, \tau) + b(\tau; u, \gamma) = 0 \quad \text{and} \quad b(\sigma; v, \eta) = -(f, v), \tag{1.7}$$

for all $(\tau, v, \eta) \in \Sigma_0 \times \mathcal{U} \times \mathcal{W}$. Here, the linear and bilinear forms and the function spaces $\Sigma_t, \mathcal{U}, \mathcal{W}$ are defined by

$$\begin{aligned} a(\sigma,\tau) &= \int_\Omega \mathbb{C}^{-1}\sigma : \tau dx, \\ b(\sigma;u,\gamma) &= \int_\Omega (\langle \operatorname{div}\sigma, u\rangle + \sigma : \gamma) dx, \\ (f,v) &= \int_\Omega \langle f, v\rangle dx, \\ \Sigma_t &= \{\sigma \in L^2(\Omega;\mathbb{M}^{d\times d}) : \operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^d),\ \sigma n = t \text{ on } \Gamma_{\mathrm{N}}\}, \\ \mathcal{U}\times\mathcal{W} &= L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{M}^{d\times d}_{\text{skew}}), \end{aligned}$$

for $t = 0$ and $t = g$. In this approach, the symmetry of the stress tensor σ is relaxed and only imposed by means of the Lagrange multiplier γ. Let $\Sigma_{t,h}$, $\mathcal{U}_h$, $\mathcal{W}_h$ be finite dimensional spaces approximating Σ_t, $\mathcal{U}$, and $\mathcal{W}$. Then the corresponding discrete solution $(\sigma_h, u_h, \gamma_h) \in \Sigma_{g,h} \times \mathcal{U}_h \times \mathcal{W}_h$ is characterised by

(1.8) $$a(\sigma_h,\tau_h) + b(\tau_h;u_h,\gamma_h) = 0 \quad \text{and} \quad b(\sigma_h;v_h,\eta_h) = -(f,v_h),$$

for all $(\tau_h, v_h, \eta_h) \in \Sigma_{0,h} \times \mathcal{U}_h \times \mathcal{W}_h$. In this formulation, σ_h satisfies only the weak symmetry condition

(1.9) $$\int_\Omega \sigma_h : \gamma_h dx = 0 \quad \forall \gamma_h \in \mathcal{W}_h,$$

which does not imply $\sigma_h = \sigma_h^{\mathrm{T}}$ if $\sigma_h - \sigma_h^{\mathrm{T}} \notin \mathcal{W}_h$. In two dimensions existence, uniqueness, and a priori estimates for several choices of discrete spaces have been proven in [St] which include the low order PEERS (plane elasticity element with reduced symmetry) constructed by ARNOLD-BREZZI-DOUGLAS [ABD] and a modification of the BREZZI-DOUGLAS-MARINI element BDM_k by STENBERG (which we will refer to as BDMS_k element). A posteriori estimates in the natural norms, on the other hand, do not seem to be available in the literature (see, however, [BKNSW] for estimates in mesh dependent norms and [RS] for results concerning stabilised dual–mixed formulations).

In this paper, we propose an a posteriori error estimator for the errors $\varepsilon = \sigma - \sigma_h$ and $e = u - u_h$ for the PEERS and the BDMS_k method (see Sect. 2 for details). Our analysis relies on a decomposition of symmetric tensors in the spirit of a generalised Helmholtz decomposition. Helmholtz decomposition was first used in [Ca,A] to prove efficiency and reliability of error estimators for mixed finite elements. The estimator accounts for the residues on the triangles T and the jumps across the element boundaries E. More precisely, we define (see Sect. 2 for the notation used below)

$$\eta_T^2 = h_T^2 \|\operatorname{div} \varepsilon\|_{2;T}^2 + h_T^2 \|\operatorname{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)\|_{2;T}^2$$
$$+ h_T^2 \inf_{v_h \in \mathcal{U}_h} \|\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla v_h\|_{2;T}^2 + \|\operatorname{Skw}(\sigma_h)\|_{2;T}^2,$$
$$\eta_E^2 = \begin{cases} h_E \|J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\|_{2;E}^2 & \text{if } E \subset \Omega \cup \Gamma_{\mathrm{D}}, \\ h_E \|(\sigma - \sigma_h)n\|_{2;E}^2 & \text{if } E \subset \Gamma_{\mathrm{N}}, \end{cases}$$
$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{E \in \mathcal{E}_h} \eta_E^2. \tag{1.10}$$

The main result of this paper states reliability and efficiency of the estimator η. All constants in the estimates are under the regularity assumption (1.4) independent of h and λ. In particular, the common locking phenomena are avoided.

Theorem 1.1. *Let $\mathcal{T}_h$ be a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$ and let $(\sigma_h, u_h, \gamma_h)$ be the solution of (1.8) for the* PEERS *or the* BDMS_k *element. Assume that the regularity assumption (1.4) holds. Then there exists a constant c_3, which depends only on Ω, μ, and the polynomial degree of the elements, such that*

$$\|u - u_h\|_{2;\Omega} + \|\gamma - \gamma_h\|_{2;\Omega} + \|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\|_{2;\Omega} \le c_3 \eta.$$

Theorem 1.2. *Assume in addition that $\operatorname{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)_{|T}$ is a polynomial for all $T \in \mathcal{T}_h$ and $(\sigma - \sigma_h)n_{|E}$ for all $E \subset \Gamma_{\mathrm{N}}$. Then there exists a constant c_4, which depends only on Ω, μ, and the polynomial degree of the elements, such that*

$$\eta \le c_4 \Big(\|u - u_h\|_{2;\Omega} + \|\mathbb{C}^{-1}(\sigma - \sigma_h) + \gamma - \gamma_h\|_{2;\Omega}$$
$$+ \|\sigma - \sigma_h\|_{2;\Omega} + \|h_{\mathcal{T}} \operatorname{div} \varepsilon\|_{2;\Omega} \Big).$$

Remarks. 1. It follows from (1.7) that $\mathbb{C}^{-1}\sigma + \gamma = \nabla u$. Therefore the terms in the estimator are natural residuals: $\operatorname{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)$ and $J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)$ are zero if $\mathbb{C}^{-1}\sigma_h + \gamma_h$ is a gradient. The distance of this term to gradients is also measured by the expression $\inf \|\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla v_h\|$.
2. The term $\inf_{v_h \in \mathcal{U}_h} \|\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla v_h\|$ can be replaced by its upper bound $\|\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla u_h\|$ which still satisfies the efficiency estimate of Theorem 1.2.
3. Since $-\operatorname{div} \varepsilon = f + \operatorname{div} \sigma_h$ is a known quantity, we can replace $\|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\|_{2;\Omega}$ by the (weighted) norm

$$\|\sigma - \sigma_h\|_{H(\operatorname{div};\Omega)} = \|\mathbb{C}^{-1/2}(\sigma - \sigma_h)\|_{2;\Omega} + \|\operatorname{div}(\sigma - \sigma_h)\|_{2;\Omega}$$

on the left hand side in Theorem 1.1, but we lose the factor h_T in the estimator above in front of the term $\|\operatorname{div}\varepsilon\|_{2;T}$.
4. The regularity assumption (1.4) is not needed for the estimate of $\|\mathbb{C}^{-1/2}(\sigma-\sigma_h)\|_{2;\Omega}$ in Theorem 1.1, but in the duality argument in the estimation of $\|u-u_h\|_{2;\Omega}$. Hence, if we suppress $\|u-u_h\|_{L^2(\Omega)}$ then Theorem 1.1 remains true even if $\operatorname{dist}(\Gamma_{\mathrm{D}};\Gamma_{\mathrm{N}})=0$.
5. According to the triangle inequality and the preceding remarks, the error

$$\|u-u_h\|_{2;\Omega}+\|\gamma-\gamma_h\|_{2;\Omega}+\|\mathbb{C}^{-1/2}(\sigma-\sigma_h)\|_{2;\Omega}$$

and the error indicator η are *equivalent* in the sense that their quotient is bounded from below and above independently of the material parameter λ and the mesh–size h. In particular, the estimates are robust with respect to $\lambda\to\infty$ for (nearly) incompressible materials.
6. The estimator justifies an adaptive finite element scheme which refines a given grid only in regions where the error is relatively large. A standard algorithm for efficient mesh-design is as follows: For each mesh T_{h_L} with a Galerkin solution (p_{h_L},u_{h_L}) and local error estimators $\eta(T)=:\eta_T+\sum_{E\subseteq\partial T}\eta_E$, we refine $T\in\mathcal{T}_{h_L}$ (e.g., by halving its largest side) if (for example)

$$\max_{T'\in T_{h_L}}\eta(T')/2\le\eta(T).$$

Then, further refinements to avoid hanging nodes lead to a new mesh $\mathcal{T}_{h_{L+1}}$ from which we start again.
7. The estimates are stated for the elements of practical importance only. The arguments used in the proofs rely only on the following properties (with $\mathcal{L}^0_0$ the piecewise constant functions on Ω and $\mathcal{L}^1_1$ the continuous piecewise affine ones)

$$\mathcal{L}^0_0\subset U_h,\quad \mathcal{L}^0_0\cap H(\operatorname{div};\Omega)^2\subseteq\Sigma_{0,h},\quad\text{and}\quad \mathcal{L}^1_1\subseteq W_h.$$

To obtain estimates for the displacements, we further require a commutation property for some (Fortin–) interpolation operator π_h (of (2.1)—(2.4) below). We refer to [Ca] for a discussion in the general framework (for Laplace's equation).

2. Preliminaries

We assume that Ω is a bounded domain in $\mathbb{R}^2$ with polygonal boundary. Let $\mathcal{T}_h$ be a regular triangulation of Ω in the sense of [Ci1], which satisfies the minimum angle condition, i.e., there exists a constant $c_5>0$ such that $c_5^{-1}h_T^2\le|T|\le c_5h_T^2$. Here, $|T|$ is the area and h_T is the diameter of $T\in\mathcal{T}_h$. The set of all element sides in $\mathcal{T}_h$ is denoted by $\mathcal{E}_h$ and h_E is the length of the edge $E \in \mathcal{E}_h$. We assume in addition that Γ_N is a finite union of connected components Γ_i, $i = 0, \ldots, M$, and that Γ_D and Γ_N have positive distance. Thus we have $\mathcal{E}_h = \mathcal{E}_\Omega \cup \mathcal{E}_\mathrm{D} \cup \mathcal{E}_\mathrm{N}$ where $\mathcal{E}_\Omega$ is the set of all interior element sides and $\mathcal{E}_\mathrm{D}$ and $\mathcal{E}_\mathrm{N}$ is the collection of all edges contained in Γ_D and Γ_N, respectively. We write $\mathcal{E}_h^0 = \mathcal{E}_\Omega \cup \mathcal{E}_\mathrm{N}$. It is useful to define a function $h_\mathcal{T}$ on Ω by $h_{\mathcal{T}|T} = h_T$ and a function $h_\mathcal{E}$ on the union of all element sides by $h_{\mathcal{E}|E} = h_E$. We write $u \in W^{m,p}(\mathcal{T}_h)$ and $v \in W^{m,p}(\mathcal{E}_h)$ if $u_{|T} \in W^{m,p}(T)$ for all $T \in \mathcal{T}_h$ and $v_{|E} \in W^{m,p}(E)$ for all $E \in \mathcal{E}_h$. For each $E \in \mathcal{E}_h$ we fix a normal n_E to E such that n_E coincides with the exterior normal to $\partial\Omega$ if $E \subset \partial\Omega$. This allows us to define a mapping $J : W^{1,2}(\mathcal{T}_h) \to L^2(\mathcal{E}_h)$ by

$$J(v)_{|E} = (v_{|T^+})_{|E} - (v_{|T^-})_{|E}$$

if $E = \bar{T}^+ \cap \bar{T}^-$ and n_E is the exterior normal to T^+ on E and

$$J(v)_{|E} = (v_{|T})_{|E}$$

if $E = \bar{T} \cap \partial\Omega$. Finally we define for $\Phi \in W^{1,2}(\Omega)$, $u = (u_1, u_2) \in W^{1,2}(\Omega; \mathbb{R}^2)$, and $\sigma \in W^{1,2}(\Omega; \mathbb{M}^{2\times 2})$

$$\operatorname{Curl} \Phi = (\Phi_{,2},\ -\Phi_{,1}),$$

$$\operatorname{Curl} u = \begin{pmatrix} u_{1,2} & -u_{1,1} \\ u_{2,2} & -u_{2,1} \end{pmatrix}, \quad \operatorname{curl} u = u_{2,1} - u_{1,2},$$
$$\operatorname{curl} \sigma = \begin{pmatrix} \sigma_{12,1} - \sigma_{11,2} \\ \sigma_{22,1} - \sigma_{21,2} \end{pmatrix}, \quad \operatorname{div} \sigma = \begin{pmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{21,1} + \sigma_{22,2} \end{pmatrix}.$$

We use the standard notation for the Lebesgue spaces $L^p(\Omega)$ with norm $\|\cdot\|_{p;\Omega}$ and the Sobolev spaces $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p;\Omega}$ and seminorm $|\cdot|_{m,p;\Omega}$. The closure of $C_\mathrm{c}^\infty(\Omega)$, the space of infinitely often differentiable functions with compact support, with respect to $\|\cdot\|_{m,p;\Omega}$ is denoted by $W_0^{m,p}(\Omega)$.
The definition of the finite element spaces involves the bubble function $b_T = \lambda_1\lambda_2\lambda_3$ on a triangle $T \in \mathcal{T}_h$, where λ_i are the barycentric coordinates of T. The PEERS is based on the following function spaces

$$\begin{aligned}
\mathcal{U}_h &= \{v_h \in \mathcal{U} :\ v_{h|T} \in \mathcal{P}_0(T; \mathbb{R}^2)\,\forall T \in \mathcal{T}_h\},\\
\mathcal{W}_h &= \{\gamma_h \in \mathcal{W} \cap C^0(\Omega; \mathbb{M}^{2\times 2}_\mathrm{skew}) :\ \gamma_{h|T} \in \mathcal{P}_1(T; \mathbb{M}^{2\times 2}_\mathrm{skew})\,\forall T \in \mathcal{T}_h\},\\
\Sigma_h &= \{\sigma_h \in L^2(\Omega; \mathbb{M}^{2\times 2}) :\\
&\qquad \operatorname{div} \sigma_h \in \mathcal{U}, \sigma_{h|T} \in \mathrm{RT}_0(T) \oplus \mathrm{B}_0(T)\,\forall T \in \mathcal{T}_h\},\\
\Sigma_{t,h} &= \{\sigma_h \in \Sigma_h :\ \sigma_h n = \tilde{t} \text{ on } \Gamma_\mathrm{N}\},
\end{aligned}$$

where $\tilde{t}$ is the orthogonal projection of t in $L^2(E)$ onto $P_0(E;\mathbb{R}^2)$ for all edges $E \subset \Gamma_{\mathrm{N}}$. Here, RT_0 is the RAVIART-THOMAS space of lowest degree, and

$$\begin{aligned}
\mathrm{RT}_0(T) &= \{\sigma \in L^2(T;\mathbb{M}^{2\times 2}) : \sigma = \tau + a \otimes x,\ \tau \in \mathbb{M}^{2\times 2}, a \in \mathbb{R}^2\},\\
\mathrm{B}_0(T) &= \{\sigma \in L^2(T;\mathbb{M}^{2\times 2}) : \sigma = a \otimes \operatorname{Curl} b_T,\ a \in \mathbb{R}^2\},\\
\mathrm{BDM}_k(\Omega) &= \{\sigma_h \in L^2(\Omega;\mathbb{M}^{2\times 2}) : \operatorname{div}\sigma_h \in \mathcal{U}, \sigma_{h|T} \in \mathcal{P}_k(T;\mathbb{M}^{2\times 2})\}.
\end{aligned}$$

The higher order methods BDMS_k are defined for $k \geq 2$ by

$$\begin{aligned}
\mathcal{U}_h &= \{v_h \in \mathcal{U} : v_{h|T} \in \mathcal{P}_{k-1}(T;\mathbb{R}^2)\,\forall\, T \in \mathcal{T}_h\},\\
\mathcal{W}_h &= \{\gamma_h \in \mathcal{W} : \gamma_{h|T} \in \mathcal{P}_k(T;\mathbb{M}^{2\times 2}_{\mathrm{skew}})\,\forall\, T \in \mathcal{T}_h\},\\
\Sigma_h &= \{\sigma_h \in L^2(\Omega;\mathbb{M}^{2\times 2}) :\\
&\qquad \operatorname{div}\sigma \in \mathcal{U},\ \sigma_{h|T} \in \mathcal{P}_k(T;\mathbb{M}^{2\times 2}) \oplus B_{k-1}(T)\},\\
\Sigma_{t,h} &= \{\sigma_h \in \Sigma_h : \sigma_h n = \tilde{t} \text{ on } \Gamma_{\mathrm{N}}\},
\end{aligned}$$

where $\tilde{t}$ is the orthogonal projection of t in $L^2(E)$ onto $\mathcal{P}_k(E;\mathbb{R}^2)$, and

$$B_{k-1}(T) = \{\sigma \in L^2(T;\mathbb{M}^{2\times 2}) : \sigma = \operatorname{Curl}(b_T w), w \in \mathcal{P}_{k-1}(T;\mathbb{R}^2)\}.$$

Using the interpolation operators for RT_0 and BDM_k (see [BF], Sect. III.3.3) we can construct an interpolation operator $\Pi_h : W^{1,2}(\Omega;\mathbb{M}^{2\times 2}) \to \Sigma_h$ such that for all $\tau \in W^{1,2}(\Omega;\mathbb{M}^{2\times 2})$

$$\int_\Omega \operatorname{div}(\Pi_h\tau - \tau) v_h dx = 0 \quad \forall\, v_h \in \mathcal{U}_h, \text{ and} \tag{2.1}$$

$$\|\Pi_h\tau - \tau\|_{2;T} \leq c_6 h_T |\tau|_{1,2;T}. \tag{2.2}$$

The projection Π_h is defined in such a way that

$$\int_\Omega (\Pi_h\tau - \tau)\nabla_h v_h dx = 0 \quad \forall\, v_h \in \mathcal{U}_h, \text{ and} \tag{2.3}$$

$$\tau n = 0 \text{ on } \Gamma_{\mathrm{N}} \quad \Rightarrow \quad \Pi_h\tau n = 0 \text{ on } \Gamma_{\mathrm{N}}. \tag{2.4}$$

If P_h^0 denotes the orthogonal projection in L^2 onto $\mathcal{L}_0^0 \subset \mathcal{U}_h$, $\mathcal{L}_0^0$, the space of piecewise constant functions, we have the estimate

$$\|v - P_h^0 v\|_{2;T} \leq c_7 h_T |v|_{1,2;T} \quad \forall\, v \in W^{1,2}(T) \quad \forall\, T \in \mathcal{T}_h.$$

Finally we use Clément's interpolation operator [Cl] $R_h : W^{1,2}(\Omega) \to \mathcal{L}_1^1$ onto the space of continuous, piecewise linear functions, which satisfies the interpolation estimates

$$\begin{aligned}
\|v - R_h v\|_{2;T} &\leq c_8 h_T \|v\|_{1,2;\omega_T},\\
\|v - R_h v\|_{2;E} &\leq c_9 h_E^{1/2} \|v\|_{1,2;\omega_E},
\end{aligned}$$

where $\omega_T = \cup\{T' \in \mathcal{T}_h : \bar{T} \cap \bar{T}' \neq \emptyset\}$ and $\omega_E = \cup\{T \in \mathcal{T}_h : E \subset \bar{T}\}$. Notice that R_h satisfies

$$v = c_i \text{ on } \Gamma_i \quad \Rightarrow \quad R_h v = c_i \text{ on } \Gamma_i. \tag{2.5}$$

The number of triangles in ω_T is uniformly bounded by some constant c_{10}, which depends only on the shape of the triangles. Throughout the paper we write $\mathrm{Sym}(\sigma)$ and $\mathrm{Skw}(\sigma)$ for the symmetric and the skew-symmetric part of a matrix σ and use little Greek letters for matrices, little Latin letters of vectors and capital Greek letters for scalars. We use the symbols ∇_h and curl_h if we apply the corresponding differential operators on each triangle to a function that is globally not smooth.

3. A Helmholtz decompostion for symmetric tensor fields

The following two results on the Helmholtz decomposition are essential for the subsequent proofs. We add a sketch of their proofs for the convenience of the reader.

Lemma 3.1. *Assume that A is a symmetric, positive definite tensor of fourth order. Let $\rho \in L^2(\Omega; \mathbb{M}^{2\times 2})$. Then there exists $q \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $q = 0$ on Γ_D and $f \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $f = c_i \in \mathbb{R}^2$ on Γ_i, $c_0 = 0$, such that*

$$\rho = \nabla q + A^{-1}\mathrm{Curl} f.$$

Proof. The classical proof for the existence of a Helmholtz decomposition for vector fields $u \in L^2(\Omega; \mathbb{R}^2)$ can be modified to yield the existence of Φ, $\Psi \in L^2(\Omega)$ such that $\Psi = 0$ on Γ_D, $\Phi = c_i$ on Γ_i and $u = \nabla\Psi + \mathrm{Curl}\,\Phi$. To do so, consider

$$I(p) = \int_\Omega \left(\frac{1}{2} A\nabla p : \nabla p - A\rho : \nabla p \right) dx.$$

It follows from the direct method in the calculus of variations that there exists a unique $q \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $q = 0$ on Γ_D such that

$$I(q) = \min\{I(p) : p \in W^{1,2}(\Omega; \mathbb{R}^2),\ p = 0 \text{ on } \Gamma_\mathrm{D}\}$$

and q satisfies the Euler–Lagrange equation

$$\int_\Omega (A\nabla q - A\rho)\nabla\phi dx = 0 \qquad \text{for all } \phi \in W^{1,2}(\Omega; \mathbb{R}^2) \text{ with } \phi = 0 \text{ on } \Gamma_\mathrm{D}.$$

It follows that $\pi = A\nabla q - A\rho$ is a divergence free vectorfield and by Green's formula

$$\int_{\partial\Omega} \langle \pi n, \phi\rangle ds = \int_\Omega (\mathrm{div}\pi\phi + \pi\nabla\phi) dx \quad \forall \phi \in W^{1,2}(\Omega; \mathbb{R}^2).$$

In view of the Euler-Lagrange equations we conclude

$$\int_{\partial\Omega} \langle \pi n, \phi \rangle ds = 0 \quad \text{for all } \phi \in W^{1,2}(\Omega;\mathbb{R}^2) \quad \text{with} \quad \phi = 0 \text{ on } \Gamma_{\mathrm{D}}.$$

With $\phi \equiv 1$ in a neighbourhood of one component of the Neumann boundary and $\phi \equiv 0$ in a neighbourhood of all the other components as well as on a neighbourhood of the Dirichlet boundary, we infer that πn has mean value zero on all connected components of the Neumann boundary. With $\phi \equiv 1$ we deduce the same property on the Dirichlet boundary and thus there exists an $f \in W^{1,2}(\Omega;\mathbb{R}^2)$ such that $\pi = \mathrm{Curl} f$ (see [GR], Chapter I, Theorem 3.1). Since $\pi n = \mathrm{Curl} f n = \nabla f t$, where t is a tangential vector, this concludes the proof. □

Furthermore, we also need a symmetric variant and define

$$\begin{aligned} X_1 &= \{v \in W^{1,2}(\Omega;\mathbb{R}^2) : \; v = 0 \text{ on } \Gamma_{\mathrm{D}}\}, \\ X_2 &= \{\Phi \in W^{2,2}(\Omega) : \\ &\qquad \int_\Omega \Phi dx = 0, \; \mathrm{Curl}\, \Phi = c_i \text{ on } \Gamma_i, \; c_i \in \mathbb{R}^2, \; c_0 = 0\}. \end{aligned}$$

Lemma 3.2. *Let* $\sigma \in L^2(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$. *Then there exists* $v \in X_1$ *and* $\Phi \in X_2$ *such that*

$$\sigma = \mathbb{C}\mathbb{E}(v) + \mathrm{Curl}\,\mathrm{Curl}\,\Phi.$$

Proof. In view of Korn's inequality there exists, by the direct method of the calculus of variations, a unique minimiser $v \in X_1$ of

$$I(v) = \int_\Omega \frac{1}{2}\mathbb{C}\mathbb{E}(v) : \mathbb{E}(v) dx - \int_\Omega \sigma : \mathbb{E}(v) dx.$$

In particular, v satisfies the corresponding Euler–Lagrange equations

$$\int_\Omega \mathbb{C}\mathbb{E}(v) : \nabla w \, dx = \int_\Omega \sigma : \nabla w \, dx \quad \forall\, w \in X_1.$$

Let $\tau = \sigma - \mathbb{C}\mathbb{E}(v) \in L^2(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$. The classical Helmholtz decomposition applied to the rows of τ yields the existence of $q \in X_1$ and $h \in W^{1,2}(\Omega;\mathbb{R}^2)$, $h = c_i$ on Γ_i with $c_0 = 0$ such that

$$\tau = \nabla q + \mathrm{Curl}\, h$$

(we refer to Lemma 3.1 for details). If we use q as a test function in the Euler Lagrange equations we deduce in view of the orthogonality of ∇q and $\mathrm{Curl}\, h$ in L^2

$$0 = \int_\Omega (\mathbb{C}\mathbb{E}(v) - \sigma) : \nabla q \, dx = \int_\Omega |\nabla q|^2 dx + \int_\Omega \mathrm{Curl}\, h : \nabla q \, dx$$

and therefore $q \equiv 0$ and $\tau = \operatorname{Curl} h$. From the symmetry of τ we deduce $-h_{1,1} = h_{2,2}$, i.e., $\operatorname{div} h = 0$. Since h is constant on the connected components of Γ_{N} which are by assumption closed Lipschitz curves, we conclude that the mean value of $\langle h, n\rangle$ vanishes on Γ_i for $i = 0, \ldots, M$. Green's formula then implies that the mean value vanishes also on Γ_{D} and hence there exists a stream function $\Phi \in W^{2,2}(\Omega)$ with $\operatorname{Curl} \Phi = h$. Subtracting from Φ a suitable constant if necessary we obtain the assertion of the lemma. □

4. An estimate for the trace of a tensor field

The following technical results are needed in the estimates below. The first estimate is a modification of well-established estimates of the trace of a tensor field by its divergence and the deviatoric part (see, e.g., [BF, Proposition 3.1 in Sect. IV.3]).

Lemma 4.1. *Let Σ_0 be a closed subspace of $H(\operatorname{div};\Omega)$ which does not contain the constant tensor* Id. *Then there exists a constant c_{11} (which depends only on Σ_0) such that*

$$\|\operatorname{tr}\tau\|_{2;\Omega} \le c_{11}\Big(\|\tau^{\mathrm{D}}\|_{2;\Omega} + \|\operatorname{div}\tau\|_{2;\Omega}\Big) \quad \forall \tau \in \Sigma_0.$$

Proof. Assume the contrary. Then there exists a sequence $(\tau_j) \in \Sigma_0$ satisfying

$$\|\operatorname{tr}\tau_j\|_{2;\Omega} = 1, \quad \|\tau_j^{\mathrm{D}}\|_{2;\Omega} + \|\operatorname{div}\tau_j\|_{2;\Omega} \to 0.$$

Thus we may choose a subsequence (again denoted by τ_j) such that $\tau_j \rightharpoonup \tau$ in $L^2(\Omega;\mathbb{M}^{d\times d})$ and $\operatorname{div}\tau_j \rightharpoonup \operatorname{div}\tau$ in $L^2(\Omega;\mathbb{R}^d)$. Clearly $\tau \in \Sigma_0$ with $\tau^{\mathrm{D}} = 0$ and therefore $\tau = \alpha \cdot \mathrm{Id}$ with $\alpha \in L^2(\Omega)$. On the other hand we have $\operatorname{div}\tau = \nabla\alpha = 0$ and hence α is constant. Since $\mathrm{Id} \notin \Sigma_0$ we conclude $\tau = 0$. It follows from the weak convergence of the sequence τ_j that

$$c_j = \frac{1}{|\Omega|}\int_\Omega \operatorname{tr}\tau_j dx \to 0$$

and thus $\sigma_j = \tau_j - \frac{c_j}{d}\cdot \mathrm{Id}$ (d being the dimension) satisfies by assumption

$$\lim_{j\to\infty}\|\operatorname{tr}\sigma_j\|_{2;\Omega} = 1. \tag{4.1}$$

We now adapt the arguments from [BF], p. 199, to obtain a contradiction. Since the integral mean of $\operatorname{tr}\sigma_j$ is zero we can solve the equation $\operatorname{div} w_j = -\operatorname{tr}\sigma_j$ for some $w_j \in W_0^{1,2}(\Omega;\mathbb{R}^d)$ which satisfies the a priori estimate

$$\|w_j\|_{1,2;\Omega} \le c_{12}\|\operatorname{tr}\sigma_j\|_{2;\Omega} \le c_{12}\|\operatorname{tr}\tau_j\|_{2;\Omega} = c_{12}.$$

Using the above identities, we calculate

$$\begin{aligned}\|\mathrm{tr}\,\sigma_j\|^2_{2;\Omega} &= -\int_\Omega \mathrm{tr}\,\sigma_j \mathrm{div}\, w_j dx = -\int_\Omega \sigma_j : \nabla w_j dx + \int_\Omega \sigma_j^{\mathrm{D}} : \nabla w_j dx \\ &\le (\|\mathrm{div}\,\sigma_j\|_{2;\Omega} + \|\sigma_j^{\mathrm{D}}\|_{2;\Omega})\|w_j\|_{1,2;\Omega} \to 0\end{aligned}$$

as $j \to \infty$. This contradicts (4.1) and proves the lemma. □

Moreover, we will use the following estimate.

Lemma 4.2. *Assume that* $\Phi \in W^{2,2}(\Omega)$ *satisfies* $\mathrm{Curl}\,\Phi = 0$ *on* Γ_{N} *if* $\Gamma_{\mathrm{N}} \neq \emptyset$ *or* $\int_\Omega \mathrm{tr}\,\mathrm{Curl}\mathrm{Curl}\Phi = 0$ *if* $\Gamma_{\mathrm{N}} = \emptyset$*. Then there exists a constant* c_{12} *which depends only on* Ω *and* Γ_{N} *such that*

$$\|\Delta\Phi\|_{2;\Omega} \le c_{12}\|(\mathrm{Curl}\,\mathrm{Curl}\,\Phi)^{\mathrm{D}}\|_{2;\Omega}.$$

Furthermore,

$$\|\mathrm{Curl}\,\mathrm{Curl}\,\Phi\|^2_{2;\Omega} \le c_{13}\|\mathrm{Curl}\,\mathrm{Curl}\,\Phi\|^2_{\mathbb{C}^{-1};\Omega},$$

where the constant c_{13} *depends only on* Ω*,* Γ_{N} *and* μ*.*

Proof. Assume first that $\Gamma_N \neq \emptyset$. Let Γ_0 be a maximal line segment contained in $\Gamma_{\mathrm{N}} \neq \emptyset$, and define

$$\Sigma_0 = \Big\{\sigma \in H(\mathrm{div};\Omega) : \int_{\Gamma_0} \sigma n ds = 0\Big\}.$$

Clearly Σ_0 is a weakly closed subspace of $H(\mathrm{div};\Omega)$ and $\mathrm{Id} \notin \Sigma_0$. From $\mathrm{div}\,\mathrm{Curl}\,\mathrm{Curl}\,\Phi = 0$ and

$$\int_{\Gamma_0} \mathrm{Curl}\,\mathrm{Curl}\,\Phi n ds = 0$$

we have $\mathrm{Curl}\,\mathrm{Curl}\,\Phi \in \Sigma_0$. If $\Gamma_N = 0$, let $\Sigma_0 := \{\sigma \in H(\mathrm{div};\Omega)$: $\int_\Omega \mathrm{tr}\sigma\, dx = 0\}$ and, by assumption, $\mathrm{Curl}\,\mathrm{Curl}\,\Phi \in \Sigma_0$. Hence, the first inequality follows from Lemma 4.1. From this,

$$\begin{aligned}&\int_\Omega |\mathrm{Curl}\,\mathrm{Curl}\,\Phi|^2 dx \\ &\quad= \frac{1}{2}\int_\Omega |\Delta\Phi|^2 dx + \int_\Omega |(\mathrm{Curl}\,\mathrm{Curl}\,\Phi)^{\mathrm{D}}|^2 dx \\ &\quad\le 2\mu(\frac{c_{12}^2}{2} + 1)\int_\Omega \mathbb{C}^{-1}\mathrm{Curl}\,\mathrm{Curl}\,\Phi : \mathrm{Curl}\,\mathrm{Curl}\,\Phi dx.\end{aligned}$$ □

5. Proof of the upper bound

We begin with the estimate for the error $\varepsilon := \sigma - \sigma_h$ in the stress variable. Lemma 3.2 implies the existence of $v \in X_1$ and $\Phi \in X_2$ such that

$$\varepsilon = \mathbb{C}\mathbb{E}(v) + \operatorname{Curl}\operatorname{Curl}\Phi + \phi, \tag{5.1}$$

where $\phi = \operatorname{Skw}(\sigma_h)$ is the skew-symmetric part of σ_h. Since

$$\begin{aligned}\int_\Omega \mathbb{E}(v) : \operatorname{Curl}\operatorname{Curl}\Phi dx &= \int_\Omega \nabla v : \operatorname{Curl}\operatorname{Curl}\Phi dx \\ &= \int_{\partial\Omega} \langle v, (\nabla \operatorname{Curl}\phi)t\rangle dx = 0,\end{aligned} \tag{5.2}$$

$\mathbb{E}(v)$ and $\operatorname{Curl}\operatorname{Curl}\Phi$ are orthogonal in $L^2(\Omega;\mathbb{M}^{2\times 2})$. Therefore, we obtain the decomposition

$$\|\varepsilon\|^2_{\mathbb{C}^{-1};2;\Omega} = \|\operatorname{Curl}\operatorname{Curl}\Phi\|^2_{\mathbb{C}^{-1};2;\Omega} + \|\mathbb{E}(v)\|^2_{\mathbb{C};2;\Omega} + \|\phi\|^2_{\mathbb{C}^{-1};2;\Omega}, \tag{5.3}$$

where we used for $A = \mathbb{C}$ and $A = \mathbb{C}^{-1}$ the notation

$$\|\tau\|^2_{A;2;\Omega} = \int_\Omega A\tau : \tau dx.$$

For $\mathbb{C}$ as in (1.3) we have with $c_{14} = 1/\sqrt{2\mu}$ and $c_{15} = \max\{1/\sqrt{2\mu}, d/\sqrt{d\lambda + 2\mu}\}$

$$\|\tau\|_{2;\Omega} \le c_{14}\|\tau\|_{\mathbb{C};2;\Omega}, \qquad \|\tau\|_{\mathbb{C}^{-1};2;\Omega} \le c_{15}\|\tau\|_{2;\Omega}.$$

In particular these constants are independent of λ for $\lambda \to \infty$. In the next lemmas we estimate the three terms on the right hand side of (5.3). All constants are independent of λ and h and depend only on μ, Ω and the shape of the triangles.

Lemma 5.1. *There exists a constant c_{16} such that we have*

$$\begin{aligned}\|\operatorname{Curl}\operatorname{Curl}\Phi\|_{\mathbb{C}^{-1};2;\Omega} \le c_{16}\Big\{&\|h_{\mathcal{T}} \operatorname{curl}_h(\mathbb{C}^{-1}\sigma_h + \gamma_h)\|^2_{2;\Omega} \\ &+ \|h_{\mathcal{E}}^{1/2} J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\|^2_{2;\mathcal{E}_h^0}\Big\}^{1/2}.\end{aligned}$$

Proof: We deduce from (5.1), (5.2) and $\mathbb{C}^{-1}\sigma = \mathbb{E}(u)$

$$\begin{aligned}\|\operatorname{Curl}\operatorname{Curl}\Phi\|^2_{\mathbb{C}^{-1};2;\Omega} &= \int_\Omega \operatorname{Curl}\operatorname{Curl}\Phi : \mathbb{C}^{-1}\varepsilon dx \\ &= -\int_\Omega \operatorname{Curl}\operatorname{Curl}\Phi : (\mathbb{C}^{-1}\sigma_h + \gamma_h)dx.\end{aligned}$$

Let $b = \operatorname{Curl} \Phi \in W^{1,2}(\Omega; \mathbb{R}^2)$ and define $b_h := R_h b \in \mathcal{L}_1^1$. Since $\Phi \in X_2$ we deduce $b = \operatorname{Curl} \Phi = c_i$ on Γ_i and therefore $(\operatorname{Curl} b)n = 0$ on Γ_{N}. In view of (2.5), $\operatorname{Curl} b_h \in \mathrm{RT}_0$ is an admissible test tensor in the discrete equation (1.8) and we obtain

$$\int_\Omega \operatorname{Curl} b_h : (\mathbb{C}^{-1}\sigma_h + \gamma_h)dx = 0. \tag{5.4}$$

Therefore, by an integration by parts on each triangle,

$$\begin{aligned}
\|\operatorname{Curl}\operatorname{Curl}\Phi\|^2_{\mathbb{C}^{-1};2;\Omega} &= -\int_\Omega \operatorname{Curl}(\operatorname{Curl}\Phi - b_h) : (\mathbb{C}^{-1}\sigma_h + \gamma_h)dx \\
&= \int_\Omega \langle \operatorname{Curl}\Phi - b_h, \operatorname{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)\rangle dx \\
&\quad + \int_{\mathcal{E}_h} \langle \operatorname{Curl}\Phi - b_h, J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\rangle ds \\
&\le \sum_{T\in\mathcal{T}_h} \|\operatorname{Curl}\Phi - b_h\|_{2;T} \|\operatorname{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)\|_{2;T} \\
&\quad + \sum_{E\in\mathcal{E}_h^0} \|\operatorname{Curl}\Phi - b_h\|_{2;E} \|J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\|_{2;E} \\
&\le c_8\sqrt{c_{10}} \|\operatorname{Curl}\Phi\|_{1,2;\Omega} \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|\operatorname{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)\|^2_{2;T} \Big)^{1/2} \\
&\quad + \sqrt{2}c_9 \|\operatorname{Curl}\Phi\|_{1,2;\Omega} \Big(\sum_{E\in\mathcal{E}_h^0} h_E \|J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\|^2_{2;E} \Big)^{1/2}.
\end{aligned}$$

If $\Gamma_N \neq \emptyset$ we conclude with Lemma 4.2 since $\operatorname{Curl}\,\Phi = 0$ on Γ_N. Otherwise we deduce from (1.8) with $\tau_h := \mathrm{Id} \in \sum_{0,h}$ and $\gamma_h = 0$ that $\int_\Omega \operatorname{tr} \sigma_h dx = 0$. Thus we obtain from (5.1)

$$\begin{aligned}
&\textstyle\int_\Omega \operatorname{tr} \operatorname{Curl}\operatorname{Curl}\Phi \, dx = \int_\Omega \operatorname{tr}(\varepsilon - \mathbb{C}\mathbb{E}(v))dx = \int_\Omega \operatorname{tr} \mathbb{C}\mathbb{E}(u - v)dx \\
&\textstyle\quad = (2\lambda + 2\mu) \int_\Omega \operatorname{div}(u - v)dx = (2\lambda + 2\mu) \int_{\partial\Omega} n(u - v)ds = 0
\end{aligned}$$

since $u, v \in W_0^{1,2}(\Omega; \mathbb{R}^2)$. In view of Poincaré's inequality and Lemma 4.2 we obtain

$$\|\operatorname{Curl}\Phi\|_{1,2;\Omega} \le c_{17} \|\nabla \operatorname{Curl}\Phi\|_{2;\Omega} \le c_{17}c_{13} \|\operatorname{Curl}\operatorname{Curl}\Phi\|_{\mathbb{C}^{-1};\Omega}.$$

The assertion of the lemma follows with $c_{16} = \sqrt{2}c_{17}c_{13}\max\{c_8\sqrt{c_{10}}, \sqrt{2}c_9\}$. □

Lemma 5.2. *There exists a constant* c_{18} *such that we have*

$$\begin{aligned}
\|\mathbb{E}(v)\|^2_{\mathbb{C};2;\Omega} + \|\phi\|^2_{\mathbb{C}^{-1};2;\Omega} \le c_{18}^2 \|h_{\mathcal{T}} \operatorname{div} \varepsilon\|^2_{2;\Omega} + \|\operatorname{Skw}(\sigma_h)\|^2_{2;\Omega} \\
+ \|h_{\mathcal{E}}^{1/2} \varepsilon n\|^2_{2;\Gamma_{\mathrm{N}}}.
\end{aligned}$$

Proof. It follows from (5.1) and (5.2) since $\varepsilon \in H(\mathrm{div};\Omega)$ and $\varepsilon+\mathrm{Skw}(\sigma_h)$ is symmetric by an integration by parts that

$$\begin{aligned}\|\mathbb{E}(v)\|^2_{\mathbb{C};2;\Omega} &= \int_\Omega \mathbb{E}(v) : (\varepsilon + \mathrm{Skw}(\sigma_h))dx\\ &= \int_\Omega \nabla v : \varepsilon dx + \int_\Omega \nabla v : \mathrm{Skw}(\sigma_h)dx\\ &= -\int_\Omega \langle v, \mathrm{div}\,\varepsilon\rangle dx + \int_{\partial\Omega}\langle v, \varepsilon n\rangle ds + \int_\Omega \nabla v : \mathrm{Skw}(\sigma_h)dx.\end{aligned}$$

The definition of the continuous and the discrete problem implies

$$\int_\Omega \langle \mathrm{div}\,\varepsilon, v_h\rangle dx = 0 \quad \forall\, v_h \in \mathcal{U}_h$$

and therefore, when c_{19} is the constant in Korn's inequality,

$$\begin{aligned}\int_\Omega \langle v, \mathrm{div}\,\varepsilon\rangle dx &= \int_\Omega \langle v - P_h^0 v, \mathrm{div}\,\varepsilon\rangle dx\\ &\le \|h_\mathcal{T}\mathrm{div}\,\varepsilon\|_{2;\Omega}\|h_\mathcal{T}^{-1}(v - P_h^0 v)\|_{2;\Omega}\\ &\le c_7|v|_{1,2;\Omega}\|h_\mathcal{T}\mathrm{div}\,\varepsilon\|_{2;\Omega}\\ &\le c_7 c_{19}\|\mathbb{E}(v)\|_{2;\Omega}\|h_\mathcal{T}\mathrm{div}\,\varepsilon\|_{2;\Omega}.\end{aligned}$$

We use the trace inequality $\|v\|_{2;E} \le c_{20} h_E^{1/2}(h_T^{-1}\|v\|_{2;T} + \|\nabla v\|_{2;T})$ to estimate the boundary integral. By definition of $\Sigma_{g,h}$

$$\begin{aligned}\int_{\Gamma_\mathrm{N}} \langle v, \varepsilon n\rangle ds = \int_{\Gamma_\mathrm{N}} \langle v - P_h^0 v, \varepsilon n\rangle ds &\le \sum_{E\in\mathcal{E}_{h,\mathrm{N}}} \|v - P_h^0 v\|_{E,2}\|\varepsilon n\|_{2;E}\\ &\le (c_7+1)c_{20}|v|_{1,2;\Omega}\|h_\mathcal{E}^{1/2}\varepsilon n\|_{2;\Gamma_\mathrm{N}}\end{aligned}$$

and the proof of the lemma follows with $c_{18} = \sqrt{3}c_{14}c_{19}(c_7+1)$. □

Throughout the rest of the section we use the notation $\rho_h := \mathbb{C}^{-1}\sigma_h + \gamma_h$, $\rho := \mathbb{C}^{-1}\sigma + \gamma = \nabla u$. Since Lemma 5.1 and Lemma 5.2 provide an estimate for $\|\sigma - \sigma_h\|_{\mathbb{C}^{-1};2;\Omega}$ it suffices to bound $\|\rho - \rho_h\|_{2;\Omega}$ in order to obtain an estimate for $\|\gamma - \gamma_h\|_{2;\Omega}$.

Lemma 5.3. *There exists a constant c_{21} such that we have*

$$\begin{aligned}||\rho - \rho_h||_{\mathbb{C};\Omega} \le c_{21}\Big(&||h_\mathcal{T}\mathrm{curl}_h\rho_h||^2_{2;\Omega} + ||h_\mathcal{T}\mathrm{div}\,\varepsilon||^2_{2;\Omega}\\ &+||\mathrm{Skw}(\sigma_h)||^2_{2;\Omega} + ||h_\mathcal{E}^{1/2}J(\rho_h t)||^2_{2;\mathcal{E}_h^0} + ||h_\mathcal{E}^{1/2}\,\varepsilon n||^2_{2;\mathcal{E}_\mathrm{N}}\Big)^{1/2}.\end{aligned} \tag{5.5}$$

Proof. In view of Lemma 3.1 there exist $f \in W^{1,2}(\Omega;\mathbb{R}^2)$, $f = c_i$ on Γ_i, $c_0 = 0$ and $q \in W^{1,2}(\Omega;\mathbb{R}^2)$, $q = 0$ on Γ_{D} such that

$$\rho - \rho_h = \mathbb{C}^{-1}\mathrm{Curl}\, f + \nabla q, \tag{5.6}$$

$$\begin{aligned}\int_\Omega \mathbb{C}(\rho-\rho_h):(\rho-\rho_h)dx &= \int_\Omega \mathbb{C}^{-1}\mathrm{Curl}\, f : \mathrm{Curl}\, f dx \\ &\quad + \int_\Omega \mathbb{C}\nabla q : \nabla q dx.\end{aligned}$$

The first term on the right hand side can be estimated by

$$\begin{aligned}\int_\Omega \mathbb{C}^{-1}\mathrm{Curl}\, f : \mathrm{Curl}\, f dx &= \int_\Omega (\rho-\rho_h) : \mathrm{Curl}\, f dx \\ &= -\int_\Omega \rho_h \mathrm{Curl}\, f dx.\end{aligned}$$

Let $R_h f \in \mathcal{L}_1^1$ be the Clément interpolation of f. Since $\mathrm{Curl}\, R_h f$ is an admissible test tensor we deduce in view of (5.4) with an integration by parts

$$\begin{aligned}-\int_\Omega \rho_h : \mathrm{Curl}\, f dx &= -\int_\Omega \rho_h : \mathrm{Curl}(f - R_h f)dx \\ &= \int_\Omega \langle \mathrm{curl}\,\rho_h, f - R_h f\rangle dx - \int_{\mathcal{E}_h} \langle J(\rho_h t), f - R_h f\rangle ds \\ &\le \sum_{T\in\mathcal{T}_h} \|\mathrm{curl}\,\rho_h\|_{2;T}\|f - R_h f\|_{2;T} + \sum_{E\in\mathcal{E}_h^0} \|J(\rho_h t)\|_{2;E}\|f - R_h f\|_{2;E} \\ &\le c_8\sqrt{c_{10}}\|f\|_{1,2;\Omega}\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathrm{curl}\,\rho_h\|_{2;T}^2\Big)^{1/2} \\ &\quad + \sqrt{2}c_9\|f\|_{1,2;\Omega}\Big(\sum_{E\in\mathcal{E}_h^0} h_E\|J(\rho_h t)\|_{2;E}^2\Big)^{1/2} \\ &\le c_{22}\|\nabla f\|_{2;\Omega}\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathrm{curl}\,\rho_h\|_{2;T}^2 + \sum_{E\in\mathcal{E}_h^0} h_E\|J(\rho_h t)\|_{2;E}^2\Big)^{1/2}\end{aligned}$$

with $c_{22} = \sqrt{2}c_{17}\max\{c_8\sqrt{c_{10}}, \sqrt{2}c_9\}$. Since $\|\nabla f\|_{2;\Omega} = \|\mathrm{Curl}\, f\|_{2;\Omega}$ we deduce

$$\begin{aligned}&\|\mathrm{Curl}\, f\|_{\mathbb{C}^{-1},2;\Omega} \\ &\quad \le c_{22}\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathrm{curl}\,\rho_h\|_{2;T}^2 + \sum_{E\in\mathcal{E}_h^0} h_E\|J(\rho_h t)\|_{2;E}^2\Big)^{1/2}.\end{aligned}$$

Taking the symmetric part in (5.1) and (5.6) we get

$$\begin{aligned}\mathrm{Sym}(\varepsilon) &= \mathbb{C}\mathbb{E}(v) + \mathrm{Curl}\,\mathrm{Curl}\,\Phi,\\ \mathrm{Sym}\mathbb{C}(\rho-\rho_h) &= \mathrm{Sym}(\varepsilon) = \mathbb{C}\mathbb{E}(q) + \mathrm{Sym}(\mathrm{Curl}\,f),\end{aligned}$$

hence

$$\mathbb{C}\mathbb{E}(v-q) = \mathrm{Sym}(\mathrm{Curl}\,f) - \mathrm{Curl}\,\mathrm{Curl}\,\Phi.$$

Thus we may estimate

$$\begin{aligned}&\|\mathbb{E}(v-q)\|^2_{\mathbb{C};2;\Omega}\\ &\quad= \int_\Omega \mathbb{C}\mathbb{E}(v-q) : \mathbb{E}(v-q)dx = \int_\Omega \mathrm{Sym}(\mathrm{Curl}\,f) : \mathbb{E}(v-q)dx\\ &\quad= \int_\Omega \mathrm{Curl}\,f : \mathbb{E}(v-q)dx \le \|\mathrm{Curl}\,f\|_{\mathbb{C}^{-1};\Omega}\|\mathbb{E}(v-q)\|_{\mathbb{C};2;\Omega},\end{aligned}$$

and hence $\|\mathbb{E}(v-q)\|_{\mathbb{C};\Omega} \le \|\mathrm{Curl}\,f\|_{\mathbb{C}^{-1};\Omega}$. By Korn's inequality we have

$$\begin{aligned}\|\nabla q\|_{\mathbb{C};\Omega} &= \int_\Omega \mathbb{C}\mathrm{Sym}(\nabla q) : \mathrm{Sym}(\nabla q)dx\\ &\qquad + \int_\Omega \mathbb{C}\mathrm{Skw}(\nabla q) : \mathrm{Skw}(\nabla q)dx\\ &\qquad\le \|\mathbb{E}(q)\|^2_{\mathbb{C};\Omega} + 2\mu\int_\Omega |\nabla q|^2 dx \le (1+2\mu c_{19}^2)\|\mathbb{E}(q)\|^2_{\mathbb{C};\Omega}\end{aligned}$$

and therefore we obtain by the triangle inequality, the estimates above, and Lemma 5.2

$$\begin{aligned}&\|\rho-\rho_h\|^2_{\mathbb{C};\Omega} = \|\nabla q\|^2_{\mathbb{C};\Omega} + \|\mathrm{Curl}\,f\|^2_{\mathbb{C}^{-1};\Omega}\\ &\quad\le (1+2\mu c_{19}^2)\|\mathbb{E}(q)\|^2_{\mathbb{C};\Omega} + \|\mathrm{Curl}\,f\|^2_{\mathbb{C}^{-1};\Omega}\\ &\quad\le 2(1+2\mu c_{19}^2)(\|\mathbb{E}(q-v)\|^2_{\mathbb{C};\Omega} + \|\mathbb{E}(v)\|^2_{\mathbb{C};\Omega}) + \|\mathrm{Curl}\,f\|^2_{\mathbb{C}^{-1};\Omega}\\ &\quad\le (2(1+2\mu c_{19}^2)+1)\|\mathrm{Curl}\,f\|^2_{\mathbb{C}^{-1};\Omega} + 2(1+2\mu c_{19}^2)\|\mathbb{E}(v)\|^2_{\mathbb{C};\Omega}\\ &\quad\le c_{22}^2(4\mu c_{19}^2+3)\Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\mathrm{curl}\,\rho_h\|^2_{2;T} + \sum_{E\in\mathcal{E}_h} h_E\|J(\rho_h t)\|^2_{2;E}\Big)\\ &\qquad + 2(2\mu c_{19}^2+1)c_{18}^2\\ &\qquad\qquad\cdot\Big(\|h_{\mathcal{T}}\mathrm{div}\,\varepsilon\|^2_{2;\Omega} + \|\mathrm{Skw}(\sigma_h)\|^2_{2;\Omega} + \|h_{\mathcal{E}}^{1/2}\varepsilon n\|^2_{2;\Gamma_{\mathrm{N}}}\Big).\end{aligned}$$

The assertion of the lemma follows with $c_{21} = \max\{c_{22}^2(4\mu c_{19}^2+3), 2(2\mu c_{19}^2+1)c_{18}^2\}^{1/2}$. □

The next step in the proof of Theorem 1.1 is an estimate for the displacement error $e = u - u_h$. The proof requires a duality argument and relies on the regularity assumption (1.4).

Lemma 5.4. *If the regularity assumption (1.4) holds, then there exists a constant c_{23} such that*

$$\|e\|_{2;\Omega} \le c_{23}\Big(\|h_{\mathcal{T}}\operatorname{div}\varepsilon\|_{2;\Omega}^2 + \|h_{\mathcal{T}}\operatorname{Skw}(\sigma_h)\|_{2;\Omega}^2 + \inf_{v_h\in\mathcal{U}_h}\|h_{\mathcal{T}}(\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla v_h)\|_{2;\Omega}^2 + \|h_{\mathcal{E}}^{1/2}\varepsilon n\|_{2;\Gamma_{\mathrm{N}}}^2\Big)^{1/2}.$$

Proof. Let $z\in W^{2,2}(\Omega)$ be the solution of the problem

$$\operatorname{div}\mathbb{C}\mathbb{E}(z) = e \quad \text{in } \Omega, \qquad z=0 \text{ on } \Gamma_{\mathrm{D}}, \quad \text{and} \quad \mathbb{C}\mathbb{E}(z)n = 0 \text{ on } \Gamma_{\mathrm{N}},$$

and let $\tau := \mathbb{C}\mathbb{E}(z)$. By assumption (1.4), $\|z\|_{2,2;\Omega}+\|\tau\|_{1,2;\Omega}\le c_1\|e\|_{2;\Omega}$. Consequently, by (2.1), (1.8), (2.5) and an integration by parts

$$\begin{aligned}\|e\|_{2;\Omega}^2 &= \int_\Omega \langle u-u_h, \operatorname{div}\tau\rangle dx = -\int_\Omega \nabla u : \tau dx - \int_\Omega \langle u_h, \operatorname{div}\Pi_h\tau\rangle dx\\ &= \int_\Omega (\nabla v_h - \nabla u) : \tau dx + \int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla v_h) : \Pi_h\tau dx\\ &\quad + \int_\Omega \nabla v_h : (\Pi_h\tau-\tau)dx.\end{aligned}$$

The last term on the right hand side vanishes according to (2.3). By the definition of τ and (2.2) we deduce

$$\begin{aligned}\|e\|_{2;\Omega}^2 &= \int_\Omega (\nabla v_h - \nabla u) : \tau dx\\ &\quad + \int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla v_h) : (\Pi_h\tau-\tau)dx\\ &\quad + \int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla v_h) : \tau dx\\ &= \int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla v_h) : (\Pi_h\tau-\tau)dx\\ &\quad + \int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla u) : \mathbb{C}\mathbb{E}(z)dx\\ &\le c_6\|h_{\mathcal{T}}(\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla v_h)\|_{2;\Omega}|\tau|_{1,2;\Omega}\\ &\quad + \int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\nabla u) : \mathbb{C}\mathbb{E}(z)dx.\end{aligned}$$

The second term on the right hand side can be rewritten as

$$\int_\Omega (\mathbb{C}^{-1}\sigma_h+\gamma_h-\mathbb{E}(u)) : \mathbb{C}\mathbb{E}(z)dx = -\int_\Omega (\sigma-\sigma_h) : \mathbb{E}(z)dx.$$

Writing $\mathbb{E}(z) = \nabla z - \mathrm{Skw}(\nabla z)$ we obtain by an integration by parts

$$\begin{aligned}\int_\Omega (\mathbb{C}^{-1}\sigma_h + \gamma_h - \mathbb{E}(u)) : \mathbb{C}\mathbb{E}(z)dx \\ = \int_\Omega \langle \mathrm{div}(\sigma - \sigma_h), z\rangle dx - \int_{\Gamma_\mathrm{N}} \langle (\sigma - \sigma_h)n, z\rangle ds \\ + \int_\Omega (\mathrm{Skw}(\sigma_h) : \mathrm{Skw}(\nabla z))dx.\end{aligned}$$

The orthogonal projection $P_h^0 z$ of z onto $\mathcal{L}_0^0$ is well defined and we deduce

$$\begin{aligned}\int_\Omega \langle \mathrm{div}\,\varepsilon, z\rangle dx = \int_\Omega \langle \mathrm{div}\,\varepsilon, z - P_h^0 z\rangle dx \\ \le \|h_\mathcal{T} \mathrm{div}\,\varepsilon\|_{2;\Omega} \|\frac{1}{h_\mathcal{T}}(z - P_h^0 z)\|_{2;\Omega} \\ \le c_7 \|h_\mathcal{T} \mathrm{div}\,\varepsilon\|_{2;\Omega} |z|_{1,2;\Omega} \le c_7 c_1 \|h_\mathcal{T} \mathrm{div}\,\varepsilon\|_{2;\Omega} \|e\|_{2;\Omega}.\end{aligned}$$

The boundary term can be estimated as in Lemma 5.2 and we obtain

$$\int_{\Gamma_h} \langle \varepsilon n, z\rangle dx \le c_1 c_{20}(c_7 + 1)\|e\|_{2;\Omega} \|h_\mathcal{E}^{1/2} \varepsilon n\|_{2;\Gamma_\mathrm{N}}.$$

In order to bound the last term we define $\xi_h = R_h \mathrm{Skw}(\nabla z) \in \mathcal{W}_h$ and infer with (1.9)

$$\begin{aligned}\int_\Omega \mathrm{Skw}(\sigma_h) : \mathrm{Skw}(\nabla z)dx = \int_\Omega \mathrm{Skw}(\sigma_h) : (\mathrm{Skw}(\nabla z) - \xi_h)dx \\ \le \|h_\mathcal{T} \mathrm{Skw}(\sigma_h)\|_{2;\Omega} \|\frac{1}{h_\mathcal{T}}(\mathrm{Skw}(\nabla z) - \xi_h)\|_{2;\Omega} \\ \le c_8 c_2 \|h_\mathcal{T} \mathrm{Skw}(\sigma_h)\|_{2;\Omega} \|e\|_{2;\Omega}.\end{aligned}$$

The estimates above imply

$$\begin{aligned}\|e\|_{2;\Omega} \le c_6 c_1 \inf_{v_h \in \mathcal{U}_h} \|\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla v_h\|_{2;\Omega} + c_7 c_1 \|h_\mathcal{T} \mathrm{div}\,\varepsilon\|_{2;\Omega} \\ + c_8 c_2 \|h_\mathcal{T} \mathrm{Skw}(\sigma_h)\|_{2;\Omega} + c_{20}(c_7 + 1)\|\varepsilon n\|_{2;\Gamma_\mathrm{N}}.\end{aligned}$$

This proves the lemma with $c_{23} = 2\max\{c_1 c_6, c_1 c_7, c_1 c_{20}, c_8 c_2\}$. □

Remark. For the higher order methods BDMS_k we have the improved estimate

$$\begin{aligned}\|e\|_{2;\Omega} \le c_{23}\Big(\|h_\mathcal{T}^2 \mathrm{div}\,\varepsilon\|_{2;\Omega} + \|h_\mathcal{T} \mathrm{Skw}(\sigma_h)\|_{2;\Omega} \\ + \inf_{v_h \in \mathcal{U}_h} \|h_\mathcal{T}(\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla v_h)\|_{2;\Omega} + \|h_\mathcal{E}^{1/2} \varepsilon n\|_{2;\Gamma_\mathrm{N}}\Big)\end{aligned}$$

since we may use the interpolation onto $\mathcal{L}_1^1$, instead of the orthogonal projection onto $\mathcal{L}_0^0$.

Proof of Theorem 1.1: Recall from (5.3) that

$$\|\varepsilon\|^2_{\mathbb{C}^{-1};2;\Omega} = \|\operatorname{Curl}\operatorname{Curl}\Phi\|^2_{\mathbb{C}^{-1};2;\Omega} + \|\mathbb{E}(v)\|^2_{\mathbb{C};2;\Omega} + \|\phi\|^2_{\mathbb{C}^{-1};2;\Omega}.$$

In view of Lemma 5.1 and 5.2 we obtain

$$\|\varepsilon\|^2_{\mathbb{C}^{-1};2;\Omega} \le 2\max\{c_{16}^2, c_{18}^2\}\eta^2 =: c_{24}^2\eta^2.$$

Moreover, by the triangle inequality and Lemma 5.3

$$\begin{aligned}\|\gamma - \gamma_h\|^2_{2;\Omega} = \frac{1}{2\mu}\|\gamma - \gamma_h\|^2_{\mathbb{C};\Omega} &\le \frac{1}{\mu}(\|\rho - \rho_h\|^2_{\mathbb{C};\Omega} + \|\sigma - \sigma_h\|^2_{\mathbb{C}^{-1};\Omega}) \\ &\le 2(c_{21}^2 + c_{24}^2)\eta^2 =: c_{25}^2\eta^2.\end{aligned}$$

The theorem follows with Lemma 5.4 and for $c_3 = c_{23} + c_{24} + c_{25}$. □

6. Proof of the lower bound

The lower bounds in Theorem 1.2 rely on two main ingredients: a localization technique introduced in [V] and classical inverse estimates in finite element spaces. We briefly summarize the relevant results (see [V] for more details). There exists an extension operator $L : C^0(E) \to C^0(T)$, $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$, which extends polynomials of degree k on E to polynomials of same degree on T and satisfies $(Lp)_{|E} = p_{|E}$ for all $p \in \mathcal{P}_k(E)$. Finally we let $\psi_T = (\max_T b_T)^{-1} b_T$ and we denote by ψ_E the uniquely determined piecewise quadratic function on ω_E which satisfies $\operatorname{supp}\psi_E \subset \omega_E$, $\psi_E \ge 0$ and $\max_E \psi_E = 1$.

Lemma 6.1. *([V],Lemma 4.1) Let $k \in \mathbb{N}$. Then there exist constants c_{26}, $\dots, c_{28}$, which depend only on k and the shape of the triangles such that we have for all $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ with $E \subset \bar{T}$ and all $u \in \mathcal{P}_k(T)$, $v \in \mathcal{P}_k(E)$*

(6.1) $$\|\psi_T u\|_{2;T} \le \|u\|_{2;T} \le c_{26}\|\psi_T^{1/2} u\|_{2;T},$$

(6.2) $$\|\psi_E v\|_{2;T} \le \|v\|_{2;E} \le c_{27}\|\psi_E^{1/2} v\|_{2;E},$$

(6.3) $$c_{26}^{-1} h_E^{1/2}\|v\|_{2;E} \le \|\psi_E^{1/2} Lv\|_{2;T} \le c_{28} h_E^{1/2}\|v\|_{2;E}.$$

Lemma 6.2. *([Ci1], Lemma 3.2.6) Assume that $v \in \mathcal{P}_k(T)$ and $0 \le \ell \le m$. Then there exists a constant c_{29}, which depends only on the shape of the triangles, k, ℓ and m such that*

(6.4) $$|v|_{m,2;T} \le c_{29} h_T^{\ell - m}|v|_{2;\ell;T}.$$

In Lemma 6.3 and 6.4 we give bounds on the different contributions in the error estimator η in (1.10). Recall that $\rho_h = \mathbb{C}^{-1}\sigma_h + \gamma_h$, $\rho = \mathbb{C}^{-1}\sigma + \gamma = \nabla u$.

Lemma 6.3. *There exists a constant c_{30} such that for all $T \in \mathcal{T}_h$*

$$h_T \|\mathrm{curl}(\mathbb{C}^{-1}\sigma_h + \gamma_h)\|_{2;T} \leq c_{30}\Big(\|\mathbb{C}^{-1}(\sigma - \sigma_h) + \gamma - \gamma_h\|_{2;T}\Big).$$

Proof. It follows from (6.1) and an integration by parts that

$$\begin{aligned} c_{26}^{-2}\|\mathrm{curl}\,\rho_h\|_{2;T}^2 \leq \|\psi_T^{1/2}\mathrm{curl}\,\rho_h\|_{2;T}^2 &= -\int_T \psi_T \langle \mathrm{curl}(\rho - \rho_h), \mathrm{curl}\,\rho_h\rangle dx \\ &= \int_T (\rho - \rho_h) : \mathrm{Curl}(\psi_T \mathrm{curl}\,\rho_h) dx \\ &\leq \|\rho - \rho_h\|_{2;T} \|\mathrm{Curl}(\psi_T \mathrm{curl}\,\rho_h)\|_{2;T}. \end{aligned}$$

From (6.4) and (6.1) we infer

$$\|\mathrm{Curl}(\psi_T \mathrm{curl}\,\rho_h)\|_{2;T} \leq c_{29} h_T^{-1} \|\psi_T \mathrm{curl}\,\rho_h\|_{2;T} \leq c_{29} h_T^{-1} \|\mathrm{curl}\,\rho_h\|_{2;T}.$$

This proves the lemma with $c_{30} = c_{26}^2 c_{29}$. □

Lemma 6.4. *There exists a constant c_{31} such that the following estimate holds for all $E \in \mathcal{E}_h^0$*

$$h_E^{1/2} \|J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\|_{2;E} \leq c_{31} \|\mathbb{C}^{-1}(\sigma - \sigma_h) + \gamma - \gamma_h\|_{2;\omega_E}.$$

Proof. Let $v_h = J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)$. We obtain from (6.2)

$$\|J((\mathbb{C}^{-1}\sigma_h + \gamma_h)t)\|_{2;E}^2 \leq c_{27}^2 \|\psi_E^{1/2} L v_h\|_{2;E}^2 = c_{27}^2 \int_E \psi_E |L v_h|^2 ds.$$

An integration by parts in each triangle of ω_E yields

$$\begin{aligned} &\int_{\omega_E} \langle \mathrm{curl}\,\rho_h, \psi_E L v_h\rangle dx + \int_{\omega_E} \rho_h : \mathrm{Curl}(\psi_E L v_h) dx \\ &\quad = \int_E \langle J(\rho_h t), \psi_E L v_h\rangle ds, \end{aligned}$$

and so

$$0 = \int_{\omega_E} (\langle \mathrm{curl}\,\rho, \psi_E L v_h\rangle + \rho : \mathrm{Curl}(\psi_E L v_h)) dx.$$

Therefore we obtain

$$\begin{aligned}
&\|\psi_E^{1/2} v_h\|_{2;E}^2 \\
&\quad = -\int_{\omega_E} \langle \operatorname{curl}(\rho - \rho_h), \psi_E L v_h \rangle dx - \int_{\omega_E} (\rho - \rho_h) : \operatorname{Curl}(\psi_E L v_h) dx \\
&\quad = \int_{\omega_E} \langle \operatorname{curl} \rho_h, \psi_E L v_h \rangle dx - \int_{\omega_E} (\rho - \rho_h) : \operatorname{Curl}(\psi_E L v_h) dx \\
&\quad \le \|\operatorname{curl} \rho_h\|_{2;\omega_E} \|\psi_E L v_h\|_{2;\omega_E} + \|\rho - \rho_h\|_{2;\omega_E} \|\operatorname{Curl}(\psi_E L v_h)\|_{2;\omega_E}.
\end{aligned}$$

Let c_{32} be a constant such that $h_E^{1/2}/h_T \le c_{32} h_E^{-1/2}$ for all $T \in \mathcal{T}_h$ with $E \subset \bar{T}$. Clearly, c_{32} depends only on the shape of the triangles in $\mathcal{T}_h$. We conclude with Lemma 6.2 and 6.3

$$\begin{aligned}
h_E^{1/2} \|v_h\|_{2;E} &\le c_{28} h_E \|\operatorname{curl} \rho_h\|_{2;\omega_E} + c_{29} c_{28} c_{32} \|\rho - \rho_h\|_{2;\omega_E} \\
&\le c_{28} c_{32} (c_{30} + c_{29}) \|\mathbb{C}^{-1}(\sigma - \sigma_h) + \gamma - \gamma_h\|_{2;\omega_E}.
\end{aligned}$$

This implies the result with $c_{31} = c_{27} c_{32} (c_{30} + c_{29})$. □

Lemma 6.5. *There exists a constant c_{33} such that the following estimate holds for all $E \in \mathcal{E}_{h,\mathrm{N}}$*

$$h_E^{1/2} \|(\sigma - \sigma_h) n\|_{2;E} \le c_{33} (\|h_{\mathcal{T}} \operatorname{div}(\sigma - \sigma_h)\|_{2;\omega_E} + \|\sigma - \sigma_h\|_{2;\omega_E}).$$

Proof. Let $v_h = (\sigma - \sigma_h) n$. Then

$$\begin{aligned}
&\int_T \langle \operatorname{div}(\sigma - \sigma_h), \psi_E L v_h \rangle dx \\
&\qquad = -\int_T (\sigma - \sigma_h) : \nabla(\psi_E L v_h) dx + \int_E |(\sigma - \sigma_h) n|^2 \psi_E ds
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_E |(\sigma - \sigma_h) n|^2 \psi_E ds \\
&\qquad \le \|\operatorname{div}(\sigma - \sigma_h)\|_{2;T} \|\psi_E L v_h\|_{2;T} + \|\sigma - \sigma_h\|_{2;T} \|\nabla(\psi_E L v_h)\|_{2;T}.
\end{aligned}$$

Hence we obtain from (6.2) and Lemma 6.2 that $\|(\sigma - \sigma_h) n\|_{2;E}^2$ is bounded from above by

$$c_{26}^2 \{ c_{28} \|\operatorname{div}(\sigma - \sigma_h)\|_{2;T} + c_{29} c_{28} h_T^{-1} \|\sigma - \sigma_h\|_{2;T} \} h_E^{1/2} \|v_h\|_{2;E}$$

and we conclude

$$h_E^{1/2} \|(\sigma - \sigma_h) n\|_{2;E} \le c_{33} (h_T \|\operatorname{div}(\sigma - \sigma_h)\|_{2;T} + \|\sigma - \sigma_h\|_{2;T}),$$

where $c_{33} = c_{26}^2 c_{28} c_{32} \max\{c_{29}, 1\}$. This implies the assertion of the lemma. □

Lemma 6.6. *There exists a constant* c_{34} *such that we have*

$$h_T\|\mathbb{C}^{-1}\sigma_h + \gamma_h - \nabla u_h\|_{2;T} \le c_{34}\Big(\|u-u_h\|_{2;T} + h_T\|\rho-\rho_h\|_{2;T}\Big).$$

Proof. It follows from (6.1) and an integration by parts that

$$\begin{aligned} c_{26}^{-2}\|\rho_h - \nabla u_h\|_{2;T}^2 &\le \int_T \psi_T(\rho_h - \nabla u_h) : (\rho_h - \nabla u_h)dx \\ &= -\int_T \psi_T(\rho-\rho_h) : (\rho_h - \nabla u_h)dx \\ &\quad + \int_T \psi_T(\rho - \nabla u_h) : (\rho_h - \nabla u_h)dx \\ &\le \Big(\|\rho-\rho_h\|_{2;T}\|\rho_h - \nabla u_h\|_{2;T} \\ &\quad + \|u-u_h\|_{2;T}\|\mathrm{div}(\psi_T(\rho_h - \nabla u_h))\|_{2;T}\Big) \\ &\le \Big(\|\rho-\rho_h\|_{2;T} + c_{29}h_T^{-1}\|u-u_h\|_{2;T}\Big)\|\rho_h - \nabla u_h\|_{2;T}. \end{aligned}$$

This proves the lemma $c_{34} = c_{26}^2 \max\{1, c_{29}\}$. □

Proof of Theorem 1.2: The proof is an immediate consequence of Lemmas 6.3 - 6.6. □

Acknowledgement. Most of the work was done while GD visited the Mathematisches Seminar at the Christian–Albrechts–Universität zu Kiel, whose hospitality is gratefully acknowledged. He was also partially supported by ARO and NSF though grants to the Center for Nonlinear Analysis at Carnegie Mellon University, Pittsburgh.

References

[A] Alonso, A. (1996): Error estimators for the mixed method. Numer. Math. **74**, 385—395

[ABD] Arnold, D.N., Brezzi, F., Douglas, J. (1984): PEERS: A new finite element for plane elasticity. Jap. J. Appl. Math **1**, 347–367

[ADG] Arnold, D.N., Douglas Jr., J., Gupta, C.P. (1984): A family of higher order mixed finite element methods for plane elasticity. Num. Math. **45**, 1–22

[AF] Arnold, D.N., Falk, R.S. (1987): Well-posedness of the fundamental boundary value problems for constrained anisotropic elastic materials. Arch. Ration. Mech. Anal. **98**, 143–190

[BKNSW] Braess, D., Klaas, O., Niekamp, R., Stein, E., Wobschal, F. (1995): Error indicators for mixed finite elements in 2-dimensional linear elasticity. Comput. Methods Appl. Engrg. **127**, 345–356

[BDM] Brezzi, F., Douglas, J., Marini, L.D. (1986): Recent results on mixed finite element methods for second order elliptic problems. In Vistas in applied mathematics. Numerical analysis, atmospheric sciences and immunology. Springer

[BF] Brezzi, F., Fortin, M. (1991): Mixed and hybride finite element methods. Springer, New York

[Ca] Carstensen, C. (1997): A posteriori error estimate for the mixed finite element method. Math. Comp. **66**, 465–476

[Ci1] Ciarlet, P.G. (1978): The finite element method for elliptic problems. North-Holland, Amsterdam

[Ci2] Ciarlet, P.G. (1988): Mathematical elasticity, Volume 1: Three-dimensional elasticity, North-Holland, Amsterdam

[Cl] Clément, P. (1975): Approximation by finite element functions using local regularization. RAIRO Sér. Rouge Anal. Numér. **R-2**, 77–84

[FdV] Fraeijs de Veubeke, B.X. (1975): Stress function appoach. In World congress on the finite element method in structural mechanics, Bournemouth

[GR] Girault, V., Raviart, P.-A. (1986): Finite element methods for Navier-Stokes equations. Springer, Berlin-Heidelberg-New York

[RS] Rannacher, R., Suttmeier, F.T. (1997): A feed-back approach to error control in finite element methods: application to linear elasticity. Comp. Mech. **19**, 434–446

[St] Stenberg, R. (1988): A family of mixed finite elements for the elasticity problem. Num. Math. **53**, 513–538

[Va] Valent, T. (1988): Boundary Value Problems of Finite Elasticity. Springer

[V] Verfürth, R. (1996): A Review of A Posteriori Error Estimation and Adaptive Mesh–Refinement Techniques. Wiley Teubner

[Vo] Vogelius, M. (1983): An analysis of the p-version of the finite element method for nearly incompressible materials. Num. Math. **41**, 39–53