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A POSTERIORI ERROR CONTROL IN LOW-ORDER FINITE ELEMENT DISCRETISATIONS OF INCOMPRESSIBLE STATIONARY FLOW PROBLEMS

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Abstract. Computable a posteriori error bounds and related adaptive mesh-refining algorithms are provided for the numerical treatment of monotone stationary flow problems with a quite general class of conforming and nonconforming finite element methods. A refined residual-based error estimate generalises the works of Verfürth; Dari, Duran and Padra; Bao and Barrett. As a consequence, reliable and efficient averaging estimates can be established on unstructured grids. The symmetric formulation of the incompressible flow problem models certain nonNewtonian flow problems and the Stokes problem with mixed boundary conditions. A Helmholtz decomposition avoids any regularity or saturation assumption in the mathematical error analysis. Numerical experiments for the partly nonconforming method analysed by Kouhia and Stenberg indicate efficiency of related adaptive mesh-refining algorithms.

1. Introduction

Adaptive finite element methods play an important practical role in computational fluid dynamics. They are often justified by a posteriori error estimates which provide computable upper and lower error bounds which then serve as error indicators. In this paper, we unify and refine the derivation of such residual error estimates for possibly nonlinear flow problems, such as the Stokes problem [V2, V3, DDP] and certain monotone nonNewtonian flow problems [BB, P]. The refinement enables a justification of averaging techniques which are quite popular in engineering applications.

In the presentation emphasis is on a unifying proof for conforming, nonconforming, and even a conforming-nonconforming scheme [KS]. Because of possible Neumann boundary conditions, we study the symmetric formulation which appears to be less frequently analysed in the mathematical literature. For notational simplicity we only give details for 2D regular triangulations but allow mixed inhomogeneous boundary data.

Given a Lipschitz continuous monotone mapping $\mathcal{A} : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$, Dirichlet data $u_D \in H^1(\Omega)^2$ and right-hand sides $f \in L^2(\Omega)^2$ and $g \in L^2(\Gamma_N)^2$ in a bounded

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Lipschitz domain $\Omega \subset \mathbb{R}^2$, find $u \in H^1(\Omega)^2$ and $p \in L^2(\Omega)$ which satisfy

$$\operatorname{div}\sigma + f = 0 \quad \text{and} \quad \operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\sigma = \mathcal{A}(\varepsilon(u)) - p\,I \quad \text{and} \quad \varepsilon(u) := (\nabla u + \nabla u^\top)/2 \qquad \text{in } \Omega, \tag{1.2}$$

$$u = u_D \qquad \text{on } \Gamma_D \quad \text{and} \quad \sigma \cdot n = g \qquad \text{on } \Gamma_N. \tag{1.3}$$

The stress-strain relation $\sigma = \mathcal{A}(\varepsilon(u)) - p\,I$ (where I denotes the 2×2-unit matrix) models Newtonian fluids for a linear function $\mathcal{A}(\alpha) = 2\mu\,\alpha$ with viscosity $\mu > 0$; $\varepsilon(u)$ is the linear Green strain rate for the velocity field u. Then (1.1)-(1.3) is the (stationary) Stokes problem in the symmetric form with mixed boundary conditions. Non-Newtonian flows, such as the Carreau law, are included as long as there are positive constants c_1 and c_2 such that the Lipschitz continuous function $\mathcal{A} : \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$ (where $\mathbb{R}^{2\times 2}_{sym}$ denotes the set of real symmetric 2×2 matrices) satisfies, for all $\alpha, \beta \in \mathbb{R}^{2\times 2}_{sym}$,

$$c_1\,|\alpha - \beta|^2 \le (\mathcal{A}(\alpha) - \mathcal{A}(\beta)) : (\alpha - \beta), \tag{1.4}$$

$$|\mathcal{A}(\alpha) - \mathcal{A}(\beta)| \le c_2\,|\alpha - \beta|. \tag{1.5}$$

(Colon denotes the scalar product in $\mathbb{R}^{2\times 2}_{sym}$, i.e., $\alpha : \beta = \sum_{j,k=1}^{2} \alpha_{jk}\,\beta_{jk}$.) The boundary $\Gamma := \partial\Omega$ of a bounded Lipschitz domain Ω in $\mathbb{R}^2$ is split into a closed Dirichlet boundary $\Gamma_D \subseteq \Gamma$ with positive surface measure and the remaining Neumann boundary $\Gamma_N := \Gamma \setminus \Gamma_D$. We mention that in the case $\Gamma_N = \emptyset$, the pressure p is defined only up to a constant and we require $\int_\Gamma u_D \cdot n\,ds = 0$.

The discrete problem is characterised by a (possibly nonconforming) discrete space $\mathcal{V} \times \mathcal{Q} \subset L^2(\Omega)^2 \times L^2(\Omega)$ with respect to an underlying regular triangulation $\mathcal{T}$ of the domain Ω. A discrete solution (u_h, p_h) in (a subspace of) $L^2(\Omega)^2 \times L^2(\Omega)$ is supposed to satisfy

$$\begin{aligned} &\int_\Omega \mathcal{A}(\varepsilon_\mathcal{T}(u_h)) : \varepsilon_\mathcal{T}(v_h)\,dx - \int_\Omega p_h \operatorname{div}_\mathcal{T} v_h\,dx \\ &\qquad = \int_\Omega f \cdot v_h\,dx + \int_{\Gamma_N} g \cdot v_h\,ds \quad (v_h \in \mathcal{V}), \end{aligned} \tag{1.6}$$

$$-\int_\Omega q_h \operatorname{div}_\mathcal{T} u_h\,dx = 0 \qquad (q_h \in \mathcal{Q}). \tag{1.7}$$

Since discrete functions may be discontinuous, a lower index $\mathcal{T}$ on differential operators (e.g., $\nabla_\mathcal{T} u_h$, $\operatorname{div}_\mathcal{T} v_h$, etc.) denotes their $\mathcal{T}$-elementwise action which may be different from their distributional meaning.

Remarks 1.1. (i) The continuity condition $\operatorname{div} u = 0$ is usually utilised in the Stokes problem to replace the term $\operatorname{div} 2\mu\,\varepsilon(u)$ in (1.1) by $\mu\,\Delta u$. In the resulting nonsymmetric formulation the natural Neumann boundary condition reads $\partial u/\partial n + p\,n = g$ and is correct from a variational point of view, but *not* from a physical perspective. Hence, if Neumann data arise in the problem, the symmetric formulation (1.1)-(1.3) is the reasonable mathematical model. However, the analysis presented below applies to the omitted nonsymmetric formulation as well.

(ii) Stability results and a priori error estimates for mixed and nonconforming finite element spaces $\mathcal{V} \times \mathcal{Q}$ can be found in [BF, BS, GR, KS]. It turns out that, in contrast to the nonsymmetric formulation, the nonconforming Crouzeix-Raviart elements are not uniformly stable for the Stokes problem [FM]. Instead, a nonconforming finite element method is stable where one component of the displacement is discretised with conforming linear elements and the other with nonconforming linear elements [KS], i.e., the trial space for the displacement field is $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ where

$$\mathcal{V}_1 := \{V \in C(\Omega) : V \text{ is affine on each } T \in \mathcal{T} \text{ and vanishes on } \Gamma_D\}, \tag{1.8}$$

$$\begin{aligned}\mathcal{V}_2 := \{V : V &\text{ is affine on each } T \in \mathcal{T}, \text{ continuous at midpoints of inner} \\ &\text{element boundaries, and vanishes at midpoints of edges } E \subset \Gamma_D\}\end{aligned} \tag{1.9}$$

(with usual modifications on Γ_D for inhomogeneous boundary conditions for the trial space) and $\mathcal{Q}$ are the $\mathcal{T}$-piecewise constants (with vanishing integral mean over Ω if $\Gamma_N = \emptyset$).

(iii) A posteriori error estimates and adaptive mesh-refining algorithms are included in [DDP, P, V1, V2, V3] for the nonsymmetric formulation without Neumann boundary data.

(iv) The unique existence of exact solutions (u, p) to (1.1)-(1.3) and discrete solutions (u_h, p_h) to (1.6)-(1.7) is discussed in the literature (see, e.g., [BB, BF, Ci, GR, KS, QV, T] and the references quoted therein). In this paper we therefore adopt the point of view that the continuous problem has a unique solution and there are (not necessarily unique) known functions (u_h, p_h) with certain $\mathcal{T}$-elementwise regularity properties given to us which satisfy the Galerkin conditions (1.6)-(1.7). There is no stability assumption on the discrete problem and indeed, in this way, unstable methods are analysed as well in their a posteriori error control (but this is not to recommend generally the application of unstable schemes).

(v) The class of finite element spaces under consideration in this paper is characterised by the fact that the integral mean of the jump $[u_h]$ vanishes (or is at least small) across interior edges. We stress that we do *not* need any a priori, saturation, or stability assumption on the discretisation or regularity of the exact solution.

(vi) The refined error estimate of this paper was (for the Stokes problem) announced in [CV]. The presented analysis results from a long term research, independent from [DDP], [P], and [BB], that started with mixed methods in [Ca1] and with the Stokes problem in [CJ].

In this paper, we establish a new residual-based efficient and (to some extent) reliable error estimate that applies to a general class of finite element discretisations. To describe the results in a simplified setting, suppose for (1.8)-(1.9), in this introduction that $f \in H^1(\Omega)$ and the (possibly discontinuous) discrete solution (u_h, p_h) is $\mathcal{T}$-piecewise smooth, satisfies $\operatorname{div}_{\mathcal{T}} u_h = 0$, $\int_{E\setminus\Gamma} [u_h]\, ds = 0$, and $\int_{E\cap\Gamma_D} (u_h - u_D)\, ds = 0$ for all $E \in \mathcal{E}$, where $\mathcal{E}$ denotes the set of all edges in $\mathcal{T}$. Suppose $\sigma_h := \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) - p_h I$ is a $\mathcal{T}$-piecewise polynomial of degree at most k.

For each $T \in \mathcal{T}$, define the element contribution to the residual-error bound $\eta_R^2 := \sum_{T\in\mathcal{T}} \eta_{R,T}^2$ by

$$\eta_{R,T}^2 := h_T^4 \|\nabla f\|_{L^2(T)}^2 + \sum_{E\in\mathcal{E}\wedge E\subset\partial T} h_E \big(\|[\sigma_h n_E]\|_{L^2(E)}^2 + \|[\partial u_h/\partial s]\|_{L^2(E)}^2\big). \tag{1.10}$$

Here h_T and h_E are diameters of an element $T \in \mathcal{T}$ and an edge $E \in \mathcal{E}$, respectively. The jump of a (possibly discontinuous) function G across the inner edge E is written $[G]$ with modifications according to boundary data, and $\partial/\partial s$ is the derivative along edges with respect to the arc-length (see Section 2 for details). All the contributions in (1.10) are computable residuals of (1.1)-(1.3) weighted with mesh-sizes.

Our first result shows that η_R is a reliable a posteriori error estimate in the sense that there exists an h-independent positive constant c_3 such that

$$\|\varepsilon_{\mathcal{T}}(u-u_h)\|^2_{L^2(\Omega)} + \|p-p_h\|^2_{L^2(\Omega)} \le c_3 \sum_{T\in\mathcal{T}} \eta^2_{R,T} = c_3\,\eta^2_R. \tag{1.11}$$

Secondly, the estimate (1.11) is efficient in the sense that the reverse inequality holds with an h-independent positive constant c_4

$$\eta_R \le c_4 \left(\|\varepsilon_{\mathcal{T}}(u-u_h)\|_{L^2(\Omega)} + \|p-p_h\|_{L^2(\Omega)}\right) + \text{ h.o.t.}, \tag{1.12}$$

up to higher order terms h.o.t. which are *known* and generically of higher order.

In the error estimator (1.11), the edge contributions dominate. This gives rise to a ZZ-type averaging estimator for the stress field as in [CB]. We prove for the conforming-nonconforming scheme (1.8)-(1.9) that

$$\|\varepsilon_{\mathcal{T}}(u-u_h)\|^2_{L^2(\Omega)} + \|p-p_h\|^2_{L^2(\Omega)} \le c_5 \|\,\sigma_h - \sigma^*_h\,\|_{L^2(\Omega)} + \text{ h.o.t.} \tag{1.13}$$

even in a more local form. Here, h.o.t. are known terms being generically of higher order and σ^*_h is a continuous (not necessarily symmetric) $\mathcal{T}$-piecewise affine approximation to the known $\mathcal{T}$-piecewise constant function σ_h which satisfies approximate Neumann boundary conditions. Taking the minimal choice defines an estimator (in practice an approximation will be computed)

$$\eta_Z := \min_{\tau_h} \|\,\sigma_h - \tau_h\,\|_{L^2(\Omega)}, \tag{1.14}$$

where τ_h is as σ^*_h above. Then, with higher order terms that depend on the smoothness of the exact solution, we have efficiency

$$\eta_Z \le \|\,\sigma - \sigma_h\,\|_{L^2(\Omega)} + \text{ h.o.t.} \tag{1.15}$$

with a constant 1 in front of the error on the right-hand side and *unknown* higher order terms. (The proof of (1.15) uses the triangle inequality and an approximation estimate of $\min_{\tau_h} \|\,\sigma - \tau_h\,\|_{L^2(\Omega)} =$ h.o.t.)

The remaining sections of the paper are organised as follows. The detailed notation as the precise statement of the reliability, namely inequality (1.11), is introduced in Section 2. The main argument in its proof in Section 3 is a Helmholtz decomposition which allows the application of Clément approximations [Cl]. In order to obtain a refined estimate we have to modify the approximation operators as in [Ca2, CV, CB]. The efficiency estimate (1.12) holds in a local form as shown in Section 4. The reliability of averaging techniques is established for unstructured grids in Section 5 where we indicate their efficiency. Numerical examples in Section 6 for the Stokes problem and the scheme (1.8)-(1.9) support our theoretical predictions and illustrate the superiority of the averaging technique in practise.

2. A RELIABLE AND EFFICIENT RESIDUAL-BASED A POSTERIORI ERROR ESTIMATE

In order to state the precise form of (1.11), we specify the hypotheses on the class of conforming and nonconforming finite elements under question.

Let $\mathcal{T}$ be a regular triangulation of $\Omega \subset \mathbb{R}^2$ in the sense of Ciarlet [Ci], i.e., $\mathcal{T}$ is a finite partition of Ω into closed triangles or parallelograms; two distinct elements T_1 and T_2 in $\mathcal{T}$ are either disjoint or $T_1 \cap T_2$ is a complete edge or a common node of both T_1 and T_2. With $\mathcal{T}$ let $\mathcal{E}$ denote the set of all edges, and we assume that $E \in \mathcal{E}$ either belongs to Γ_D or $E \cap \Gamma_D$ has vanishing surface measure, so there is no change of boundary conditions within one edge $E \subseteq \Gamma$.

Furthermore, let $P_k(T)$ and $Q_k(T)$ denote the set of the algebraic polynomials of total and partial degree $\leq k$, respectively, and define $\mathcal{P}_k(T) := P_k(T)$ if T is a triangle and $\mathcal{P}_k(T) := Q_k(T)$ if T is a parallelogram.

The discrete solution (u_h, p_h) satisfies (1.6)-(1.7) and is supposed to belong to $H^2(\mathcal{T})^2 \times H^1(\mathcal{T})$, where $H^k(\mathcal{T}) := H^k(\bigcup_{T \in \mathcal{T}} \operatorname{int} T)$. The test function space $\mathcal{V} \times \mathcal{Q}$ in (1.6)-(1.7) is supposed to satisfy

(2.1)
$$\mathcal{S} := \mathcal{S}_1(\mathcal{T})^2 \cap H^1_D(\Omega) \subseteq \mathcal{V} \subseteq H^2(\mathcal{T}) \quad \text{and} \quad \mathcal{L}_0(\mathcal{T}) \subseteq \mathcal{Q} \subset L^2(\Omega).$$

Here, the Lebesgue and Sobolev spaces $L^2(\Omega)$ and $H^1(\Omega)$ are defined as usual [Ho, LM] and

$$\begin{aligned}
\mathcal{L}_k(\mathcal{T}) &:= \{V \in L^\infty(\overline{\Omega}) : \forall T \in \mathcal{T}, V|_T \in \mathcal{P}_k(T)\},\\
\mathcal{S}_1(\mathcal{T}) &:= \mathcal{L}_1(\mathcal{T}) \cap C(\overline{\Omega}),\\
H^1_D(\Omega) &:= \{v \in H^1(\Omega)^2 : v|_{\Gamma_D} = 0\}.
\end{aligned}$$

Since the test and trial functions are possibly discontinuous, we define their jumps across the edges as follows. If $E \in \mathcal{E}$ is an inner edge, i.e., $E \not\subset \Gamma$, then $E = T_1 \cap T_2$ for two different $T_1, T_2 \in \mathcal{T}$, and $[\partial u_h/\partial s]$ denotes the difference of the traces of the tangential derivatives of u_h in T_1 and T_2. Similarly $[\sigma_h n_E]$ denotes the jump of the stress vectors, i.e., for $x \in E$ and a normal n_E on E,

$$[\sigma_h n_E](x) := \lim_{\delta \to 0^+} \big(\sigma_h(x + \delta n_E) - \sigma_h(x - \delta n_E)\big) n_E.$$

If $E \subset \Gamma_D$ belongs to the Dirichlet part of the boundary, then $[\partial u_h/\partial s] := \partial(u_D - u_h)/\partial s$ and $[\sigma_h n_E] := 0$. If $E \subset \overline{\Gamma}_N$ belongs to the Neumann part of the boundary, then $[\partial u_h/\partial s] := 0$ and $[\sigma_h n_E] := g - \sigma_h n$.

Let $\mathcal{N}$ denote the set of all nodes in $\mathcal{T}$, and denote the set of free nodes by $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$. Let $\mathcal{M}$ denote the set of all midpoints of edges in $\mathcal{T}$. Let $\varphi_z \in \mathcal{S}^1(\mathcal{T})$ denote a hat function for $z \in \mathcal{N}$ defined by $\varphi_z(x) = 0$ if $x \in \mathcal{N}$ and $x \neq z$ and $\varphi_z(z) = 1$. Let $\omega_z := \{x \in \Omega : \varphi_z(x) > 0\}$ denote the patch of $z \in \mathcal{N}$.

For a fixed node $y \in \mathcal{N} \setminus \mathcal{K}$ we choose a neighbouring free node $\zeta(y) \in \mathcal{K}$ and set $\zeta(y) := y$ if $y \in \mathcal{K}$ such that $I(y) = \{z' \in \mathcal{N} : y = \zeta(z')\}$ yields a partition $(I(z) : z \in \mathcal{K})$ of $\mathcal{N}$ and the connected and open enlarged patch $\Omega_z := \bigcup_{z \in I(z)} \omega_z$ with diameter $h_z := \operatorname{diam}(\Omega_z)$ for $z \in \mathcal{K}$.

Theorem 2.1 implies the estimate (1.11) as a particular case.

Theorem 2.1. *Let* $(u,p) \in H^1(\Omega)^2 \times L^2(\Omega)$ *solve* (1.1)-(1.3) *and let* $(u_h, p_h) \in H^2(\mathcal{T})^2 \times L^2(\Omega)$ *solve* (1.6)-(1.7). *Suppose that* Γ_D *is connected. Then there exist* h*-independent constants* $c_6, \ldots, c_9$ *that depend on the shape of the elements and the* *patches* $(\Omega_z : z \in \mathcal{K})$ *only such that*

$$\begin{aligned}\|\varepsilon_{\mathcal{T}}(u-u_h)\|^2_{L^2(\Omega)} &+ \|p-p_h\|^2_{L^2(\Omega)} \\ \le c_6 &\sum_{T\in\mathcal{T}} \|\operatorname{div} u_h\|^2_{L^2(T)} + c_7 \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}^2} \|\operatorname{div}\sigma_h + f - f_z\|^2_{L^2(\Omega_z)} \\ &+ c_8 \sum_{E\in\mathcal{E}} h_E(\|[\sigma_h n_E]\|^2_{L^2(E)} + \|[\partial u_h/\partial s]\|^2_{L^2(E)}) \\ &+ c_9 \inf_v \|\nabla_{\mathcal{T}}(u_h - v)\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.2}$$

The infimum in (2.2) *is taken over all* $v \in H^1(\mathcal{T})^2$ *such that* $\int_E [v]\,ds = 0$ *if* $E \in \mathcal{E}$ *and* $E \not\subseteq \Gamma_D$ *and* $\int_E v\,ds = \int_E u_D\,ds$ *if* $E \in \mathcal{E}$ *and* $E \subseteq \Gamma_D$.

Remarks 2.1. (i) We refer to [QV] for discussion and references on mixed boundary values in the Stokes problem. It seems not to be clear how a change of boundary conditions affects the regularity of the solution in the general case. As a consequence, any type of a priori estimate is avoided in Theorem 2.1 (a saturation assumption, for instance, is disputable to indicate efficiency, but not reliability).

(ii) In the case of pure Dirichlet boundary conditions, i.e., $\Gamma = \Gamma_D$ and $\int_\Gamma u_D \cdot n\,ds = 0$, we normalise $p, p_h \in L^2(\Omega)/\mathbb{R}$ by $\int_\Omega p\,dx = 0 = \int_\Omega p_h\,dx$.

(iii) Our analysis is partly based on the observation of a Galerkin orthogonality for continuous test functions, i.e.,

$$\int_\Omega (\sigma - \sigma_h) : \varepsilon(v_h)\,dx = 0 \qquad (v_h \in \mathcal{S}). \tag{2.3}$$

To prove (2.3), we use integration by parts to infer from (1.1)-(1.3) that for $v_h \in \mathcal{S}$ (with $\mathcal{S}$ as in (2.1)),

$$\begin{aligned}\int_\Omega \mathcal{A}(\varepsilon(u)) : \varepsilon(v_h)\,dx &- \int_\Omega p \operatorname{div} v_h\,dx \\ &= \int_\Omega \mathcal{A}(\varepsilon_{\mathcal{T}})(u_h) : \varepsilon(v_h)\,dx - \int_\Omega p_h \operatorname{div} v_h\,dx. \quad \square\end{aligned} \tag{2.4}$$

(iv) For triangles, the condition $\mathcal{S}_1(\mathcal{T})^2 \cap H^1_D(\Omega) \subseteq \mathcal{V}$ in (2.1) is satisfied for all standard conforming or nonconforming finite element spaces. On parallelograms, the nonconforming nodal basis functions do not include the conforming Q_1-finite elements; our a posteriori error estimates require nonconforming ansatz and trial spaces on parallelograms of higher order.

(v) The main argument in the proof of Theorem 2.1 is a Helmholtz decomposition which was first utilised in [A, Ca1, CD, DDP] in the context of a posteriori error estimates.

(vi) The refinement in Theorem 2.1 over [BB, CJ, DDP, P] concerns the second term on the right-hand side with the factor c_7. Since the open cover $(\Omega_z : z \in \mathcal{K})$ of Ω has finite overlap, we have the estimate

$$\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}^2} \|\operatorname{div}\sigma_h + f - f_z\|^2_{L^2(\Omega_z)} \le c_{10} \|\, h_{\mathcal{T}}(\operatorname{div}_{\mathcal{T}}\sigma_h + f)\,\|^2_{L^2(\Omega)}, \tag{2.5}$$

where $h_{\mathcal{T}}$ is the $\mathcal{T}$-piecewise constant defined on Ω by $h_{\mathcal{T}}|_T = h_T$ for $T \in \mathcal{T}$. If $\sigma_h := \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) - p_h I$ is $\mathcal{T}$-piecewise constant, $\operatorname{div}_{\mathcal{T}}\sigma_h = 0$, and by using a

Poincaré inequality we obtain

$$\sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}^2} \|f-f_z\|^2_{L^2(\Omega_z)} \leq c_{10}\, \| \, h_{\mathcal{T}}^2 \nabla f \, \|^2_{L^2(\Omega)}\,, \tag{2.6}$$

which leads to the volume contributions in (1.10).

(vii) The term $\inf_v \|\nabla_{\mathcal{T}}(u_h-v)\|^2_{L^2(\Omega)}$ in the a posteriori estimate can be estimated once the treatment of the Dirichlet boundary conditions in the discrete problem is specified. Suppose that $\int_E [u_h]\, ds = 0$ if $E\in\mathcal{E}$ and $E\not\subset\Gamma$ and that u_D-u_h vanishes at one point of each edge. Then, with an h-independent constant $c_{11}>0$,

$$\inf_v \|\nabla_{\mathcal{T}}(u_h-v)\|_{L^2(\Omega)} \leq c_{11} \| \, h_{\mathcal{E}}^{1/2} \partial(u_D-u_h)/\partial s \, \|_{L^2(\Gamma_D)}, \tag{2.7}$$

from where the term $\inf_v \|\nabla_{\mathcal{T}}(u_h-v)\|_{L^2(\Omega)}$ is bounded by the jump terms $\|[\partial u_h/\partial s]\|_{L^2(E)}$ in (2.2) and so may be neglected in the a posteriori estimate.

To see (2.7), choose $v|_T := u_h|_T$ on each element which has no edge $E\subset\Gamma_D$. On each of the remaining elements T_m we have $m\in T_m\cap\Gamma_D\cap\mathcal{M}$, where $\mathcal{M}$ denotes the set of all midpoints of edges in $\mathcal{E}$. Choose $v=u_h-a_m\,\psi_m$, where ψ_z describes the nonconforming hat function for the midpoint m on T_m. The coefficient $a_m=\int_E(u_D-u_h)\,ds/h_E\in\mathbb{R}^2$ guarantees that v is admissible in (2.7). Then

$$\|\nabla_{\mathcal{T}}(u_h-v)\|^2_{L^2(\Omega)} = \sum_{z\in\mathcal{M}\cap\Gamma_D} |a_m|^2 \, \| \, \nabla\psi_m \, \|^2_{L^2(T_m)} \leq c_{11} \sum_{z\in\mathcal{M}\cap\Gamma_D} |a_m|^2, \tag{2.8}$$

where $c_{11}=\max_z \| \, \nabla\psi_m \, \|^2_{2,T_m}$ depends on the shape of the elements only. Since u_D-u_h has a zero on E, $\| \, u_D-u_h \, \|_{2,E} \leq h_E \, \| \, \partial(u_D-u_h)/\partial s \, \|_{2,E}$ and with Cauchy's inequality,

$$|a_m|^2 \leq \Big(\int_E |u_D-u_h|\, ds\Big)^2/h_E^2 \leq h_E \, \| \, \partial(u_D-u_h)/\partial s \, \|^2_{L^2(E)}. \qquad \square \tag{2.9}$$

(viii) In the example of the conforming-nonconforming finite element space (1.8)-(1.9), the discrete solution u_h satisfies the Dirichlet boundary condition at the boundary nodes in the first component, i.e., $e_1\cdot u_h(z)=e_1\cdot u_D(z)$ for all $z\in\mathcal{N}\cap\Gamma_D$, and in the second component the discrete midpoint value equals the integral mean of the exact boundary value at a boundary edge, i.e., $e_2\cdot u_h(m)=e_2\cdot\int_E u_D\,ds/h_E$ for the midpoint m of $E\in\mathcal{E}$, $E\subset\Gamma_D$. Then, for the same h-independent constant c_{11} as in (2.8) the arguments in (vii) plus a finer estimate of the affine interpolation error show that the term $\inf_v \|\nabla_{\mathcal{T}}(u_h-v)\|^2_{L^2(\Omega)}$ is of higher order. Let $\mathcal{I}u_D\in C(\Gamma_D)^2$ denote the $\mathcal{E}$-piecewise affine nodal interpolant to u_D on Γ_D. If $u_D|_E\in H^2(E)$ for all $E\in\mathcal{E}$ with $E\subset\Gamma_D$, then

$$\inf\nolimits_v \|\nabla_{\mathcal{T}}(u_h-v)\|_{L^2(\Omega)} \leq c_{11} \| \, h_{\mathcal{E}}^{1/2} \partial(u_D-\mathcal{I}u_D)/\partial s \, \|_{L^2(\Gamma_D)}, \tag{2.10}$$

$$\inf\nolimits_v \|\nabla_{\mathcal{T}}(u_h-v)\|_{L^2(\Omega)} \leq c_{11} \| \, h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} u_D/\partial s^2 \, \|_{L^2(\Gamma_D)}. \tag{2.11}$$

(ix) The constants in Theorem 2.1 depend on the shape of the patches by the overlap of $(\Omega_z : z\in\mathcal{K})$. The further assumption that each element contains at least one free node reduces this dependence to usual dependence on the elements.

3. Proof of reliability

To put emphasis on possible Neumann data, let us suppose in this section that Γ_N has positive surface measure. ($\Gamma = \Gamma_D$ requires the exact and discrete pressure to have a vanishing integral mean on Ω, but, apart from this modification, appears less technical.)

Let (u,p) solve (1.1)-(1.3), let (u_h,p_h) solve (1.6)-(1.7), and consider the errors

$$e := u - u_h \in H^1(\mathcal{T})^2 \quad \text{and} \quad \epsilon := p - p_h \in L^2(\Omega). \tag{3.1}$$

For abbreviation, we frequently write $\|\cdot\|_{2,\Omega} := \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{1,2,\Omega} := \|\cdot\|_{H^1(\Omega)}$ and neglect the domain Ω if there is no risk of confusion. Furthermore, we define the following residuals, which contribute to (1.10) or to the infimum in (2.2),

$$\eta_1^2 := \sum_{T\in\mathcal{T}} \|\operatorname{div} u_h\|^2_{L^2(T)}, \tag{3.2}$$

$$\eta_2^2 := \sum_{z\in\mathcal{K}} h_z^2 \min_{f_z\in\mathbb{R}^2} \|\operatorname{div}\sigma_h + f - f_z\|^2_{L^2(\Omega_z)}, \tag{3.3}$$

$$\eta_3^2 := \sum_{E\in\mathcal{E}} h_E \|[\sigma_h n_E]\|^2_{L^2(E)}, \tag{3.4}$$

$$\eta_4^2 := \inf\Big\{\|\nabla_{\mathcal{T}}(u_h - v)\|^2_{L^2(\Omega)} \,:\, v\in H^1(\mathcal{T}) \text{ and for all } E\in\mathcal{E} \text{ there holds} \\ \int_E [v]\,ds = 0 \text{ if } E\not\subset\Gamma \text{ and } \int_E v\,ds = \int_E u_D\,ds \text{ if } E\subset\Gamma_D\Big\}, \tag{3.5}$$

$$\eta_5^2 := \sum_{E\in\mathcal{E}} h_E \|[\partial u_h/\partial s]\|^2_{L^2(E)}. \tag{3.6}$$

In the first step of the proof, we define an auxiliary function v which allows us to control the error by an energy integral.

Lemma 3.1. *There exist a constant $c_{12} = c_{12}(\Omega)$ and a function $w \in H^1_D(\Omega)$ with*

$$\operatorname{div} w = \epsilon \qquad \text{and} \qquad \|w\|_{H^1(\Omega)} \le c_{12}\|\epsilon\|_{L^2(\Omega)}. \tag{3.7}$$

Furthermore, the function $v := c_1^{-1}c_2^2c_{12}^2\, e - w$ satisfies (*with c_1, c_2 from* (1.4)-(1.5))

$$\frac{c_2^2c_{12}^2}{2}\|\varepsilon_{\mathcal{T}}(e)\|^2_{L^2(\Omega)} + \frac14\|\epsilon\|^2_{L^2(\Omega)} \le \int_\Omega (\sigma-\sigma_h) : \varepsilon_{\mathcal{T}}(v)\,dx + c_1^{-2}c_2^4c_{12}^4\,\eta_1^2. \tag{3.8}$$

Proof. Since Ω has a polygonal boundary, we can enlarge Ω to $\hat\Omega$ such that the open surface piece (or finite collection of pieces) $\Gamma_N \subset \partial\Omega$ belongs to the interior of the bounded Lipschitz domain $\hat\Omega$. The function ϵ is extended to $\hat\epsilon$ by a constant $\epsilon_0 := -\int_\Omega \epsilon\,dx/\operatorname{meas}(\hat\Omega\setminus\Omega)$ so that $\int_{\hat\Omega}\hat\epsilon\,dx = 0$. The Stokes problem

$$\Delta\hat w - \nabla q = 0 \quad \text{and} \quad \operatorname{div}\hat w = \hat\epsilon \qquad \text{in } \hat\Omega \tag{3.9}$$

has a unique solution $\hat{w} \in H^1_0(\hat{\Omega})^2$ and $q \in L^2(\hat{\Omega})$, which satisfies the a priori estimate [GR, T, BF]

$$\|\hat{w}\|_{H^1(\hat{\Omega})} \leq c_{13}\|\hat{\epsilon}\|_{L^2(\hat{\Omega})} \leq c_{14}\|\epsilon\|_{L^2(\Omega)} \tag{3.10}$$

with constants c_{13} and c_{14} that depend only on the geometry of $\hat{\Omega}$ and Ω, but not on ϵ. The restriction $w := \hat{w}|_\Omega$ satisfies (3.7).

According to the definition of v, σ, and σ_h, we calculate

$$\begin{aligned}\int_\Omega (\sigma - \sigma_h) : \varepsilon_{\mathcal{T}}(v)\, dx &= c_1^{-1}c_2^2c_{12}^2 \int_\Omega (\mathcal{A}(\varepsilon(u)) - \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h))) : \varepsilon_{\mathcal{T}}(e)\, dx \\ - c_1^{-1}c_2^2c_{12}^2 \int_\Omega \epsilon : \operatorname{div}_{\mathcal{T}} e\, dx &- \int_\Omega (\mathcal{A}(\varepsilon(u)) - \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h))) : \varepsilon_{\mathcal{T}}(w)\, dx + \|\epsilon\|^2_{2,\Omega}.\end{aligned} \tag{3.11}$$

The symmetry of $\mathcal{A}(\varepsilon(u)) - \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h))$ and the estimates (1.5) and (3.7) yield

$$\begin{aligned}&\int_\Omega (\mathcal{A}(\varepsilon(u)) - \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h))) : \varepsilon(w)\, dx \\ &\qquad \leq c_2\|\varepsilon_{\mathcal{T}}(e)\|_{2,\Omega}\|\nabla w\|_{2,\Omega} \leq \frac{c_2^2c_{12}^2}{2}\|\varepsilon_{\mathcal{T}}(e)\|^2_{2,\Omega} + \frac{1}{2}\|\epsilon\|^2_{2,\Omega}.\end{aligned} \tag{3.12}$$

Because $\operatorname{div} u = 0$, $\|\operatorname{div}_{\mathcal{T}} e\|_{2,\Omega} = \eta_1$, and (1.4), (3.11)-(3.12), we calculate

$$\frac{c_2^2c_{12}^2}{2}\|\varepsilon_{\mathcal{T}}(e)\|^2_{2,\Omega} + \frac{1}{2}\|\epsilon\|^2_{2,\Omega} \leq \int_\Omega (\sigma - \sigma_h) : \varepsilon_{\mathcal{T}}(v)\, dx + c_1^{-1}c_2^2c_{12}^2\, \eta_1\, \|\epsilon\|_{2,\Omega}. \tag{3.13}$$ □

Now we consider a Helmholtz decomposition of $\varepsilon_{\mathcal{T}}(v)$ from [CD] (see also [A, Ca1, CJ]). Recall $\operatorname{Curl}\beta = (\partial\beta/\partial x_2, -\partial\beta/\partial x_1) \in L^2(\Omega; \mathbb{R}^{d\times 2})$ if $\beta \in H^1(\Omega)^d$ for $d = 1, 2$.

Lemma 3.2 ([CD, FM]). *There exist* $\alpha \in H^1_D$ *and* $\beta \in H^2(\Omega)$ *with* $(\operatorname{Curl}\operatorname{Curl}\beta)n = 0$ *on* Γ_N *satisfying*

$$\varepsilon_{\mathcal{T}}(v) = \varepsilon(\alpha) + \operatorname{Curl}\operatorname{Curl}\beta \quad \textit{a.e. in } \Omega. \tag{3.14}$$

Proof. We sketch a proof for convenient reading. Let α solve the elliptic problem $\operatorname{div}\varepsilon(\alpha) = \operatorname{div}_{\mathcal{T}}\varepsilon_{\mathcal{T}}(v)$ with boundary conditions $\alpha = 0$ on Γ_D and $\varepsilon(\alpha)n = \varepsilon_{\mathcal{T}}(v)n$ on Γ_N, i.e., $\int_\Omega(\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v)) : \varepsilon(\eta)\, dx = 0$ for all $\eta \in H^1_D(\Omega)$. Since $\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v)$ is symmetric, the solution α satisfies

$$\int_\Omega (\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v)) : \nabla\eta\, dx = 0 \quad (\eta \in H^1_D(\Omega)^2). \tag{3.15}$$

Hence, each row of $\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v)$ is divergence-free and $(\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v))n$ vanishes on Γ_N. Hence there exists some $\hat{\beta} \in H^1(\Omega)^2$ with $\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v) = \operatorname{Curl}\hat{\beta}$ (see [GR, Sect. 3] for proofs). Moreover, since $\varepsilon(\alpha) - \varepsilon_{\mathcal{T}}(v)$ is symmetric, $\operatorname{Curl}\hat{\beta} : \varepsilon(\eta) = \operatorname{Curl}\hat{\beta} : \nabla\eta$, and by integration by parts

$$0 = \int_\Omega \operatorname{Curl}\hat{\beta} : \varepsilon(\eta)\, dx = -\int_{\Gamma_N} \operatorname{Curl}\hat{\beta}\, n \cdot \eta\, ds \quad (\eta \in H^1_D(\Omega)). \tag{3.16}$$

We conclude $\operatorname{Curl}\hat{\beta}n = 0$ whence $\partial\hat{\beta}/\partial s = 0$ on Γ_N and so $\hat{\beta}$ is constant on each component of Γ_N. If Γ_j is a connectivity component of Γ that does not include Γ_D, then $\hat\beta = \hat\beta_j$ is constant there and we deduce

$$\int_{\Gamma_j} \hat\beta\, n\, ds = \hat\beta_j \int_{\Gamma_j} n\, ds = 0 \tag{3.17}$$

from the divergence theorem. Since Γ_D is connected, there is exactly one component Γ_0 that includes Γ_D.

The symmetry of $\operatorname{Curl}\hat\beta$ reads $\hat\beta_{2,2} = -\hat\beta_{1,1}$, i.e., $\operatorname{div}\hat\beta = 0$. The divergence theorem on Ω and (3.17) then show that for all $j \neq 0$

$$0 = \int_\Omega \operatorname{div}\hat\beta\, dx = \int_{\Gamma_0} \hat\beta\, n\, ds. \tag{3.18}$$

Hence, $\hat\beta\, n$ has a vanishing integral over all connectivity components of Γ and is divergence free. Thus, there exists some $\beta \in H^2(\Omega)/\mathbb{R}$ that satisfies $\hat\beta = \operatorname{Curl}\beta$ [T, GR]. □

The Helmholtz decomposition (3.14) of $\varepsilon_{\mathcal{T}}(v)$ from Lemma 3.1 leads to

$$\begin{aligned}\frac{c_2^2 c_{12}^2}{2}\|\varepsilon_{\mathcal{T}}(e)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\epsilon\|^2_{L^2(\Omega)} \le& \int_\Omega (\sigma-\sigma_h) : \varepsilon(\alpha)\, dx \\ &+ \int_\Omega (\sigma-\sigma_h) : \operatorname{Curl}\operatorname{Curl}\beta\, dx + c_1^{-2} c_2^4 c_{12}^4\, \eta_1^2.\end{aligned} \tag{3.19}$$

The two integrals on the right-hand side in (3.19) will be estimated in Lemma 3.4 and 3.5 below. Therein, we require a Clément-type approximation operator [Cl, Ci, BS] in a refined form (see also [CB, CV]).

For a regular triangulation $\mathcal{T}$ with set of edges $\mathcal{E}$ we associate mesh-size weights $h_{\mathcal{T}}$ and $h_{\mathcal{E}}$, which are $\mathcal{T}$-piecewise and $\mathcal{E}$-piecewise constant defined on Ω and the skeleton $\bigcup\mathcal{E}$ of all points on edges by $h_{\mathcal{T}}|_T = h_T$ for $T \in \mathcal{T}$ and $h_{\mathcal{E}}|_E = h_E$ for $E \in \mathcal{E}$, respectively.

Lemma 3.3 ([Ca2])**.** *There exists a linear mapping $\mathcal{J} : H^1_D(\Omega)^2 \to \mathcal{S}$, bounded if domain and range space are endowed with H^1-semi norms, which satisfies*

$$\|h_{\mathcal{T}}^{-1}(\varphi - \mathcal{J}\varphi)\|_{L^2(\Omega)} + \|h_{\mathcal{E}}^{-1/2}(\varphi - \mathcal{J}\varphi)\|_{L^2(\bigcup\mathcal{E})} \le c_{15}\, \|\nabla\varphi\|_{L^2(\Omega)} \tag{3.20}$$

for all $\varphi \in H^1_D(\Omega)$. In addition, there holds for all $R \in L^2(\Omega)^2$

$$\int_\Omega R\cdot(\varphi - \mathcal{J}\varphi)\, dx \le c_{16}\, \|\nabla\varphi\|_{L^2(\Omega)} \left(\sum_{z\in\mathcal{K}} h_z^2 \min_{R_z\in\mathbb{R}^2} \int_{\Omega_z} |R - R_z|^2 dx\right)^{1/2}. \tag{3.21}$$

The positive constants c_{15}, c_{16} do not *depend on the mesh-sizes $h_{\mathcal{T}}$ and $h_{\mathcal{E}}$, but on the shape of the elements only.* □

The first integral on the right-hand side in (3.19) is studied in the next lemma.

Lemma 3.4. *We have*

$$\int_\Omega (\sigma-\sigma_h) : \varepsilon(\alpha)\, dx \le \max\{c_{15}, c_{16}\}\, (\eta_2^2 + \eta_3^2)^{1/2} \|\nabla\alpha\|_{L^2(\Omega)}. \tag{3.22}$$

Proof. Utilising (2.3), $A := \mathcal{J}\alpha \in \mathcal{S} \subset \mathcal{V}$, $\alpha - A \in H^1_D(\Omega)$, and elementwise integration by parts, we infer

$$\begin{aligned}\int_\Omega (\sigma - \sigma_h) : \varepsilon(\alpha)\, dx &= \int_\Omega (\sigma - \sigma_h) : \varepsilon(\alpha - A)\, dx \\ &= \int_\Omega (\operatorname{div} \sigma_h + f) \cdot (\alpha - A)\, dx + \sum_{E \in \mathcal{E}} \int_E [\sigma_h n_E](\alpha - A)\, ds.\end{aligned} \tag{3.23}$$

Recall that $[\sigma_h n_E]$ is the jump of $\sigma_h n_E$ across an interior element boundary $E \in \mathcal{E}$, and is defined by $g - \sigma_h n$ on Γ_N and $[\sigma_h n_E] = 0$ on Γ_D. From Cauchy's inequality and (3.20)-(3.21) we conclude

$$\begin{aligned}&\int_\Omega (\sigma - \sigma_h) : \varepsilon_{\mathcal{T}}(\alpha)\, dx \\ &\quad \le \Big(c_{16} \sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathbb{R}^2} \| \operatorname{div} \sigma_h + f - f_z \|^2_{2,\Omega_z} + c_{15} \sum_{E \in \mathcal{E}} h_E \| [\sigma_h n_E] \|^2_{2,E} \Big)^{1/2} \| \nabla \alpha \|^2_{2,\Omega} \\ &\quad \le \max\{c_{15}, c_{16}\}\, (\eta_2^2 + \eta_3^2)^{1/2} \| \nabla \alpha \|_{2,\Omega}. \quad \square\end{aligned} \tag{3.24}$$

The second integral on the right-hand side in (3.19) is studied in the next lemma, where c_{17} will be characterised below in (3.28) as an analogue to c_{15}.

Lemma 3.5. *We have*

$$\int_\Omega (\sigma - \sigma_h) : \operatorname{Curl}\operatorname{Curl} \beta\, dx \le c_1^{-1} c_2^2 c_{12}^2 c_{17} (\eta_4^2 + \eta_5^2)^{1/2}\, \| \sigma - \sigma_h \|_{L^2(\Omega)}. \tag{3.25}$$

Proof. Let $a \in H^1_D(\Omega)$ and $b \in H^2(\Omega)$ define a Helmholtz decomposition of $\sigma - \sigma_h$ as in Lemma 3.2, i.e.,

$$\sigma - \sigma_h = \varepsilon(a) + \operatorname{Curl}\operatorname{Curl} b \quad \text{a.e. in } \Omega. \tag{3.26}$$

Since $(\operatorname{Curl}\operatorname{Curl} b)n = 0$ on Γ_N we have L^2-orthogonality of $\operatorname{Curl}\operatorname{Curl}\beta$ and $\varepsilon(a)$ and deduce from $v := c_1^{-1} c_2^2 c_{12}^2\, e - w$ and an integration by parts that

$$\begin{aligned} c_1 c_2^{-2} c_{12}^{-2} \int_\Omega (\sigma - \sigma_h) : \operatorname{Curl}\operatorname{Curl} \beta\, dx &= \int_\Omega \operatorname{Curl}\operatorname{Curl} b : \varepsilon_{\mathcal{T}}(e)\, dx \\ &= \int_{\Gamma_D} u_D \cdot \operatorname{Curl}\operatorname{Curl} b\, n\, ds - \int_\Omega \operatorname{Curl}\operatorname{Curl} b : \varepsilon_{\mathcal{T}}(u_h)\, dx. \end{aligned} \tag{3.27}$$

Let $B \in \mathcal{S}_1(\mathcal{T})^2$ denote an approximation to $\operatorname{Curl} b$ as in Lemma 3.3 where the role of the Dirichlet and Neumann boundaries is interchanged; here Γ_N acts as the Dirichlet boundary, i.e., $B \in C(\overline{\Omega})^2$, $\operatorname{Curl} b = B$ on Γ_N. Recall that $0 = \operatorname{Curl}\operatorname{Curl} b\, n = \partial \operatorname{Curl} b / \partial s$ such that $\operatorname{Curl} b$ is constant on each component of Γ_N and so the interpolation indeed yields $\operatorname{Curl} b = B$ on Γ_N. As in Lemma 3.3 we have

$$\| \operatorname{Curl} B \|_2 + \| h_{\mathcal{E}}^{-1/2} (\operatorname{Curl} b - B) \|_{2, \cup\mathcal{E}} \le c_{17}\, \| D^2 b \|_2. \tag{3.28}$$

Note that $\operatorname{Curl} B\, n_E = 0$ on Γ_N and, furthermore, $\operatorname{Curl} B\, n_E$ is constant on each $E \in \mathcal{E}$. Thus, for $v \in H^1(\mathcal{T})^2$ as in Theorem 2.1,

$$\begin{aligned}(3.29)\quad \int_{\Gamma_D} u_D \cdot \operatorname{Curl} B\, n\, ds &= \int_{\Gamma_D} v \cdot \operatorname{Curl} B\, n\, ds \\ &\quad + \sum_{E\in\mathcal{E}} \int_{E\setminus\Gamma} [v] \cdot \operatorname{Curl} B\, n_E\, ds = \int_\Omega \nabla_{\mathcal{T}} v : \operatorname{Curl} B\, dx,\end{aligned}$$

where we utilised an elementwise integration by parts. From the symmetry of $\operatorname{Curl}\operatorname{Curl} b$, (3.27), and (3.29), we infer

$$\begin{aligned}(3.30)\quad c_1 c_2^{-2} c_{12}^{-2} \int_\Omega (\sigma - \sigma_h) : \operatorname{Curl}\operatorname{Curl}\beta\, dx &= \int_{\Gamma_D} u_D \cdot \operatorname{Curl}(\operatorname{Curl} b - B)\, n\, ds \\ &\quad + \int_\Omega \operatorname{Curl}(B - \operatorname{Curl} b) : \nabla_{\mathcal{T}} u_h\, dx - \int_\Omega \operatorname{Curl} B : \nabla_{\mathcal{T}}(u_h - v)\, dx.\end{aligned}$$

From $\operatorname{Curl} b = B$ on Γ_N and the integration by parts formula on the closed Lipschitz curve (or curves of) Γ we deduce

$$\begin{aligned}(3.31)\quad \int_{\Gamma_D} u_D \cdot \operatorname{Curl}(\operatorname{Curl} b - B)\, n\, ds &= \int_{\Gamma_D} u_D \cdot \partial(\operatorname{Curl} b - B)/\partial s\, ds \\ &= -\int_{\Gamma_D} \partial u_D/\partial s \cdot (\operatorname{Curl} b - B)\, ds.\end{aligned}$$

Elementwise integration by parts, Cauchy's inequality, (3.28), (3.30), and (3.31) result in

$$\begin{aligned}(3.32)\quad & c_1 c_2^{-2} c_{12}^{-2} \int_\Omega (\sigma - \sigma_h) : \operatorname{Curl}\operatorname{Curl}\beta\, dx \le \|\operatorname{Curl} B\|_2 \,\|\nabla_{\mathcal{T}}(u_h - v)\|_2 \\ &\quad + \int_{\cup\mathcal{E}} [\partial u_h/\partial s] \cdot (\operatorname{Curl} b - B)\, ds \le c_{17}\, \|D^2 b\|_2 \max\{\eta_5, \|\nabla_{\mathcal{T}}(u_h - v)\|_2\}. \quad \square\end{aligned}$$

Proof of Theorem 2.1. Combining Lemmas 3.4 and 3.5 we obtain from (3.19) and $\|\sigma - \sigma_h\|_2 \le c_2 \|\varepsilon_{\mathcal{T}}(e)\|_2 + \|\epsilon\|_2$ and $\|\varepsilon(\alpha)\|_2 \le \|\varepsilon_{\mathcal{T}}(e)\|_2$ that

$$\begin{aligned}(3.33)\quad \frac{c_2^2 c_{12}^2}{2} \|\varepsilon_{\mathcal{T}}(e)\|_2^2 + \frac{1}{4}\|\epsilon\|_2^2 &\le \max\{c_{15}, c_{16}\}\, (\eta_2^2 + \eta_3^2)^{1/2} \|\varepsilon_{\mathcal{T}}(e)\|_2 \\ &\quad + c_1^{-1} c_2^2 c_{12}^2 c_{17} (\eta_4^2 + \eta_5^2)^{1/2}\, (c_2 \|\varepsilon_{\mathcal{T}}(e)\|_2 + \|\epsilon\|_2) + c_1^{-2} c_2^4 c_{12}^4\, \eta_1^2.\end{aligned}$$

With Young's inequality, the terms $\|\varepsilon_{\mathcal{T}}(e)\|_2$ and $\|\epsilon\|_2$ on the left-hand side can be absorbed. This concludes the proof of Theorem 2.1. $\square$

4. EFFICIENCY

This section is devoted to efficiency investigations whose aim is to prove the converse estimate up to higher order terms. Some of the technical results of the section are preliminary to the proof of reliability of the averaging techniques in the subsequent section.

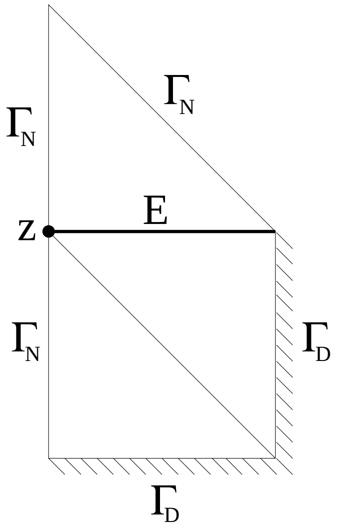

FIGURE 1. Coarse triangulation where the inner edge E violates geometric restrictions

To indicate the efficiency of the a posteriori error bound (1.10) we follow Verfürth [V1] and refine a corresponding estimate in [DDP].

To cover the conforming-nonconforming scheme (1.8)-(1.9) as well as conforming finite element schemes, we suppose that u_h belongs to $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$ where the jumps on interior element edges or Dirichlet edges satisfy different continuity conditions in each component:

$$\mathcal{W}_1 := \{V \in \mathcal{L}_k(\mathcal{T}) : \forall E \in \mathcal{E},\ [V] \text{ vanishes at the endpoints of } E\}, \tag{4.1}$$

$$\mathcal{W}_2 := \{V \in \mathcal{L}_k(\mathcal{T}) : \forall E \in \mathcal{E},\ \int_E [V]\, ds = 0\}. \tag{4.2}$$

Here, $1 \leq k$ and the jump on boundary edges is understood as $[V] := 0$ on $\overline{\Gamma}_N$ and $[V] := u_D - V$ on Γ_D.

According to the different role of the two components, we need different mild restrictions on the coarseness of the mesh: Assume first that each connectivity component of Γ_D (Γ_D is no longer necessarily connected) contains more than one edge in $\mathcal{E}$. Second assume that each edge $E \not\subset \Gamma$ has at least one endpoint which is an interior node (see Figure 1 where the second condition is violated).

Suppose that $u_D \in C(\Gamma_D)^2$ satisfies $u_D|_E \in H^1(E)^2$ for all $E \in \mathcal{E}$ with $E \subset \Gamma_D$, and write $\partial^m_{\mathcal{E}} u_D/\partial s^m$ for the $\mathcal{E}$-piecewise derivative of u_D on Γ_D with respect to the arc-length. Recall that $\mathcal{I}u_D \in C(\Gamma_D)^2$ denotes the $\mathcal{E}$-piecewise affine nodal interpolant to u_D on Γ_D (i.e., $\mathcal{I}u_D(z) = u_D(z)$ for all $z \in \mathcal{N} \cap \Gamma_D$, and $\mathcal{I}u_D|_E$ is affine on $E \in \mathcal{E}$ with $E \subset \Gamma_D$). Let $f_{\mathcal{T}} \in \mathcal{S}_0(\mathcal{T})^2$ be the L^2-projection of f on $\mathcal{S}_0(\mathcal{T})^2$ and let $g_{\mathcal{E}}$ denote an $\mathcal{E}$-piecewise polynomial approximation to g on Γ_N.

Theorem 4.1 implies the estimate (1.12) as a particular case.

Theorem 4.1. *Let* $(u, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ *solve* (1.1)-(1.3), $(u_h, p_h) \in H^2(\mathcal{T})^2 \times L^2(\Omega)$ *solve* (1.6)-(1.7), $u_h \in \mathcal{W}$, *and set* $\sigma_h := \mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) - p_h I$. *Then there exists an* h*-independent constant* $c_{19} > 0$ *such that, for all* $T \in \mathcal{T}$ *and the patch*

$\omega_T := \bigcup\{T' \in \mathcal{T} :\ T \cap T' \in \mathcal{E} \cup \mathcal{T}\}$ *of neighbouring elements*

$$\tag{4.3} \begin{aligned} \sum_{E\in\mathcal{E}\wedge E\subset\partial T} & h_E\big(\|[\sigma_h n_E]\|^2_{L^2(E)} + \|[\partial u_h/\partial s]\|^2_{L^2(E)}\big) \\ & \le c_{19}\Big(\inf_{v=u_D \text{ on } \Gamma_D} \|\varepsilon_{\mathcal{T}}(u_h - v)\|^2_{L^2(\omega_T)} \\ & \qquad + \inf_{\tau\in H(\mathrm{div};\omega_T)^2} (\|\sigma_h - \tau\|^2_{2,\omega_T} + h_E^2\|\operatorname{div}_{\mathcal{T}}(\sigma_h - \tau)\|^2_{2,\omega_T}) \\ & \qquad + \|h_{\mathcal{E}}^{1/2}(g - g_{\mathcal{E}})\|^2_{L^2(\Gamma_N\cap\partial\omega_T)} + h_T^2\|f - f_{\mathcal{T}}\|^2_{L^2(\omega_T)} \\ & \qquad\qquad + \|\, h_{\mathcal{E}}^{1/2}\partial(u_D - \mathcal{I}u_D)/\partial s\,\|^2_{L^2(\Gamma_D\cap\partial\omega_T)}\Big). \end{aligned}$$

In particular, we have the efficiency estimate

$$\tag{4.4} \begin{aligned} \sum_{E\in\mathcal{E}\wedge E\subset\partial T} & h_E\big(\|[\sigma_h n_E]\|^2_{L^2(E)} + \|[\partial u_h/\partial s]\|^2_{L^2(E)}\big) \\ & \quad + h_T^2\,\|\,f + \operatorname{div}\sigma_h\,\|^2_{L^2(T)} + \|\,\operatorname{div} u_h\,\|^2_{L^2(T)} \\ & \le c_{19}\Big(\|\varepsilon_{\mathcal{T}}(u - u_h)\|^2_{L^2(\omega_T)} + \|p - p_h\|^2_{L^2(\omega_T)} + \|h_{\mathcal{E}}^{1/2}(g - g_{\mathcal{E}})\|^2_{L^2(\Gamma_N\cap\partial\omega_T)} \\ & \qquad + h_T^2\|f - f_{\mathcal{T}}\|^2_{L^2(\omega_T)} + \|\, h_{\mathcal{E}}^{1/2}\partial(u_D - \mathcal{I}u_D)/\partial s\,\|^2_{L^2(\Gamma_D\cap\partial\omega_T)}\Big). \end{aligned}$$

Remarks 4.1. (i) The constant c_{19} in Theorem 4.1 depends on the shape and on the degree of the finite elements (not on their diameters).

(ii) The condition $u_h \in \mathcal{W}$ is satisfied for all conforming finite element methods as well as for the conforming-nonconforming scheme (1.6)-(1.7).

(iii) The compatibility condition in (4.1) could be further relaxed. The proof shows that any compatibility condition which guarantees an affine function $e_1 \cdot [v]$ vanishes is indeed sufficient.

(iv) The mild restrictions on the mesh are violated in the example indicated in Figure 1. Note that successive red-refinements cannot change that the top triangle can rigidly move around the midpoint of E. A green-refinement of E in the top triangle cures the failure (see, e.g., [V1] for the definition of red-green-blue refinement).

(v) In corresponding results in [BB, CJ] the error term $\|\,\nabla(u - u_h)\,\|_{2,\Omega_T}$ arises which is replaced here by the Green strain error $\|\,\varepsilon(u - u_h)\,\|_{2,\Omega_T}$. For conforming schemes, $u - u_h \in H^1_D(\Omega)$ and this improvement is not important since Korn's inequality provides global equivalence. For nonconforming schemes, Korn's inequality is not available [FM]. For the conforming-nonconforming finite element scheme with (1.8)-(1.9), Korn's inequality is globally available for u_h and u but not necessarily for $u - u_h$. (The different statement in [BB] is still unproven.)

The remaining part of the section is devoted to the proof of Theorem 4.1 preceded by several technical lemmas.

The first lemma provides a version of Korn's inequality (certainly known to the experts but not easily found in textbooks).

Lemma 4.1. *Let $\mathcal{RM}(\Omega)$ denote the rigid body motions in $\mathbb{R}^d$ $(d = 2, 3)$. Then, $\|\cdot\|_{H^1(\Omega)}$ and $\|\,\varepsilon(\cdot)\,\|_{L^2(\Omega)}$ are two equivalent norms on $H^1(\Omega)^d/\mathcal{RM}(\Omega)$.*

Proof. The standard version of Korn's inequality is the estimate

$$\| \cdot \|_2 + \| \nabla \cdot \|_2 \leq c_{20} \Big(\| \cdot \|_2 + \| \varepsilon(\cdot) \|_2 \Big) \text{ in } H^1(\Omega)^d \tag{4.5}$$

and can be found in textbooks. The main point in the lemma is the related estimate

$$\min_{r \in \mathcal{RM}(\Omega)} \| v - r \|_{1,2} \leq c_{12} \| \varepsilon(v) \|_2 \tag{4.6}$$

for $v \in H^1(\Omega)^d$. We sketch an indirect proof for convenient reading. If this inequality was false, we could find a sequence (w_j) in $H^1(\Omega)^2/\mathcal{RM}(\Omega)$ with

$$\lim_{j\to\infty} \| \varepsilon(w_j) \|_2 = 0 \quad \text{and} \quad \min_{r \in \mathcal{RM}} \| w_j - r \|_{1,2} = 1.$$

Banach-Alaoglu's theorem yields a weakly convergent subsequence in $H^1(\Omega)^d$ and by Rellich's theorem there exists a strongly convergent subsequence (v_j) in $L^2(\Omega)^d$. Hence we may and will assume without loss of generality that there exists a weak limit v in $H^1(\omega)^d$ with $\lim_{j\to\infty} \| v - v_j \|_2 = 0$. Since $\| \varepsilon(\cdot) \|_2$ is sequentially weakly lower semi-continuous, we deduce $\varepsilon(v) = 0$, i.e., $v \in \mathcal{RM}(\Omega)$. Korn's inequality (4.5) shows

$$1 = \min_{r \in \mathcal{RM}(\Omega)} \| v_j - r \|_{1,2} \leq \| \nabla(v_j - v) \|_{1,2} \leq c_{20} \Big(\| \varepsilon(v_j - v) \|_2 + \| v_j - v \|_2 \Big), \tag{4.7}$$

but the right-hand side in (4.7) tends toward zero. This contradiction proves (4.6). We omit the proof of the remaining assertions. ☐

Lemma 4.2. *Let $T_1, \dots, T_J \in \mathcal{T}$ be a sequence of neighbouring elements such that $E = T_j \cap T_{j+1} \in \mathcal{E}$ is an edge which is not parallel to the x_1-axis. Then, all $r \in \mathcal{W}$ with $r_{T_j} \in \mathcal{RM}(T_j)$, $j = 1, \dots, J$ are rigid, i.e., $r \in \mathcal{RM}(\bigcup_{j=1}^J T_j)$.*

Proof. It suffices to prove the assertion for $J = 2$; the general case follows by induction. Consider a common edge E of two distinct neighbouring elements T_1 and T_2 where there exist real numbers a_T, b_T, c_T with

$$r(x) = (a_T - c_T\, x_2, b_T + c_T\, x_1) \quad \text{for } x \in T. \tag{4.8}$$

The condition (4.1) shows that the first component $[r_1]$ of the jump $[r]$ of r vanishes at two distinct points. In case E is not aligned with the x_1-axis, this shows that the affine function $[r_1]$ vanishes, i.e., $a_{T_1} = a_{T_2}$ and $c_{T_1} = c_{T_2}$. Condition (4.2) implies that $[r_2]$ vanishes at the midpoint of E. Knowing $c_{T_1} = c_{T_2}$ already, $b_{T_1} = b_{T_2}$ follows from this and (4.8). Thus, if E is not aligned with the x_1-axis, $[r] = 0$, i.e., $r \in \mathcal{RM}(T_1 \cup T_2)$. ☐

Lemma 4.3. *Let ω_z be the patch of $z \in \mathcal{N}$. Suppose that either z is an interior point of Ω or that z does not belong to exactly one edge $E \in \mathcal{E}$ parallel to the x_1-axis (cf. Figure 1). Let $r \in \mathcal{W}$ such that $r|_T \in \mathcal{RM}(T)$ for all $T \in \mathcal{T}$ with $T \subset \overline{\omega_z}$. Then $r|_{\omega_z} \in \mathcal{RM}(\omega_z)$.*

Proof. In two dimensions, there are either zero, one or two edges $E \sqsubset \omega_z$ with $E \subset \mathbb{R} \times \{z_2\}$, where "$E \sqsubset \omega_z$" stands for all edges $E \in \mathcal{E}$ with $z \in E$. If there is no such edge, the assertion follows from Lemma 4.2.

In case there is one such edge $E \subset \mathbb{R} \times \{z_2\}$ and z is an interior point, Lemma 4.2 reveals $r \in \mathcal{RM}(\omega_z \setminus \{E\})$ and, since the set $\omega_z \setminus \{E\}$ is connected, $r \in \mathcal{RM}(\omega_z)$.

In case there are two such edges E_1 and $E_2 \subset \mathbb{R} \times \{z_2\}$ we find that r is a rigid body motion on either of the two components of $\omega_z \setminus \mathbb{R} \times \{z_2\}$. The jump $[r_2]$ across

$E_j \subset \mathbb{R}\times\{z_2\}$ is affine on $\mathbb{R}\times\{z_2\}$ and vanishes at the midpoints of E_j. Hence, $[r_2]=0$. From this and $[r_1]=0$ we then deduce $r\in\mathcal{RM}(\omega_z)$. $\square$

Lemma 4.4. *Let ω_z be a patch of $z\in\mathcal{N}\cap\Gamma_D$ such that $\Gamma_D\cap\partial\omega_z = E_1\cup E_2$ for two distinct edges $E_1, E_2\in\mathcal{E}$ which are parallel. Suppose $u_D=0$ and $r\in\mathcal{W}$ with $\varepsilon_{\mathcal{T}}(r)=0$ on ω_z. Then $r=0$ on ω_z.*

Proof. Lemma 4.2 shows that r is a rigid body motion on each connectivity component of $\omega_z\setminus\mathbb{R}\times\{z_2\}$. If there is no edge E with $z\in E$ parallel to the x_1-axis, then r is a global rigid body motion which is zero at the midpoint m_j of E_j and so $r=0$. The same conclusion can be drawn if E_1 and E_2 are parallel to the x_1-axis. In the remaining case, E_j is not parallel to the x_1-axis but possibly one other edge. As in the proof of Lemma 4.2 we deduce from the boundary conditions in $\mathcal{W}$ that $r|_{E_j}=0$. Hence, $r=0$ on that component of $\omega_z\setminus\mathbb{R}\times\{z_2\}$, to which E_j belongs. E_1 and E_2 belong to different components and so $r=0$ on ω_z. $\square$

Lemma 4.5. *Let ω_z be a patch of $z\in\mathcal{N}$ which is either an interior node or belongs to a straight part of Γ_D in the sense that $\{z\}=E_1\cap E_2$, $E_1, E_2\in\mathcal{E}$ for parallel $E_1, E_2\subset\Gamma_D$. Then, there exists an h-independent constant $c_{22}>0$ such that for all $v_h\in\mathcal{W}$,*

$$\begin{aligned}(4.9)\quad &\Big(\sum_{E\sqsubset\omega_z} h_E\,\|[\partial v_h/\partial s]\|^2_{L^2(E)}\Big)^{1/2}\\ &\le c_{22}\Big(\|\, h_{\mathcal{E}}^{1/2}\partial(u_D-\mathcal{I}u_D)/\partial s\,\|_{L^2(\Gamma_D\cap\partial\omega_z)} + \inf_{v=u_D \text{ on } \Gamma_D}\|\,\varepsilon(v_h-v)\,\|_{L^2(\omega_z)}\Big).\end{aligned}$$

In the infimum, "$v=u_D$ on Γ_D" stands for all $v\in H^1(\omega_z)^2$ if z is an interior node and otherwise for all $v\in H^1(\omega_z)^2$ with $v|_{E_1\cup E_2}=u_D|_{E_1\cup E_2}$.

Proof. In the first step, we prove the lemma for the homogeneous case $u_D=0$. The left- and right-hand sides of (4.9) are semi-norms on the spaces

$$(4.10)\qquad \mathcal{C}_z := \mathcal{W}_z\cap H^1_D(\omega_z)\subset\mathcal{W}_z := \{V|_{\omega_z} : V\in\mathcal{W}\},$$

where $H^1_D(\omega_z) := H^1(\omega_z)^2$ if z is an interior node and $H^1_D(\omega_z) := \{v\in H^1(\omega_z)^2 : v=0 \text{ on } E_1\cup E_2\}$ if $z\in E_1\cup E_2\subset\Gamma_D$. We claim that the right-hand side of (4.9) is a norm on $\mathcal{W}_z/\mathcal{C}_z$.

Suppose that $v_h\in\mathcal{W}_z$ satisfies $\inf_{v\in H^1_D(\omega_z)}\|\,\varepsilon_{\mathcal{T}}(v_h-v)\,\|_{L^2(\omega_z)}=0$. Then, there exists a sequence (v_j) in $H^1_D(\omega_z)$ with $\lim_{j\to\infty}\|\,\varepsilon_{\mathcal{T}}(v_h-v_j)\,\|_{L^2(\omega_z)}=0$. Hence, $\varepsilon(v_j)$ is bounded in $L^2(\omega_z)$ and so is (v_j) in $H^1_D(\omega_z)/\mathcal{RM}(\omega_z)$ owing to Korn's inequality in the form of Lemma 4.1. Banach-Alaoglu's theorem yields a weakly convergent subsequence (v_k) with weak limit v in $H^1(\Omega)^2$. Then $\|\,\varepsilon_{\mathcal{T}}(v_h-v)\,\|_{2,\omega_z}=0$, and so $r:=v_h-v$ belongs to $\mathcal{RM}(T)$ for all $T\in\mathcal{T}$ with $T\subset\overline{\omega_z}$. Note that $v=v_h-r$ is a piecewise polynomial in $H^1_D(\omega_z)$ and so $v\in\mathcal{C}_z$ and $r\in\mathcal{W}_z$.

For $z\notin\Gamma_D$, Lemma 4.3 shows that r is a rigid body motion on ω_z and this implies $v_h=r+v\in\mathcal{C}_z$. If z lies on the Dirichlet boundary, Lemma 4.4 shows $r=0$ and so $v_h\in\mathcal{C}_z$. Thus in any case $v_h=0$ in the quotient space $\mathcal{W}_z/\mathcal{C}_z$.

The left- and right-hand sides of (4.9) are norms on the finite-dimensional space $\mathcal{W}_z/\mathcal{C}_z$ and hence equivalent. This proves (4.9) with a constant c_{22} that depends on ω_z provided $u_D=0$. A scaling argument shows that the weights h_E are chosen properly so that c_{22} is independent from h_z but merely dependent on the shape of the elements and so the shape of the patch.

In the second part of the proof, we have $z \in \Gamma_D \cap E_1 \cap E_2$ and allow $u_D \neq 0$. We extend the nodal interpolant $\mathcal{I}u_D$ to Ω by taking the remaining nodal values equal to zero. Then, $\mathcal{I}u_D \in \mathcal{S}^1(\mathcal{T})^2$. Let ψ_j be the nonconforming nodal basis function related to the midpoint m_j of the edge E_j. Then the discrete function

$$\begin{aligned} w_h &:= v_h - \mathcal{I}u_D - a_1\,\psi_1\,e_2 - a_2\,\psi_2\,e_2, \\ a_j &:= \frac{e_2}{h_{E_j}} \cdot \int_{E_j} (u_D - \mathcal{I}u_D)\,ds, \quad j = 1, 2, \end{aligned} \tag{4.11}$$

satisfies the compatibility conditions for homogeneous Dirichlet data as considered in the first step of this proof. Hence, we obtain in particular

$$h_{E_j}^{1/2}\,\|\partial w_h/\partial s\|_{2,E_j} \leq c_{22} \inf_{w \in H^1_D(\omega_z)} \|\,\varepsilon_{\mathcal{T}}(w_h - w)\,\|_{2,\omega_z}. \tag{4.12}$$

As in the proof of (2.11) and similar to (2.9) we have

$$|a_j| \leq \|\,h_{\mathcal{E}}^{1/2}\,\partial(u_D - \mathcal{I}u_D)/\partial s\,\|_{2,E_j}, \quad j = 1, 2. \tag{4.13}$$

Note also that with $\bigcup \mathcal{E}$ denoting the skeleton of all points on an edge, we have

$$h_{E_j}^{1/2}\,\|\,\partial_{\mathcal{E}}\psi_j/\partial s\,\|_{2,\bigcup\mathcal{E}} + \|\,\nabla_{\mathcal{T}}\psi_j\,\|_{2,\omega_z} \leq c_{23}\,. \tag{4.14}$$

Combining (4.12)-(4.14) we conclude

$$\begin{aligned} &\|\,\partial(u_D - v_h)/\partial s\,\|_{2,E_j} \\ &\quad\leq \|\,\partial w_h/\partial s\,\|_{2,E_j} + \|\,\partial(u_D - \mathcal{I}u_D)/\partial s\,\|_{2,E_j} + c_{23}(|a_1|/h_{E_1}^{1/2} + |a_2|/h_{E_2}^{1/2}) \\ &\quad\leq c_{22} h_{E_j}^{-1/2} \inf_{w = \mathcal{I}u_D \text{ on } \Gamma_D} \|\,\varepsilon_{\mathcal{T}}(v_h - w - a_1\,\psi_1\,e_2 - a_2\,\psi_2\,e_2)\,\|_{2,\omega_z} \\ &\qquad\qquad + c_{24}\,\|\,\partial(u_D - \mathcal{I}u_D)/\partial s\,\|_{2,\Gamma_D \cap \partial\omega_z} \\ &\quad\leq c_{22}\,h_{E_j}^{-1/2} \inf_{w = \mathcal{I}u_D \text{ on } \Gamma_D} \|\,\varepsilon_{\mathcal{T}}(v_h - w)\,\|_{2,\omega_z} + c_{25}\,\|\,\partial(u_D - \mathcal{I}u_D)/\partial s\,\|_{2,\Gamma_D \cap \partial\omega_z}. \end{aligned} \tag{4.15}$$

To change "$w = \mathcal{I}u_D$ on Γ_D" to "$w = u_D$ on Γ_D" in the infimum, we will prove that

$$\inf_{w = u_D - \mathcal{I}u_D \text{ on } \Gamma_D} \|\,\varepsilon(w)\,\|_{2,\omega_z} \leq c_{26}\,\|\,h_{\mathcal{E}}^{1/2}\,\partial(u_D - \mathcal{I}u_D)/\partial s\,\|_{2,\Gamma_D \cap \partial\omega_z}. \tag{4.16}$$

Since $u_D - \mathcal{I}u_D$ vanishes at the endpoints of each edge E_j, it suffices to prove for an edge $E = E_j$ of $T \in \mathcal{T}$ that

$$\inf_{w = u_D - \mathcal{I}u_D \text{ on } E} \|\,\varepsilon(w)\,\|_{2,T} \leq c_{26}\,h_E^{1/2}\,\|\,\partial(u_D - \mathcal{I}u_D)/\partial s\,\|_{2,E}. \tag{4.17}$$

Several explicit constructions of sufficient w are possible. For instance, let $w \in H^1(T)^2$ be the harmonic extension of the boundary values $w = u_D - \mathcal{I}u_D$ on E and $w = 0$ on $\partial T \setminus E$. Then,

$$\|\,\nabla w\,\|_{2,T} \leq c_{27}\,\|\,w\,\|_{H^{1/2}(\partial T)}. \tag{4.18}$$

From a characterisation of the trace space $H^{1/2}(\partial T)$ by interpolation of $H^1(\partial T)$ and $L^2(\partial T)$ we deduce

$$\|\,w\,\|_{H^{1/2}(\partial T)} \leq c_{28}\,\|\,w\,\|_{1,2,\partial T}^{1/2}\|\,w\,\|_{2,\partial T}^{1/2} = c_{28}\,\|\,w\,\|_{1,2,E}^{1/2}\|\,w\,\|_{2,E}^{1/2}. \tag{4.19}$$

With Young's inequality we infer from (4.18)-(4.19) that

$$\| \nabla w \|_{2,T} \le c_{29} \big(h_E^{1/2} \| \partial w/\partial s \|_{2,E} + h_E^{-1/2} \| w \|_{2,E} \big). \tag{4.20}$$

A transformation argument shows that the constant c_{29} is h_E-independent and only depends on the shape of T. From (4.20) and a standard estimate of $w = u_D - \mathcal{I} u_D$, we deduce (4.17) and then (4.16).

The remaining terms such as $h_E^{1/2} \| [\partial v_h/\partial s] \|_{2,E}$ for an inner edge in ω_z can be treated similarly utilising an estimate for $h_E^{1/2} \| [\partial w_h/\partial s] \|_{2,E}$ as in (4.12). We omit the details. □

Lemma 4.6. *Let $\omega_E := \bigcup \{T \in \mathcal{T} : E \subset T\}$ denote the neighbourhood $E \in \mathcal{E}$ with $E \not\subset \Gamma_D$. Then, there exist h_E-independent constants $c_{30}, c_{31}, c_{32}, c_{33}$ such that*

$$h_E^{1/2} \| [\sigma_h \, n_E] \|_{2,E} \le c_{30} \inf_{\tau \in H(\mathrm{div};\omega_E)^2} \Big(\| \sigma_h - \tau \|_{2,\omega_E} + h_E \| \operatorname{div}_{\mathcal{T}} (\sigma_h - \tau) \|_{2,\omega_E} \Big) \tag{4.21}$$

if E is an inner edge. If $E \subset \overline{\Gamma}_N$ and $\omega_E = T_E$ for some $T_E \in \mathcal{T}$, there holds, for each $g_E \in P_k(E)$,

$$h_E^{1/2} \| \sigma_h \, n_E - g_E \|_{2,E} \le c_{31} \inf_{\tau n = g_E} \Big(\| \sigma_h - \tau \|_{2,T_E} + h_E \| \operatorname{div}_{\mathcal{T}} (\sigma_h - \tau) \|_{2,T_E} \Big), \tag{4.22}$$

$$\begin{aligned} h_E^{1/2} \| \sigma_h \, n_E - g \|_{2,E} &\le h_E^{1/2} \| g_E - g \|_{2,E} \\ &\quad + c_{32} \inf_{\tau n = g} \Big(\| \sigma_h - \tau \|_{2,T_E} + c_{33} h_E \| \operatorname{div}_{\mathcal{T}} (\sigma_h - \tau) \|_{2,T_E} \Big), \end{aligned} \tag{4.23}$$

where in the infima in (4.22) and (4.23), "$\tau n = g_E$" and "$\tau n = g$" stand for all functions $\tau \in H(\mathrm{div};\omega_E)^2$ with $\tau n = g_E$ and $\tau n = g$ on E, respectively.

Proof. The proof follows [V1] and considers the $\mathcal{T}$-piecewise quadratic bubble function b_E for the edge $E \subset \partial T$; b_E vanishes on $\partial T \backslash E$ and is normalized by $\max b_E = 1$. The norms $\| \cdot \|_{2,E}$ and $\| b_E^{1/2} \cdot \|_{2,E}$ are equivalent, with equivalent constant c_{34}, on the finite dimensional space which $[\sigma_h \, n_E]$ belongs to. Let $E \in \mathcal{E}$ be an inner edge, $E = T_1 \cap T_2$ for some $T_1, T_2 \in \mathcal{T}$. Then using the extension operator $P \colon C(E) \to C(T_1 \cup T_2)$ of [V1] and

$$\int_{\omega_E} \big\{ \nabla (b_E P([\sigma_h \, n_E])) : \tau + b_E P([\sigma_h n_E]) \cdot \operatorname{div} \tau \big\} \, dx = 0, \tag{4.24}$$

(owing to integration by parts) we infer

$$\begin{aligned} \| [\sigma_h \, n_E] \|_{2,E}^2 &\le c_{34} \int_E b_E P([\sigma_h \, n_E]) \cdot [\sigma_h \, n_E] \, ds \\ &= c_{34} \int_{\omega_E} \big\{ \nabla (b_E P([\sigma_h \, n_E]))(\sigma_h - \tau) + b_E P([\sigma_h \, n_E]) \operatorname{div}_{\mathcal{T}} (\sigma_h - \tau) \big\} \, dx. \end{aligned} \tag{4.25}$$

Cauchy's inequality, the inverse estimate

$$\| \nabla (b_E P([\sigma_h \, n_E])) \|_{2,\omega_E} \le c_{35} h_E^{-1} \| b_E P([\sigma_h \, n_E]) \|_{2,\omega_E},$$

and

$$\| b_E P([\sigma_h \, n_E]) \|_{2,\omega_E} \le c_{36} h_E^{1/2} \| [\sigma_h \, n_E] \|_{2,E}$$

lead to

(4.26)
$$h_E^{1/2}\|[\sigma_h\, n_E]\|_{2,E} \le c_{34}c_{36}\big\{c_{35}\|\sigma_h-\tau\|_{2,\omega_E} + h_E\|\operatorname{div}_{\mathcal{T}}(\sigma_h-\tau)\|_{2,\omega_E}\big\}.$$

The proof of (4.21) is finished. The same arguments prove (4.22) for an edge on the boundary Γ_N as well by straightforward modifications. We omit the details which lead to (4.22) and mention in the proof of (4.23) only that instead of (4.25) we now study

$$\|\sigma_h\, n_E - g_E\|_{2,E}^2 \le c_{34}\int_E b_E P(\sigma_h\, n_E - g_E)\cdot(\sigma_h\, n_E - g_E)\, ds \tag{4.27}$$
$$\le c_{34}\int_E b_E P(\sigma_h\, n_E - g_E)\cdot(\sigma_h - \tau)\, n_E\, ds + c_{34}\int_E b_E P(\sigma_h\, n_E - g_E)\cdot(g - g_E)\, ds$$

and integrate by parts only in the second to last term. □

Proof of Theorem 4.1. We prove (4.3) by combining the estimates of Lemma 4.5 and 4.6. One needs to notice carefully that the conditions on the mesh allow, for any edge E even on the boundary, the choice of some endpoint $z \in \mathcal{N}$ such that E is involved in the patch-oriented estimates.

Notice that $v = u$ and $\tau = \sigma$ can be included and that $\mathcal{A}$ is Lipschitz continuous. To prove (4.4) it remains to observe (u is divergence free)

$$\|\operatorname{div} u_h\|_{2,T} \le \|\varepsilon_{\mathcal{T}}(u - u_h)\|_{2,T} \tag{4.28}$$

and estimate the term $\operatorname{div}\sigma_h + f_T$ following [V1]. To do so, consider the nonnegative cubic bubble function b_T on $T \in \mathcal{T}$ with $\max b_T = 1$. Then, the norms $\|\cdot\|_{2,T}$ and $\|b_T^{1/2}\cdot\|_{2,T}$ are equivalent on the finite dimensional space to which $\operatorname{div}\sigma_h + f_T$ belongs, and so

(4.29)
$$\begin{aligned}
&c_{37}^{-1}\|\operatorname{div}\sigma_h + f_T\|_{2,T}^2\\
&\quad\le \int_T b_T(\operatorname{div}\sigma_h + f_T)\cdot(\operatorname{div}(\sigma_h - \sigma) - f + f_T)\, dx\\
&\quad\le \int_T \nabla(b_T(\operatorname{div}\sigma_h + f_T)) : (\sigma_h - \sigma)\, dx + \|\operatorname{div}\sigma_h + f_T\|_{2,T}\|f - f_T\|_{2,T}\\
&\quad\le \|\nabla(b_T(\operatorname{div}\sigma_h + f_T))\|_{2,T}\|\sigma - \sigma_h\|_{2,T} + \|\operatorname{div}\sigma_h + f_T\|_{2,T}\|f - f_T\|_{2,T}.
\end{aligned}$$

Utilising the inverse estimate $\|\nabla(b_T(\operatorname{div}\sigma_h + f_T))\|_{2,T} \le c_{38}h_T^{-1}\|\operatorname{div}\sigma_h + f_T\|_{2,T}$, we obtain

(4.30)
$$h_T\|\operatorname{div}\sigma_h + f_T\|_{2,T} \le c_{37}c_{38}\|\sigma - \sigma_h\|_{2,T} + c_{37}h_T\|f - f_T\|_{2,T}. \quad \square$$

5. Averaging techniques for a posteriori error control

The ZZ-estimator [ZZ] and estimators often based on gradient recovery techniques can be justified on arbitrary shape-regular meshes by the refined estimate of the previous sections. The first result shows the reliability of low order conforming schemes; below we discuss estimators for the lowest order conforming-nonconforming finite element scheme (1.8)-(1.9).

Let $(u,p) \in H^1(\Omega)^2 \times L^2(\Omega)$ solve (1.1)-(1.3) and let $(u_h, p_h) \in \mathcal{W}$ with $k = 1$ satisfy (1.6)-(1.7). As in Stokes' problem suppose that $\mathcal{A}$ maps deviatoric strains onto deviatoric ones, i.e., $\operatorname{tr}\mathcal{A}(E) = 0$ for all $E \in \mathbb{R}^{2\times 2}_{sym}$ with $\operatorname{tr} E = 0$.

Let $\mathcal{N}^*$ denote the set of all $z \in \mathcal{N}$ which are either free nodes or belong to two aligned edges on Γ_D. If $z \in \mathcal{N}^* \cap \Gamma_D$ such that $\Gamma_D \cap \partial\omega_z = E_1 \cup E_2 =: \gamma_z$ for two aligned distinct edges $E_1, E_2 \in \mathcal{E}$ and such that there exists a third edge $E_3 \in \mathcal{E} \setminus \{E_1, E_2\}$ through z and parallel to the x_1-axis, then for some interior node $\zeta \in \mathcal{N}$ with $E_3 = \operatorname{conv}\{z, \zeta\} \subset \mathbb{R} \times \{z_2\}$ (e.g., $\zeta = \zeta(z)$ as in Theorem 2.1) we define $\Omega_z := \omega_z \cup \omega_\zeta$. In all remaining cases of $z \in \mathcal{N}^*$ we define $\Omega_z := \omega_z$ and $\gamma_z := \emptyset$.

Theorem 5.1 implies the estimate (1.13) and the reliability of (1.14).

Theorem 5.1. *Let* $(u, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ *solve* (1.1)-(1.3) *and* $(u_h, p_h) \in H^2(\mathcal{T})^2 \times L^2(\Omega)$ *solve* (1.6)-(1.7). *Suppose that* Γ_D *is connected and* $u_D \in C(\Gamma_D) \cap H^1(\Gamma_D)$ *is piecewise* $H^2(\Gamma_D)$ *in the sense that* $u_D|_{\gamma_z} \in H^2(\gamma_z)^2$ *for* $z \in \mathcal{N}^* \cap \Gamma_D$.

Then, there exists an h-independent constant $c_{39} > 0$ *that depends on the shape of the elements and the patches* $(\omega_z : z \in \mathcal{K})$ *only such that*

$$
\begin{aligned}
(5.1)\quad & \|\varepsilon_{\mathcal{T}}(u - u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\
& \leq c_{39} \Biggl(\sum_{z \in \mathcal{N}^*} \Bigl(\min_{\tau_z \in \mathcal{S}^1(\mathcal{T}|_{\hat\omega_z})^{2\times 2}} \bigl(\| \sigma_h - \tau_z \|^2_{L^2(\Omega_z)} + \| h_{\mathcal{E}}^{1/2} (g - \tau_z\, n) \|^2_{L^2(\Gamma_N \cap \partial\omega_z)} \bigr) \\
& \quad + \| h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} u_D / \partial s^2 \|^2_{L^2(\Gamma_D \cap \partial\omega_z)} \Bigr) \Biggr)^{1/2} + c_7 \Biggl(\sum_{z \in \mathcal{K}} \min_{f_z \in \mathbb{R}^2} \| h_{\mathcal{T}} (f - f_z) \|^2_{L^2(\Omega_z)} \Biggr)^{1/2}.
\end{aligned}
$$

Moreover, suppose $g \in C(\Gamma_N)^2$ *is* $\mathcal{E}$*-piecewise in* $H^1(\Gamma_N)^2$, *i.e.,* $g|_E \in H^1(E)^2$ *for* $E \in \mathcal{E}$ *with* $E \subset \overline{\Gamma}_N$. *Assume that* $\sigma_h^* \in \mathcal{S}^1(\mathcal{T})^{2\times 2}$ *satisfies* $g(z) = \sigma_h^*(z) n_E(z)$ *for each endpoint* z *of an edge* E *on* $\overline{\Gamma}_N$. *Then,*

$$
\begin{aligned}
(5.2)\quad & \\
& \|\varepsilon_{\mathcal{T}}(u - u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\
& \quad \leq c_{39} \Bigl(\| \sigma_h - \sigma_h^* \|_{L^2(\Omega)} + \| h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} u_D / \partial s^2 \|_{L^2(\Gamma_D)} + \| h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}} g / \partial s \|_{L^2(\Gamma_D)} \Bigr) \\
& \qquad + c_7 \Bigl(\sum_{z \in \mathcal{K}} \min_{f_z \in \mathbb{R}^2} \| h_{\mathcal{T}} (f - f_z) \|^2_{L^2(\Omega_z)} \Bigr)^{1/2}.
\end{aligned}
$$

Remarks 5.1. (i) If g is $\mathcal{E}$-piecewise in $H^2(\Gamma_N)^2$, the perturbation term $\|h_{\mathcal{E}}^{3/2} \partial g / \partial s\|_2$ in (5.2) can be improved to $\| h_{\mathcal{E}}^{5/2} \partial^2_{\mathcal{E}} g / \partial s^2 \|_2$.

(ii) The discrete Neumann boundary conditions on the nonsymmetric σ_h^* can be satisfied exactly even at corner points (with two different normals); see (6.2.i)-(6.2.ii) below for details.

(iii) The choice of the remaining degrees of freedom in σ_h^* is arbitrary: any averaging scheme is reliable. The efficiency of the averaging process is a different topic and has to be checked separately.

(iv) It is interesting to notice that the higher order terms in the reliability estimate depend on the smoothness of the data while the the higher order terms in the efficiency estimate depend on the smoothness of the exact solution.

Lemma 5.1. *For any* $z \in \mathcal{N}^*$ *there exists an* h*-independent constant* $c_{40} > 0$ *such that, for all* $v_h \in \mathcal{W}$ *(for* $k = 1$*),*

$$\begin{aligned} &\Big(\sum_{E \sqsubset \Omega_z} h_E \, \| [\partial v_h / \partial s] \|^2_{L^2(E)} \Big)^{1/2} \\ &\quad \le c_{40} \Big(\| \, h_{\mathcal{E}}^{3/2} \partial^2 (u_D - \mathcal{I} u_D) / \partial s^2 \, \|_{L^2(\Gamma_D \cap \partial\omega_z)} + \min_{V_h, \tau_h} \| \, \varepsilon(v_h - V_h) - \tau_h \, \|_{L^2(\Omega_z)} \Big). \end{aligned} \tag{5.3}$$

In the minimum of (5.3), V_h *is arbitrary in* $\mathcal{S}^1_D(\mathcal{T}|_{\Omega_z}) := \{ V \in C(\Omega_z)^2 \cap \mathcal{S}^1(\mathcal{T}|_{\Omega_z})^2 : V = 0 \text{ on } \gamma_z \}$ *and* τ_h *is arbitrary in* $\mathcal{S}^1(\mathcal{T}|_{\Omega_z})^{2\times 2} = \{ (\tau_{jk}) \in C(\Omega_z)^{2\times 2} : \forall T \in \mathcal{T},\ \tau_{jk}|_T \in P_1(T) \text{ provided } T \subset \overline{\Omega_z} \}$.

Proof. In the first step of the proof we consider $u_D = 0$ and show that the right-hand side of (5.3) is a norm on the space $\mathcal{W}_z / \mathcal{C}_z$ from (4.10). To check definiteness, suppose that $v_h \in \mathcal{W}_z$, $V_h \in \mathcal{S}^1_D(\mathcal{T}|_{\Omega_z})$, and $\tau_h \in \mathcal{S}^1(\mathcal{T}|_{\Omega_z})^{2\times 2}$ satisfies $\tau_h = \varepsilon_{\mathcal{T}}(v_v - V_h)$ and so the $\mathcal{T}$-piecewise constant function $\varepsilon_{\mathcal{T}}(v_v - V_h)$ is continuous, where it is constant on Ω_z. Thus we can find an affine mapping $Ax + b$ such that $r(x) := v_h(x) - V_h(x) - Ax - b$ satisfies $\varepsilon_{\mathcal{T}}(r) = 0$ on Ω_z, i.e., $r|_T \in \mathcal{RM}(T)$ for all $T \in \mathcal{T}$ with $T \subset \overline{\Omega_z}$. The compatibility conditions on the edges for v_h imply the same for r and so Lemma 4.4 shows $r \in \mathcal{RM}(\omega_z)$ if z is an interior node or if no edge $E \sqsubset \omega_z$ is parallel to the x_1-axis. In the case that $z \in \Gamma_D$ we have $\Gamma_D \cap \partial\omega_z = E_1 \cup E_2$ for two aligned distinct edges $E_1, E_2 \in \mathcal{E}$. If they are parallel to the x_1-axis we have $r \in \mathcal{RM}(\omega_z)$ from Lemma 4.4. If there exists another edge $E_3 = \operatorname{conv}\{z, \zeta\} \subset \mathbb{R} \times \{z_2\}$ we have $r \in \mathcal{RM}(\omega_\zeta)$. Since r is the same rigid body motion on both elements joining E_3 we find $r \in \mathcal{RM}(\Omega_z)$ using Lemma 4.2. The interior jumps in the left-hand side of (5.3) vanish in any case. The boundary contributions vanish as well since the affine function $v_h - V_h$ vanishes at the midpoint m_j of E_j for $j = 1, 2$ and so on the straight line through γ_z.

We have seen that the left-hand side of (5.3) vanishes if the right-hand side does. A compactness and a scaling argument concludes the proof of the lemma if $u_D = 0$.

In the second part of the proof, we have $z \in \Gamma_D \cap E_1 \cap E_2$ for two aligned distinct edges $E_j = \operatorname{conv}\{z, \zeta_j\} \subset \Gamma_D$, $j = 1, 2$, and allow $u_D \neq 0$. Extend $\mathcal{I} u_D$ (prescribing remaining nodal values) to $\mathcal{I} u_D \in \mathcal{S}^1(\mathcal{T})^2$ and define the affine functions

$$a_1(x) := \tfrac{(x-\zeta_1)\cdot(\zeta_2-\zeta_1)}{|\zeta_2-\zeta_1|^2} \, e_1 \cdot u_D(\zeta_2) + \tfrac{(\zeta_2-x)\cdot(\zeta_2-\zeta_1)}{|\zeta_2-\zeta_1|^2} \, e_1 \cdot u_D(\zeta_1), \tag{5.4}$$

$$a_2(x) := \tfrac{(x-m_1)\cdot(m_2-m_1)}{h_{E_2}|m_2-m_1|^2} \int_{E_2} e_2 \cdot u_D \, ds + \tfrac{(m_2-x)\cdot(m_2-m_1)}{h_{E_1}|m_2-m_1|^2} \int_{E_1} e_2 \cdot u_D \, ds. \tag{5.5}$$

The discrete function

$$w_h := v_h - (e_1 \cdot \mathcal{I} u_D) \, e_1 - a_2 \, e_2 \tag{5.6}$$

satisfies the compatibility conditions for homogeneous Dirichlet data considered in the first step of this proof. Hence, we obtain in particular, for $j = 1, 2$,

$$h_{E_j}^{1/2} \, \| \partial w_h / \partial s \|_{2, E_j} \le c_{22} \min_{V_h, \tau_h} \| \, \varepsilon(w_h - V_h) - \tau_h \, \|_{2, \Omega_z}. \tag{5.7}$$

Notice that $\tau_h = \varepsilon(a_1 \, e_1 + a_2 \, e_2)$ is constant and so allowed in the minimum in (5.7). Hence we may replace w_h in (5.7) by $\tilde{w}_h := w_h + a_1 \, e_1 + a_2 \, e_2 - V_h$ where V_h is such that $(a_1 - e_1 \cdot \mathcal{I} u_D) \, e_1 - V_h$ vanishes at all nodes different from z ($a_1 - e_1 \cdot \mathcal{I} u_D$ has zeros ζ_1, ζ_2 by construction). This shows $\tilde{w}_h = v_h + (a_1(z) - e_1 \cdot u_D(z)) \, \varphi_z \, e_1$ with φ_z being the nodal basis function at z. From $\| \, \nabla \varphi_z \, \|_2 \le c_{41}$ and a triangle inequality we deduce

$$\begin{aligned}&\min_{V_h,\tau_h} \| \,\varepsilon(w_h - V_h) - \tau_h \,\|_{2,\Omega_z} \\ &\qquad \le \min_{V_h,\tau_h} \| \,\varepsilon(v_h - V_h) - \tau_h \,\|_{2,\Omega_z} + c_{41}|a_1(z) - e_1 \cdot u_D(z)|.\end{aligned} \tag{5.8}$$

As in (4.15), we infer

$$\begin{aligned} h_{E_j}^{1/2} \| \,\partial(u_D - v_h)/\partial s \,\|_{2,E_j} &\le h_{E_j}^{1/2} \| \,\partial w_h/\partial s \,\|_{2,E_j} \\ &+ h_{E_j}^{1/2} \| \,\partial e_1 \cdot (u_D - \mathcal{I}u_D)/\partial s \,\|_{2,E_j} + h_{E_j}^{1/2} \| \,\partial(a_2 - e_2 \cdot u_D)/\partial s \,\|_{2,E_j}.\end{aligned} \tag{5.9}$$

Standard arguments in one dimension show

$$\begin{aligned}&|a_1(z) - e_1 \cdot u_D(z)| + h_{E_j}^{1/2} \| \,\partial e_1 \cdot (u_D - \mathcal{I}u_D)/\partial s \,\|_{2,E_j} \\ &\quad + h_{E_j}^{1/2} \| \,\partial(a_2 - e_2 \cdot u_D)/\partial s \,\|_{2,E_j} \le c_{42} \| \,h_{\mathcal{E}}^{3/2} \partial^2 (u_D - \mathcal{I}u_D)/\partial s^2 \,\|_{2,E_1 \cup E_2}.\end{aligned} \tag{5.10}$$

Combining (5.7)-(5.10) we conclude that (5.3) holds. ☐

Proof of Theorem 5.1. Some terms in Theorem 2.1 simplify because u_h is $\mathcal{T}$-piecewise affine. For instance, $\operatorname{div}_{\mathcal{T}} \sigma_h = 0$ and $\operatorname{div}_{\mathcal{T}} u_h = 0$. The term $\inf_v \|\nabla_{\mathcal{T}}(u_h - v)\|_2$ can be bounded as in (2.7) of Remark 2.1. The remaining edge contributions are estimated with Lemmas 4.6 and 5.1 (with $V_h = 0$) in a manner similar to the proof of Theorem 4.1. In this way, we obtain

$$\begin{aligned}&\|\varepsilon_{\mathcal{T}}(u - u_h)\|_2^2 + \|p - p_h\|_2^2 \le c_7^2 \sum_{z \in \mathcal{K}} \min_{f_z \in \mathbb{R}^2} \| \,h_{\mathcal{T}}(f - f_z) \,\|_{2,\Omega_z}^2 \\ &\quad + c_{43} \sum_{z \in \mathcal{N}^*} \Big(\min_{\tau_z, \tilde{\tau}_z \in \mathcal{S}^1(\mathcal{T}|_{\hat{\omega}_z})^{2\times 2}} (\| \,\sigma_h - \tau_z \,\|_{2,\Omega_z}^2 + h_z^2 \| \,\operatorname{div}_{\mathcal{T}}(\sigma_h - \tau_z) \,\|_{2,\Omega_z}^2 \\ &+ \| \,\varepsilon_{\mathcal{T}}(u_h) - \tilde{\tau}_z \,\|_{2,\Omega_z}^2 + \| \,h_{\mathcal{E}}^{1/2}(g - \tau_z \, n) \,\|_{2,\Gamma_N \cap \partial\omega_z}^2 + \| \,h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D/\partial s^2 \,\|_{2,\Gamma_D \cap \partial\omega_z}^2 \Big).\end{aligned} \tag{5.11}$$

An inverse estimate shows for one summand in (5.11) that

$$h_z \| \,\operatorname{div}_{\mathcal{T}}(\sigma_h - \tau_z) \,\|_{2,\Omega_z} \le \| \,\sigma_h - \tau_z \,\|_{2,\Omega_z}. \tag{5.12}$$

The equivalence of norms and a scaling argument for the h_z-independent constant $c_{45} > 0$ shows for $\mathcal{T}$-piecewise constants $\mathcal{L}_0(\mathcal{T}|_{\Omega_z})$ and the continuous $\mathcal{T}$-piecewise affine functions $\mathcal{S}^1(\mathcal{T}|_{\Omega_z})$ that, for all $\alpha \in \mathcal{L}_0(\mathcal{T}|_{\Omega_z})^{2\times 2}$,

$$c_{45} \min_{\gamma \in \mathbb{R}_{sym}^{2\times 2}} \| \,\alpha - \gamma \,\|_{2,\Omega_z} \le \min_{\beta \in \mathcal{S}^1(\mathcal{T}|_{\Omega_z})_{sym}^{2\times 2}} \| \,\alpha - \beta \,\|_{2,\Omega_z} \le \min_{\gamma \in \mathbb{R}_{sym}^{2\times 2}} \| \,\alpha - \gamma \,\|_{2,\Omega_z}. \tag{5.13}$$

A Cauchy inequality and (1.4) reveal that for all $\alpha, \beta \in \mathbb{R}_{sym}^{2\times 2}$,

$$c_1 \,|\alpha - \beta| \le |\mathcal{A}(\alpha) - \mathcal{A}(\beta)| \le c_2 \,|\alpha - \beta|. \tag{5.14}$$

Owing to monotonicity arguments, the mapping $\mathcal{A}$ is a bijection on $\mathbb{R}_{sym}^{2\times 2}$ and so we deduce from (5.13)-(5.14) for the piecewise constant $\varepsilon_{\mathcal{T}}(u_h)$ that

$$\begin{aligned}&c_1 \min_{\tilde{\tau}_z \in \mathcal{S}^1(\mathcal{T}|_{\Omega_z})_{sym}^{2\times 2}} \| \,\varepsilon_{\mathcal{T}}(u_h) - \tilde{\tau}_z \,\|_{2,\Omega_z} \le c_1 \min_{\gamma \in \mathbb{R}_{sym}^{2\times 2}} \| \,\varepsilon_{\mathcal{T}}(u_h) - \gamma \,\|_{2,\Omega_z} \\ &\quad \le \min_{\gamma \in \mathbb{R}_{sym}^{2\times 2}} \| \,\mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) - \mathcal{A}(\gamma) \,\|_{2,\Omega_z} = \min_{\gamma \in \mathbb{R}_{sym}^{2\times 2}} \| \,\mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) - \gamma \,\|_{2,\Omega_z} \\ &\quad \le c_{45}^{-1} \min_{\tau_h \in \mathcal{S}^1(\mathcal{T}|_{\Omega_z})_{sym}^{2\times 2}} \| \,\mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) - \tau_h \,\|_{2,\Omega_z} \\ &= c_{45}^{-1} \min_{\tau_h \in \mathcal{S}^1(\mathcal{T}|_{\Omega_z})_{sym}^{2\times 2}} \| \,\operatorname{dev}(\sigma_h - \tau_h) \,\|_{2,\Omega_z} \le c_{45}^{-1} \min_{\tau_h \in \mathcal{S}^1(\mathcal{T}|_{\Omega_z})_{sym}^{2\times 2}} \| \,\sigma_h - \tau_h \,\|_{2,\Omega_z}.\end{aligned} \tag{5.15}$$

Here we used $\operatorname{dev} A = A - \operatorname{tr}(A)/2\, I$ and, in the second to last step, that $\mathcal{A}(\varepsilon_{\mathcal{T}}(u_h)) = \sigma_h - p_h I$ is pointwise almost everywhere deviatoric (since $0 = \operatorname{div}_{\mathcal{T}} u = \operatorname{tr} \varepsilon_{\mathcal{T}}(u_h)$) and so is the optimal τ_h.

A combination of (5.11), (5.12), and (5.15) concludes the proof of Theorem 5.1. $\square$

6. NUMERICAL EXPERIMENTS

In this section we report on the numerical performance of the two a posteriori estimates established in this paper (Theorems 2.1 and 5.1) and give two strategies to refine a given mesh automatically.

The proposed estimator in Theorem 5.1 is based on a function $\sigma_h^* \in \mathcal{S}^1(\mathcal{T})^{2\times 2}$ which satisfies $g(z) = \sigma_h^*(z) n_E(z)$ for each endpoint z of an edge E on $\overline{\Gamma}_N$. We define

$$\sigma_h^* := \sum_{z\in\mathcal{N}} \mathcal{I}_z \sigma_h\, \varphi_z, \tag{6.1}$$

where $\mathcal{I} : L^2(\Omega)^{2\times 2} \to \mathcal{S}^1(\mathcal{T})^{2\times 2}$ for $z \in \mathcal{N} \setminus \overline{\Gamma}_N$ is, with $\fint_{\omega_z} \sigma_h\, dx$ denoting the integral mean of σ_h over Ω_z,

$$\mathcal{I}_z \sigma_h := \fint_{\omega_z} \sigma_h\, dx\,.$$

For $z \in \overline{\Gamma}_N$ we distinguish between the following cases (i) and (ii) to fulfill the discrete Neumann condition $g(z) = \sigma_h^*(z) n_E$ at z.

(6.2.i) In case $z \in E_1 \cap E_2$ for two distinct edges $E_1, E_2 \subset \Gamma_N$ with linearly independent outer unit normals n_{E_1} and n_{E_2} on E_1 and E_2, respectively, we choose $\mathcal{I}_z\sigma_h$ to be the unique solution $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$ of the linear system

$$\begin{pmatrix} n_{1,E_1} & n_{2,E_1} & 0 & 0 \\ 0 & 0 & n_{1,E_1} & n_{2,E_1} \\ n_{1,E_2} & n_{2,E_2} & 0 & 0 \\ 0 & 0 & n_{1,E_2} & n_{2,E_2} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} g_1|_{E_1}(z) \\ g_2|_{E_1}(z) \\ g_1|_{E_2}(z) \\ g_2|_{E_2}(z) \end{pmatrix}.$$

(6.2.ii) In the remaining cases $z \in E_1 \cap \Gamma_D$ or $z \in E_1 \cap E_2$ with two parallel outer unit normals n_{E_1}, n_{E_2}, we choose t_{E_1} to be the unit tangent to Ω at z that is perpendicular to n_{E_1}, and let $\mathcal{I}_z\sigma_h$ be the solution $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$ of the uniquely solvable system

$$\begin{pmatrix} n_{1,E_1} & n_{2,E_1} & 0 & 0 \\ 0 & 0 & n_{1,E_1} & n_{2,E_1} \\ t_{1,E_2} & t_{2,E_2} & 0 & 0 \\ 0 & 0 & t_{1,E_2} & t_{2,E_2} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} g_1|_{E_1}(z) \\ g_2|_{E_1}(z) \\ \left(\fint_{\omega_z} (\sigma_{h,11}, \sigma_{h,12})\, dx\right) t_{E_1} \\ \left(\fint_{\omega_z} (\sigma_{h,21}, \sigma_{h,22})\, dx\right) t_{E_1} \end{pmatrix}.$$

In the error indicator $\eta_{Z,T}$ this amounts to, for each $T \in \mathcal{T}$,

$$\eta_{Z,T} := \|\sigma_h - \sigma_h^*\|_{L^2(T)}\,. \tag{6.3}$$

Since the symmetric formulation with $P_1^2 \times P_0$-finite elements is unstable for the conforming and the nonconforming case, we considered the conforming-nonconforming scheme from (1.8)-(1.9).

The implementation is performed on triangles in Matlab in the spirit of [ACF] using analytic formulae in the calculation of the stiffness matrix. Since $\mathcal{A}(\alpha) = 2\mu\alpha$

in (1.2) is a linear operator in our examples, the linear system of equations can be solved directly. In order to approximate the right-hand side for a given function $g \in L^2(\Gamma_N)^2$, we compute $\int_{\Gamma_N} g v_h \, ds$ via a three-point Gaussian quadrature rule on any edge E. The Dirichlet boundary conditions are implemented as in Remark 2.1(viii).

In the comparison of uniform mesh-refinement with adaptive refinement techniques we use the following adaptive Algorithms (A_R) and (A_Z). Both algorithms are different in the error indicators only.

Algorithm 6.1. (A_R) resp. (A_Z) (a) Start with a coarse mesh $\mathcal{T}_0$, $k = 0$.

(b) Solve the discrete problem with respect to the actual mesh $\mathcal{T}_k$.

(c) Compute η_T for all $T \in \mathcal{T}_k$, where, for (A_R),

$$\eta_T^2 = \eta_{R,T}^2 := h_T^4 \|\nabla f\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E} \wedge E \subset \partial T} h_E \big(\|[\Sigma n_E]\|_{L^2(E)}^2 + \|[2\mu \partial U/\partial s]\|_{L^2(E)}^2 \big)$$

and, for (A_Z), with an averaged function σ_h^* of the discrete stress field σ_h as in (6.1),

$$\eta_T = \eta_{Z,T} := \|\sigma_h - \sigma_h^*\|_{L^2(T)}.$$

(d) Compute a given stopping criterion and decide to terminate or to go to (e).

(e) Refine the element T (*red* refinement) provided,

$$\frac{1}{2} \max_{T' \in \mathcal{T}_k} \eta_{T'} \leq \eta_T.$$

(f) Refine further elements (*red-green-blue* refinement) to avoid hanging nodes. Define the resulting mesh as the actual mesh $\mathcal{T}_{k+1}$, update k and go to (b).

Remarks 6.1. (i) Details on the so-called *red-green-blue* refinement strategies can be found in [V2].

(ii) Stopping criteria for termination in step (d) can be based on $\eta_{\mathcal{T}} := (\sum_{T \in \mathcal{T}} \eta_T^2)^{1/2}$. For instance, we can terminate in (d) if $\eta_{\mathcal{T}_k}$ is less then a certain percentage of $\eta_{\mathcal{T}_0}$. If f is sufficiently smooth and the mesh is sufficiently fine, η_Z might be regarded as a very good guess for the exact error.

(iii) Utilising the initial mesh displayed in Figure 2, Algorithm (A_R) (resp. (A_Z)) generates a sequence of meshes which satisfy the assumptions of Section 4.

Example 6.1. The first numerical example for the Stokes problem is on the L-shaped domain $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ with $f = 0$ and $\mathcal{A}(\alpha) = 2\alpha$ [V2]. The geometries of Ω, Γ_D and Γ_N are depicted in Figure 2, where $\mathcal{T}_0$ is shown as well. The boundary values u_D, g are taken from the exact solution (u, p) which reads, in polar coordinates for $\alpha = 856399/1572864 \approx .54448$, $\omega = 3\pi/2$,

$$\begin{aligned} u(r,\varphi) =& r^\alpha \left((1+\alpha)\big(\sin(\varphi), -\cos(\varphi)\big) w(\varphi) + \big(\cos(\varphi), \sin(\varphi)\big) w_\varphi(\varphi) \right), \\ p(r,\varphi) =& - r^{\alpha-1} \left((1+\alpha)^2 w_\varphi(\varphi) + w_{\varphi\varphi\varphi}(\varphi) \right) / (1-\alpha), \\ w(\varphi) =& \big(\sin((1+\alpha)\varphi)\cos(\alpha\omega)\big)/(1+\alpha) - \cos\big((1+\alpha)\varphi\big) \\ & - \big(\sin((1-\alpha)\varphi)\cos(\alpha\omega)\big)/(1-\alpha) + \cos((1-\alpha)\varphi). \end{aligned}$$

A plot of the mesh $\mathcal{T}_9$ generated by Algorithm (A_R) as some magnified detail near the re-entrant corner (zoom of $(-0.1, 0.1)^2$) is given in Figure 3 and shows a high refinement of the mesh near the singularity at the origin.

The resulting improvement of the convergence is outlined in Figure 4, where the error $e_N := \|2\varepsilon(u - u_h) - (p - p_h)I\|_{L^2(\Omega)}$ is plotted versus the number of degrees of

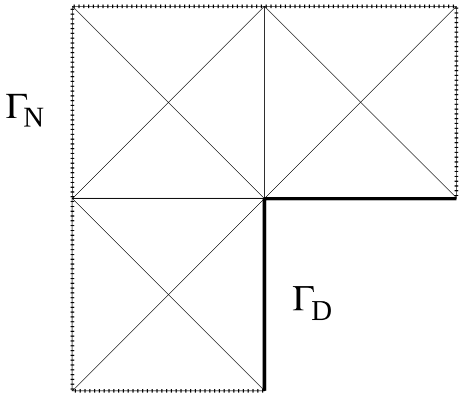

FIGURE 2. Initial mesh $\mathcal{T}_0$ of the domain Ω and boundary Γ_D, resp. Γ_N in Example 6.1.

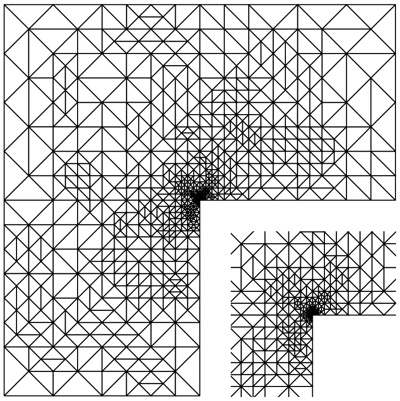

FIGURE 3. Mesh $\mathcal{T}_9$ and magnified detail at the re-entrant corner in Example 6.1.

freedom N on a log/log-scale. (A slope $-1/2$ in Figures 4 and 5 corresponds to an experimental convergence rate 1 owing to $N \propto h^{-2}$ in two dimensions.) Figure 4 shows the convergence rates for the uniform refinement in comparison with the mentioned adaptive Algorithm (A_R) or (A_Z); eta_R (eta_Z adapted) corresponds to η_R for a sequence of meshes generated by Algorithm (A_Z). According to the re-entrant corner, the uniform refinement yields a convergence rate of approximately 0.544 which coincides with the theoretically expected rate. The adaptive mesh-refining Algorithms (A_Z) and (A_R) improve this experimental convergence order to 1 which is expected to be optimal for $\mathcal{V}_1 \times \mathcal{V}_2$-elements.

In Table 1 we displayed the errors and the bounds for different meshes computed with uniform refinements. Here, N is the number of unknowns, e_N is the error-norm (evaluated by using a 7-point Gauss quadrature formula of order 6 on each element), and η_R (resp. η_Z) is the computed upper bound of the a posteriori estimate. From Table 1 and Figure 4 we observe that the quotients η_R/e_N (resp. η_Z/e_N) remain bounded from above in agreement with our theoretical results. Moreover, the quotient of overestimation η_R/e_N is approximately 2.5 for uniform refinements (Table 1) and becomes slightly larger (≈ 3.3) for the adaptive strategies (A_R) and (A_Z). The error estimator η_Z estimates the error asymptotically exactly for uniform refinements and both adaptive strategies which could result from local symmetries

TABLE 1. Errors e_N and error estimates η_R, η_Z for uniform meshes of Example 6.1.

N	e_N	η_R	η_R/e_N	η_Z	η_Z/e_N
45	5.1346	9.4007	1.8308	4.8630	0.9471
161	4.0600	8.7255	2.1491	4.0150	0.9889
609	2.9155	6.8511	2.3499	2.8915	0.9918
2369	2.0357	4.9026	2.4082	2.0232	0.9939
9345	1.4075	3.4146	2.4260	1.4003	0.9949
37121	0.9690	2.3575	2.4327	0.9646	0.9954
147969	0.6658	1.6219	2.4356	0.6629	0.9956

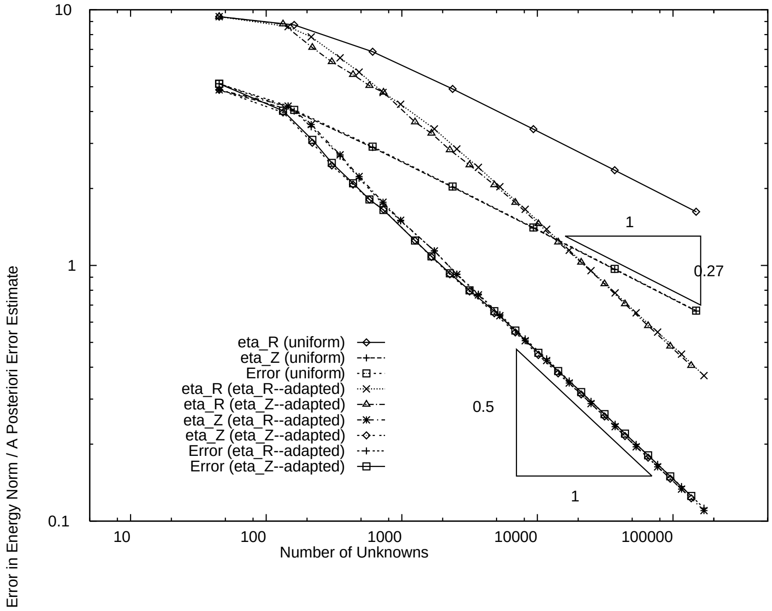

FIGURE 4. Errors e_N vs N for uniform and adaptive meshes of Example 6.1.

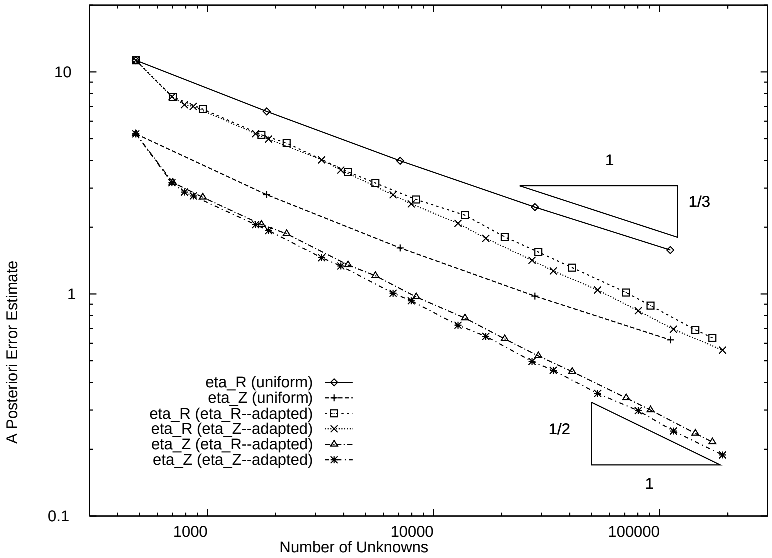

FIGURE 5. A posteriori error indicator η_R and η_Z vs N for uniform and adaptive meshes of Example 6.2.

in the mesh and superconvergence. Preasymptotically we obtain by Algorithm (A_Z) meshes with slightly smaller errors e_N and quantities η_R and η_Z. This numerical example supports the assertion that the estimates are reliable and efficient.

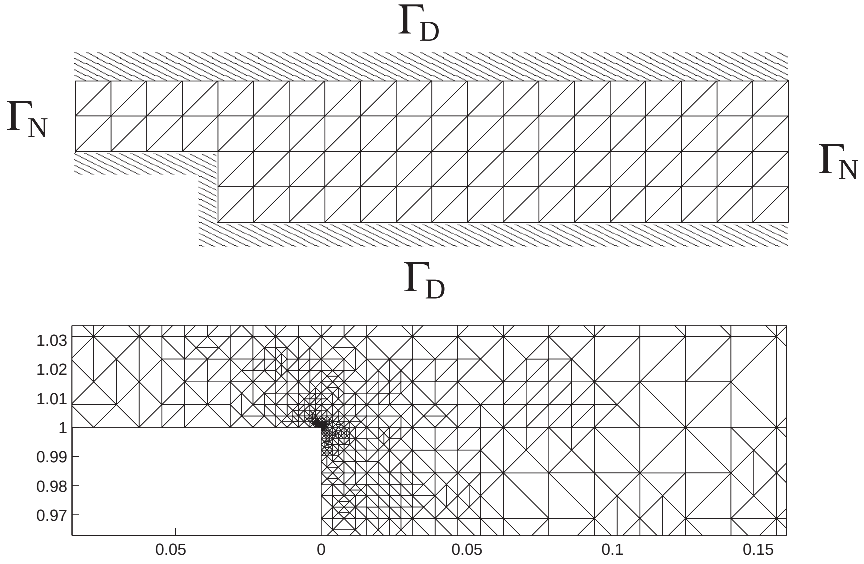

Figure 6. Mesh $\mathcal{T}_0$ and magnified detail at re-entrant corner of Mesh $\mathcal{T}_9$ of Example 6.2.

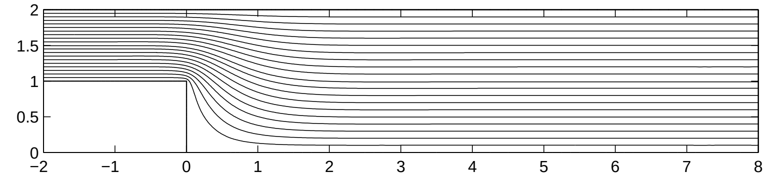

Figure 7. Approximation to streamlines based on $\mathcal{T}_{12}$ of Example 6.2.

Example 6.2. Finally, we report on a benchmark example. Here, we consider the backward facing step with initial mesh and a magnified detail at re-entrant corner after nine iterations using Algorithm (A_Z) as plotted in Figure 6 (cf. [BW]). Here, we choose $\mathcal{A}(\alpha) = \alpha/50$. Neumann boundary conditions are $g := (68, (2y-3)/1100)$ for $x = -2$, $1 \leq y \leq 2$ and $g := (17, (1-y)/4400)$ for $x = 8$, $0 \leq y \leq 2$. On the remaining boundary we define homogeneous Dirichlet conditions.

In Figure 5 we plot the a posteriori error estimates η_R and η_Z for uniform and adaptive meshes. The convergence rate of η_R and η_Z is approximately 1 for the adaptive meshes and 0.66 for uniform meshes. As expected, the a posteriori estimates η_R and η_Z decrease faster for adaptively refined meshes with an optimal convergence rate. If we supposed that η_z is almost exact, then the error estimator η_R would overestimate by a factor ≈ 3.1. The quantities η_R and η_Z are smaller on meshes obtained by Algorithm (A_Z) than Algorithm (A_R).

The approximate streamlines based on $\mathcal{T}_{12}$ and Algorithm (A_Z) are plotted in Figure 7 in agreement with corresponding pictures in the literature.

In all examples, the meshes are highly nonuniform and the experimental convergence rates of the true and estimated error have been improved to the optimal order which supports the assertion that our adaptive schemes are very useful in practise.

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