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MERGING THE BRAMBLE-PASCIAK-STEINBACH AND THE CROUZEIX-THOMÉE CRITERION FOR H^1-STABILITY OF THE L^2-PROJECTION ONTO FINITE ELEMENT SPACES

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Abstract. Suppose $\mathcal{S} \subset H^1(\Omega)$ is a finite-dimensional linear space based on a triangulation $\mathcal{T}$ of a domain Ω, and let $\Pi : L^2(\Omega) \to L^2(\Omega)$ denote the L^2-projection onto $\mathcal{S}$. Provided the mass matrix of each element $T \in \mathcal{T}$ and the surrounding mesh-sizes obey the inequalities due to Bramble, Pasciak, and Steinbach or that neighboring element-sizes obey the global growth-condition due to Crouzeix and Thomée, Π is H^1-stable: For all $u \in H^1(\Omega)$ we have $\|\Pi u\|_{H^1(\Omega)} \le C\,\|u\|_{H^1(\Omega)}$ with a constant C that is independent of, e.g., the dimension of $\mathcal{S}$.

This paper provides a more flexible version of the Bramble-Pasciak-Steinbach criterion for H^1-stability on an abstract level. In its general version, (i) the criterion is applicable to *all* kind of finite element spaces and yields, in particular, H^1-stability for nonconforming schemes on arbitrary (shape-regular) meshes; (ii) it is *weaker than* (i.e., implied by) *either* the Bramble-Pasciak-Steinbach *or* the Crouzeix-Thomée criterion for regular triangulations into triangles; (iii) it guarantees H^1-stability of Π a priori for a class of *adaptively-refined* triangulations into right isosceles triangles.

1. The L^2-projection in a finite element space

Suppose the bounded Lipschitz domain Ω in $\mathbb{R}^d$ is partitioned into a triangulation $\mathcal{T}$, i.e., $\overline{\Omega} = \bigcup \mathcal{T}$ for a finite set $\mathcal{T}$ of elements T which are closed and whose interiors are Lipschitz domains. The intersection of two distinct elements has zero d-dimensional Lebesgue measure. To describe nonconforming finite elements, let H be a closed subset of $H^1(\mathcal{T})$,

$$H^1_0(\Omega) \subseteq H \subset H^1(\mathcal{T}) := \{u \in L^2(\Omega) : \forall T \in \mathcal{T},\ u|_T \in H^1(T)\}, \tag{1}$$

closed with respect to the semi-norm $\|\nabla_{\mathcal{T}} \cdot\|$, where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm and $\nabla_{\mathcal{T}}$ is the $\mathcal{T}$-piecewise action of the gradient ∇ (different from the distributional gradient for discontinuous arguments). For instance, in the conforming setting, the choice of $H = H^1_0(\Omega)$ or $H = H^1(\Omega)$ is a typical example.

Suppose that $\mathcal{S} \subset H$ is an n-dimensional subspace with a (not necessarily nodal) basis $(\varphi_1, \varphi_2, \dots, \varphi_n)$, and let Π denote the $L^2(\Omega)$-projection defined, for all $u \in H$,

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by

$$\Pi\, u \in \mathcal{S} \quad \text{and} \quad \int_\Omega (u - \Pi\, u)\varphi_j \, dx = 0 \quad \text{for all } j = 1, \ldots, n. \tag{2}$$

In this context, the L^2-projection Π is called H^1-stable if there exists a constant $c_1 > 0$ with

$$\|\nabla_\mathcal{T}\, \Pi\, u\| \le c_1 \|\nabla_\mathcal{T}\, u\| \quad \text{for all } u \in H. \tag{3}$$

Two sets of parameters, the n positive parameters $(d_1, d_2, \ldots, d_n)$ and the $\mathcal{T}$-piecewise constant weight $h_\mathcal{T}$, defined on $T \in \mathcal{T}$ by $h_T > 0$, will provide the link between the triangulation $\mathcal{T}$ and the discrete space $\mathcal{S}$. Their choice is arbitrary up to the severe restriction of inequality (7) below.

To verify H^1-stability of the L^2-projection (3) we suppose that there exist a (possibly nonlinear) mapping $P : H \to \mathcal{S}$ and a constant $c_2 > 0$ that satisfy, for all $u \in H$,

$$\|\nabla_\mathcal{T}\, P(u)\| + \|h_\mathcal{T}^{-1}(u - P(u))\| \le c_2 \|\nabla_\mathcal{T}\, u\|. \tag{4}$$

Remark 1. In Sections 4, 5, and 6, h_T will be the element-size and d_ℓ a measure for the size of supp φ_ℓ.

Remark 2. Approximation operators which satisfy (4) for $h_T = \operatorname{diam}(T)$ can be found in [Ca, CF, Cl].

2. Mass matrices and two inequalities

To define the mass matrix for a given $T \in \mathcal{T}$, let $\ell(T,1), \ell(T,2), \ldots, \ell(T, m(T))$ denote exactly those indices of basis functions whose restrictions $\psi_{T,j} := \varphi_{\ell(T,j)}|_T \in H^1(T)$, $1 \le j \le m(t)$, on T are nonzero. Then the shape functions $(\psi_{T,j} : j = 1, \ldots, m(T))$ on T satisfy an inverse inequality (by equivalence of norms),

$$\Big\| \sum_{j=1}^{m(T)} \xi_j \, \nabla \psi_{T,j} \Big\|_{L^2(T)} \le c_3 \, h_T^{-1} \Big\| \sum_{j=1}^{m(T)} \xi_j \, \psi_{T,j} \Big\|_{L^2(T)} \quad \text{for all } (\xi_1, \ldots, \xi_{m(T)}) \in \mathbb{R}^{m(T)}. \tag{5}$$

The (local) $m(T) \times m(T)$-dimensional mass matrix $M(T)$ and the diagonal matrix $\Lambda(T)$,

$$\Lambda(T)_{jk} = \frac{h_T}{d_{\ell(T,j)}} \, \delta_{jk} \quad \text{and} \quad M(T)_{jk} = \int_T \psi_{T,j} \psi_{T,k} \, dx \quad \text{for all } j,k = 1, \ldots, m(T), \tag{6}$$

($\delta_{jk} \in \{0,1\}$ denotes Kronecker's symbol) are supposed to satisfy, for constants $c_4, c_5 > 0$,

$$c_4^2 \, x \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x \le x \cdot M(T) x \le c_5 \, x \cdot \Lambda(T)^2 M(T) x \quad \text{for all } x \in \mathbb{R}^{m(T)}. \tag{7}$$

Remark 3. Inverse estimates [BS, Ci] provide (5) for a size-independent constant c_3 if $h_T = \operatorname{diam}(T)$.

Remark 4. The first inequality of (7) merely reflects a proper scaling of $d_{\ell(T,j)}$ and h_T.

Remark 5. The second inequality of (7) implies that $\Lambda(T)^2 M(T)$ has positive definite symmetric part. This is the crucial condition and relates the mass-matrix $M(T)$ to neighboring mesh-sizes.

Remark 6. We stress that (7) can *always* be satisfied even with $c_4 = c_5 = 1$ if we let $h_T = d_{\ell(T,j)}$ be equal to a global discretization parameter. For quasi-uniform meshes this implies (7).

Remark 7. In the original version [BPS, S], d_j is fixed as the arithmetic mean of all h_T with $T \subset \operatorname{supp} \varphi_j$, where h_T^d is the d-dimensional volume of an element $T \in \mathcal{T}$. Then, the Bramble-Pasciak-Steinbach criterion [BPS, (4.2)] implies the crucial second inequality in (7) (and is, in particular situations, equivalent).

3. A modified Bramble-Pasciak-Steinbach criterion for H^1-stability

Under the present assumptions (1)-(2) and (4)-(7) we have H^1-stability of Π.

Theorem 1. *We have* (3) *with* $c_1 = c_2 \max\{1, c_3\, c_5/c_4\}$.

The proof is a review of arguments in [BPS] in an abstract setting, and is included here for completeness. Theorem 1 implies the Bramble-Pasciak-Steinbach criterion [BPS] for a special choice of h_T and d_j (of Remark 7).

Proof. Given $u \in H$, define $q_h := P(u) - \Pi\, u = \sum_{\ell=1}^n q_\ell\, \varphi_\ell \in \mathcal{S}$ and $p_h := \sum_{\ell=1}^n q_\ell\, d_\ell^{-2}\, \varphi_\ell \in \mathcal{S}$ so that

$$q_h|_T = \sum_{\ell=1}^n q_\ell\, \varphi_\ell|_T = \sum_{j=1}^{m(T)} \xi_{T,j}\, \psi_{T,j} \quad \text{on } T \in \mathcal{T} \tag{8}$$

for certain coefficient vectors $x_T = (\xi_{T,1}, \dots, \xi_{T,m(T)}) = (q_{\ell(T,1)}, \dots, q_{\ell(T,m(T))})$. The triangle inequality for $\Pi\, u = P(u) - q_h$ and (4)-(5) show that

$$\|\nabla_{\mathcal{T}}\, \Pi u\| \le \|\nabla_{\mathcal{T}}\, P(u)\| + \|\nabla_{\mathcal{T}}\, q_h\| \le c_2\, \|\nabla_{\mathcal{T}}\, u\| + c_3\, \|h_{\mathcal{T}}^{-1}\, q_h\|. \tag{9}$$

According to direct calculations with coefficients from (8), the second inequality in (7) yields

$$\begin{aligned} c_5^{-1}\, \|h_{\mathcal{T}}^{-1}\, q_h\|^2 &= c_5^{-1} \sum_{T\in\mathcal{T}} h_T^{-2}\, x_T \cdot M(T) x_T \le \sum_{T\in\mathcal{T}} h_T^{-2}\, x_T \cdot \Lambda(T)^2 M(T) x_T \\ &= \sum_{T\in\mathcal{T}} \sum_{j=1}^{m(T)} \frac{q_{\ell(T,j)}}{d_{\ell(T,j)}^2} \int_T \varphi_{\ell(T,j)}\, q_h\, dx = \int_\Omega p_h\, q_h\, dx \\ &= \int_\Omega p_h (P(u) - u) dx \le c_2\, \|h_{\mathcal{T}}\, p_h\|\, \|\nabla_{\mathcal{T}}\, u\| \end{aligned} \tag{10}$$

because of (2), Cauchy's inequality, and (4). Similar arguments and (7) lead to

$$\begin{aligned} c_4^2\, \|h_{\mathcal{T}}\, p_h\|^2 &= c_4^2 \sum_{T\in\mathcal{T}} h_T^{-2}\, x_T \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x_T \\ &\le \sum_{T\in\mathcal{T}} h_T^{-2} x_T \cdot M(T) x_T = \|h_{\mathcal{T}}^{-1}\, q_h\|^2. \end{aligned}$$

Utilizing this in (10), we obtain a bound of $\|h_{\mathcal{T}}^{-1}\, q_h\|$, which we need in (9) to see (4). □

4. Examples for Courant triangles

Suppose $\mathcal{T}$ is a regular triangulation (in the sense of Ciarlet [BS, Ci]) of the bounded Lipschitz domain Ω in the plane into triangles. Homogeneous Dirichlet conditions may apply on a (relatively closed and possibly empty) boundary part Γ_D (matched exactly by edges). Each node $z \in \mathcal{N}$ with nodal basis function φ_z involves a positive real number d_z such that $h_T/d_z + d_z/h_T \leq c_6$ for all triangles $T \in \mathcal{T}$ of diameter h_T with vertex z. Let $\mathcal{S} := \operatorname{span}\{\varphi_z : z \in \mathcal{K}\}$, where $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ denotes the set of free nodes, and for the preceding notation identify $(\varphi_z : z \in \mathcal{K})$ and the parameters $(d_z : z \in \mathcal{K})$ with $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $(d_1, d_2, \ldots, d_n)$, respectively.

Theorem 2. *Suppose that $d_z/d_\zeta \leq \kappa < \sqrt{2} + \sqrt{3} \approx 3.1462$ for all vertices z and ζ of some triangle $T \in \mathcal{T}$. Then we have* (3).

Proof. The mass-matrix of a fixed $T \in \mathcal{T}$ is a multiple of the 3×3 matrix M with $M_{jk} = 1 + \delta_{jk}$ and $\Lambda(T)$ has diagonal entries $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_j/\lambda_k \leq \kappa$. The eigenvalues of $\Lambda(T)^{-1} A \Lambda(T)^{-1}$ for $A := (\Lambda(T)^2 M + M\Lambda(T)^2)/2$ can be calculated [BPS, S], and their smallest value is $(5 - \mu)$ for $\mu^2 := \sum_{j,k=1}^{3} \lambda_j^2/\lambda_k^2$. A straightforward analysis reveals that $\mu^2 \leq 3 + 2(1 + \kappa^2 + 1/\kappa^2) < 25$, which shows that A is positive definite. Therefore, $(x \cdot Ax)^{1/2}$ defines a norm which is equivalent to $|x|$ in $\mathbb{R}^3$. This and $h_T/d_z \leq c_6$ yield (7). □

Remark 8. The proof shows that $\sum_{j,k=1}^{3} \lambda_j^2/\lambda_k^2 \leq \nu < 22$ for some constant ν suffices for (3). Given d_j as in Remark 7, this is the a posteriori criterion of [BPS, S] for two dimensions.

The technical assumption on the artificial, extended triangulation in the following theorem merely reduces the consideration to interior triangles for brevity.

Theorem 3. *Suppose $\mathcal{T} \subset \hat{\mathcal{T}}$ for some regular triangulation $\hat{\mathcal{T}}$ of a Lipschitz domain $\hat{\Omega} \supset \Omega$ such that $\hat{\mathcal{T}}$ consists of right isosceles triangles only, there are no hanging nodes, and each free node on the boundary is an interior node of $\hat{\Omega}$. Then we have* (3).

Proof. Theorem 2 yields the assertion if we take

$$d_z = \min\{|z - \zeta| : \zeta \in \mathcal{N},\, \delta(z, \zeta) = 1\},$$

where $\delta(z, \zeta) = 1$ characterizes neighboring vertices z and ζ, i.e., $z, \zeta \in T \cap \mathcal{N}$ for at least one $T \in \mathcal{T}$. Since (up to scaling, transition, and rotation) there are only

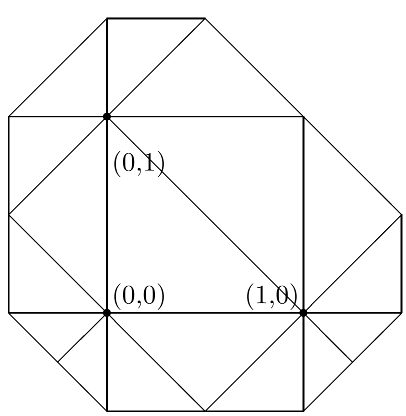

Figure 1: Part of a mesh as a smallest neighborhood of the reference triangle.

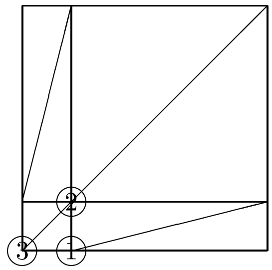

Figure 2: Reference mesh for comparisons in Example 1.

a finite number of possible configurations, it can be checked by a finite number of figures that $d_z/d_\zeta \leq \sqrt{8}$. Figure 1 illustrates a deduction: Suppose T has the vertices $(0,0)$, $(1,0)$, and $(0,1)$. Then, the patch $\operatorname{supp}\varphi_z$ of $z = (0,0)$ must include the polygonal domain with vertices $(0,1)$, $(-.5,.5)$, $(-.5,0)$, $(0,-.5)$, $(.5,-.5)$, $(1,0)$. This shows that $1/\sqrt{8} \leq d_{(1,0)} \leq 1$. Similarly, the patch $\operatorname{supp}\varphi_{(1,0)}$ must include the polygon $(0,0)$, $(.5,-.5)$, $(1,-.5)$, $(1.5,0)$, $(1.5,3)$, $(1,1)$, $(0,1)$, whence $1/\sqrt{8} \leq d_{(1,0)}, d_{(0,1)} \leq 1$. Consequently, $d_z/d_\zeta \leq \sqrt{8}$ for any choice of two vertices z and ζ of T. □

Example 1. Let $\mathcal{T}$ be the mesh of Figure 2 that consists of 8 triangles in a regular pattern that match the square $\Omega := (0,H)^2$ for positive $H = 1+\lambda$, where non-diagonals' lengths are either $\lambda < 1$ or 1. For the nodes 1, 2, and 3 of Figure 2 the choice of (d_1, d_2, d_3) from [BPS], mentioned in Remark 7, is

$$((\lambda + 2\lambda^{1/2})/3, (\lambda^2 + \lambda^{1/2} + 1)/3, \lambda)/\sqrt{2}.$$

The conditions of the Bramble-Pasciak-Steinbach criterion (cf. Remark 8) and those of Theorem 2 are violated for $\lambda < .1349$, which corresponds to an aspect ratio larger than 7.4122 However, Theorem 1 with the parameters from Remark 6 guarantees (3) for any positive λ (with a λ-dependent constant $c_1 = c_1(\lambda)$).

Example 2. Take a scaled copy of Ω and the mesh from Example 1 and extend it by reflection about the x_1-axis, the x_2-axis, and about the anti-diagonal through the origin to $h(-1,1)^2$; and then extend it $2h$-periodically to the entire plane. The calculations of Example 1 remain valid and we conclude that, for a fixed $\lambda < .1349$, the Bramble-Pasciak-Steinbach criterion is not applicable, but Remark 6 (or the Crouzeix-Thomée criterion) guarantees (3) with an h-independent constant $c_1 = c_1(\lambda)$.

The nonconforming Crouzeix-Raviart finite element (cf., e.g., [BS, Ci]) concludes our first series of applications.

Theorem 4. *Suppose T is an arbitrary shape-regular triangulation into triangles and $\mathcal{S}$ denotes the $\mathcal{T}$-piecewise affine functions which are continuous at midpoints of edges. Then we have* (3).

Proof. The mass-matrices are diagonal, so (7) is a consequence of shape-regularity. The operator P can be chosen exactly as in the conforming case. □

5. Weakening of the Crouzeix-Thomée criterion for H^1-stability

Part of the Crouzeix-Thomée criterion [CT] is the existence of c_7 and $1 \le \kappa := \sqrt{\alpha} < \sqrt{2} + \sqrt{3}$ such that

$$|T_1|/|T_2| \le c_7 \alpha^{l(T_1,T_2)} \quad \text{for all } T_1, T_2 \in \mathcal{T}. \tag{11}$$

Here, $|T_j|$ is the area of $T_j \in \mathcal{T}$ and the neighbor-index $l(T_1,T_2)$ might be defined via a metric δ on the nodes $\mathcal{N}$: For two distinct nodes z and ζ, $\delta(z,\zeta)$ is the smallest integer j such that there exists a polygon $(z_1, z_2, \ldots, z_j)$ of nodes $z_1, \ldots, z_j \in \mathcal{N}$ which connects $z = z_1$ with $\zeta = z_j$ along edges, i.e., $\{z_i, z_{i+1}\} \subset \partial T_i$ for some $T_i \in \mathcal{T}$ and all $i = 1, \ldots, j-1$; $\delta(z,z) := 0$. For any $T, K \in \mathcal{T}$ and $z \in \mathcal{N}$, let $\delta(z,T) := \min_{\zeta \in T \cap \mathcal{N}} \delta(z,\zeta)$ and $\delta(K,T) = \min_{z \in K \cap \mathcal{N}} \delta(z,T)$. Then, $l(T_1,T_2) = \delta(T_1,T_2) + 1$ if $T_1 \ne T_2$, while $l(T_1,T_2) = 0$ if and only if $T_1 = T_2$.

At first glance, the *local* Bramble-Pasciak-Steinbach and the *global* Crouzeix-Thomée criteria appear incomparable: a large constant c_7 prohibits a direct application of (11) in the spirit of Theorems 2 and 3 (as $d_z/d_\zeta \le c_8 \left(|T_1|/|T_2|\right)^{1/2} \le c_7^{1/2} c_8 \kappa \not< \sqrt{2} + \sqrt{3}$ for $\delta(z,\zeta) = 1$). However, all necessities are provided by

$$d_z := \min_{T \in \mathcal{T}} h_T \, \kappa^{\delta(z,T)} \quad \text{for all } z \in \mathcal{N} \quad \text{and} \quad h_T := |T|^{1/2} \quad \text{for all } T \in \mathcal{T}. \tag{12}$$

Theorem 5. *Suppose* (11)-(12) *hold for a planar regular triangulation $\mathcal{T}$. Then, the conditions of Theorem 2 are satisfied and we have* (3).

Proof. Given $z \in K \in \mathcal{T}$, we have $d_z \le h_K$ (K is allowed in the minimization (12), and $\delta(z,K) = 0$) and $l(T,K) - \delta(z,T) \le 1$. With a minimizing $T \in \mathcal{T}$ in (12), (11) shows that

$$h_K/d_z = \frac{h_K}{h_T \, \kappa^{\delta(z,T)}} = \frac{|K|^{1/2}}{|T|^{1/2}} \kappa^{-\delta(z,T)} \le \sqrt{c_7} \kappa^{l(T,K) - \delta(z,T)} \le \sqrt{c_7} \kappa. \tag{13}$$

To bound d_z/d_ζ for $z, \zeta \in \mathcal{N}$ with $\delta(z,\zeta) = 1$, let $K \in \mathcal{T}$ satisfy $d_\zeta = h_K \, \kappa^{\delta(z,K)}$. The definition (12) and $\delta(z,K) - \delta(\zeta,K) \le 1$ show that

$$d_z/d_\zeta = \frac{d_z}{h_K \, \kappa^{\delta(\zeta,K)}} \le \frac{h_K \, \kappa^{\delta(z,K)}}{h_K \, \kappa^{\delta(\zeta,K)}} = \kappa^{\delta(z,K) - \delta(\zeta,K)} \le \kappa. \quad \square \tag{14}$$

Example 3. There exists an adaptively-refined mesh [CV, Figure 1] of right isosceles triangles where the modified Bramble-Pasciak-Steinbach criterion guarantees H^1-stability (cf. [S] or Theorem 3) while the Crouzeix-Thomée criterion is not applicable.

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