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Inhomogeneous Dirichlet conditions in a priori and a posteriori finite element error analysis

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Summary. The numerical solution of elliptic boundary value problems with finite element methods requires the approximation of given Dirichlet data u_D by functions $u_{D,h}$ in the trace space of a finite element space on Γ_D. In this paper, quantitative a priori and a posteriori estimates are presented for two choices of $u_{D,h}$, namely the nodal interpolation and the orthogonal projection in $L^2(\Gamma_D)$ onto the trace space. Two corresponding extension operators allow for an estimate of the boundary data approximation in global H^1 and L^2 a priori and a posteriori error estimates. The results imply that the orthogonal projection leads to better estimates in the sense that the influence of the approximation error on the estimates is of higher order than for the nodal interpolation.

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1 Introduction

In this paper we investigate the influence of approximation errors in the Dirichlet boundary data for finite element approximations of elliptic partial differential equations. We restrict our attention to the model problem

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$$(1.1)\quad -\Delta u = f \text{ in } \Omega,\;\; u = u_D \text{ on } \Gamma_D,\;\; \partial_n u = \nabla u \cdot n = g \text{ on } \Gamma_N$$

but the results have an important impact also for the analysis of nonlinear problems, see, e.g., [BC2] and [B] for an a posteriori error analysis of variational inequalitites and time-dependent Ginzburg-Landau equations, respectively. Here, Ω is an open and bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2$ or $d = 3$, with polygonal or polyhedral boundary, respectively, and n is the unit outer normal to $\partial\Omega$. We suppose that $\partial\Omega = \Gamma_D \cup \Gamma_N$ where the Dirichlet boundary Γ_D is a closed subset of $\partial\Omega$ with positive surface measure and the Neumann boundary is given by $\Gamma_N = \partial\Omega \setminus \Gamma_D$. The basis for finite element approximations of (1.1) is its weak form given by

$$(1.2)\qquad (\nabla u; \nabla v) = (f; v) + \int_{\Gamma_N} g v \, \mathrm{d}s \quad \text{for all } v \in H^1_D(\Omega)$$

where $H^1_D(\Omega)$ is the subspace of the Sobolev space $H^1(\Omega)$ given by

$$H^1_D(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\},$$

and where $(\cdot;\cdot)$ denotes the L^2 scalar product. Let $\mathcal{T}$ be a regular triangulation of Ω into triangles ($d = 2$) or tetrahedra ($d = 3$) in the sense of Ciarlet [Ci]. We consider the simplest conforming finite element space $\mathcal{S}^1(\mathcal{T})$ of all continuous and piecewise affine functions on $\mathcal{T}$. Moreover, we denote by $\mathcal{S}^1_D(\mathcal{T})$ the subspace of all functions in $\mathcal{S}^1(\mathcal{T})$ that vanish on Γ_D, and by $\mathcal{S}^1(\Gamma_D)$ the trace space of functions in $\mathcal{S}^1(\mathcal{T})$ on Γ_D. In order to formulate the finite element approximation of (1.1), we fix a function $u_{D,h} \in \mathcal{S}^1(\Gamma_D)$. Then $u_h \in \mathcal{S}^1(\mathcal{T})$ is the finite element solution of (1.1) if $u_h = u_{D,h}$ on Γ_D and

$$(1.3)\quad (\nabla u_h; \nabla v_h) = (f; v_h) + \int_{\Gamma_N} g v_h \, \mathrm{d}s \quad \text{for all } v_h \in \mathcal{S}^1_D(\mathcal{T}).$$

We assume that $f \in L^2(\Omega)$, that $g \in L^2(\Gamma_N)$, and that u_D is continuous on Γ_D. More regularity of u_D on the faces of the elements in Γ_D will be required for some of the estimates. Then the model problem (1.1) and its finite element approximation (1.3) have unique solutions u and u_h, respectively.

The focus of this paper is to provide appropriate tools which allow to analyze the influence of approximated boundary data on a priori and a posteriori estimates for the error $e = u - u_h$ in H^1 and L^2 and to analyze in particular the effect of the choice of the discrete Dirichlet data $u_{D,h}$ on the estimates. A standard choice for the Dirichlet data $u_{D,h}$ in finite element spaces is the nodal interpolation $I_D u_D$ of the given function u_D. This does not influence the a priori H^1 error estimates since the approximation error does not appear explicitly. As for a posteriori estimates, the results in [BC1,CB,CBJ,Ca1] show that this choice affects the estimates only in a higher order term. Surprisingly, this situation is different for the corresponding L^2 error estimates based on duality techniques (Aubin-Nitsche trick). As a remedy we propose to use $u_{D,h} = \Pi_D u_D$ where Π_D is the L^2 projection of the Dirichlet data u_D onto $\mathcal{S}^1(\Gamma_D)$. A surprising consequence of the analysis in this paper is that the resulting contributions to a priori and a posteriori error estimates are always of higher order, see Section 2 for an informal overview of our results and Sections 6 and 7 for the precise statements.

2 Formulation of the problem and main results

In order to formulate our results, we recall the following general framework for a posteriori error estimates. Let w be the solution of

$$\begin{aligned} -\Delta w &= 0 && \text{in } \Omega, \\ w &= u_D - u_{D,h} && \text{on } \Gamma_D, \\ \partial_n w &= 0 && \text{on } \Gamma_N. \end{aligned}$$

Equivalently, w is the minimizer of the Dirichlet integral subject to the given Dirichlet conditions. Thus

$$\|\nabla e\|^2_{L^2(\Omega)} = \mathrm{Res}(e-w) + \int_\Omega \nabla e \cdot \nabla w \, \mathrm{d}x$$

where we define for a function $v \in H^1_D(\Omega)$ the residual by

$$\mathrm{Res}(v) = \int_\Omega \nabla e \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x + \int_{\Gamma_N} g v \, \mathrm{d}s - \int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x.$$

This identity and the orthogonality $\|\nabla e\|^2_{L^2} = \|\nabla w\|^2_{L^2} + \|\nabla(e-w)\|^2_{L^2}$ allow us to estimate

$$\|\nabla e\|_{L^2(\Omega)} \le \|\,\mathrm{Res}\,\|_{-1} + \eta_D^{(1/2)} \tag{2.1}$$

where

$$\eta_D^{(1/2)} = \|\nabla w\|_{L^2(\Omega)} = \inf_{\substack{v \in H^1(\Omega) \\ v|_{\Gamma_D} = u_D - u_{D,h}}} \|\nabla v\|_{L^2(\Omega)} \tag{2.2}$$

and

$$\|\,\mathrm{Res}\,\|_{-k} = \sup_{\substack{v \in H^k(\Omega)\setminus\{0\} \\ v|_{\Gamma_D} = 0}} \frac{\mathrm{Res}(v)}{\|v\|_{H^k(\Omega)}} \quad \text{for } k = 1,\ 2. \tag{2.3}$$

The L^2 estimates are based on duality techniques and we define correspondingly z to be the solution of the dual problem

$$
\begin{aligned}
-\Delta z &= e, && \text{in } \Omega,\\
z &= 0 && \text{on } \Gamma_D,\\
\partial_n z &= 0 && \text{on } \Gamma_N.
\end{aligned}
\tag{2.4}
$$

We obtain by integration by parts

$$
\|e\|^2_{L^2(\Omega)} = \operatorname{Res}(z) - \int_{\Gamma_D} (u_D - u_{D,h})\, \partial_n z \,\mathrm{d}s. \tag{2.5}
$$

For the derivation of L^2 error estimates we assume that the dual problem is H^2 regular, i.e., that $z \in H^2(\Omega)$ and that the elliptic estimate

$$
\|z\|_{H^2(\Omega)} \le c_1 \|e\|_{L^2(\Omega)} \tag{2.6}
$$

holds with a constant c_1 that only depends on Ω and Γ_D. A sufficient condition for this estimate to hold is that the domain Ω is convex and that $\Gamma_D = \partial\Omega$. Then

$$
\|e\|_{L^2(\Omega)} \le c_1 \big(\| \operatorname{Res} \|_{-2} + \eta_D^{(-1/2)} \big).
$$

Here

$$
\eta_D^{(-1/2)} = \sup_{\phi \in H^2_{DN}(\Omega)\setminus\{0\}} \frac{1}{\|\phi\|_{H^2(\Omega)}} \int_{\Gamma_D} (u_D - u_{D,h})\, \partial_n \phi \,\mathrm{d}s \tag{2.7}
$$

where

$$
H^2_{DN}(\Omega) = \{\phi \in H^2(\Omega) \cap H^1_D(\Omega) : \partial_n \phi|_{\Gamma_N} = 0\}.
$$

The foregoing representations of the error in L^2 and H^1 have the important feature that they separate terms that depend on the given boundary data, the expressions $\eta_D^{(\pm 1/2)}$, and terms that only depend on Γ_D, namely the negative norms of the residuals given by (2.3). Estimates for these residuals are well established and can be found in the literature, see [AO, BS, BC1, CB, CBJ, CV, EEHJ, V] for details. For completeness of the presentation, we quote a few results in Section 3. The main focus of this paper is therefore to establish bounds on the additional error terms that reflect directly the influence of the boundary data and their approximation. We give an overview of our results in Table 1. The surprising observation is that the L^2 projection of the given Dirichlet data onto the trace space of the finite element functions on the boundary gives always higher order contributions in a posteriori estimates

Table 1. Overview of main results: approximation of the Dirichlet data by nodal interpolation (I_D) and by L^2 projection (Π_D) and the corresponding contributions $\eta_D^{(\pm 1/2)}$ in the global L^2 and H^1 a priori and a posteriori error estimates

Error Estimate	$\eta_D^{(\pm 1/2)}$ for I_D	$\eta_D^{(\pm 1/2)}$ for Π_D
A priori H^1	0	$\mathcal{O}(h^{3/2})$
A posteriori H^1	$\mathcal{O}(h^{3/2})$	$\mathcal{O}(h^{3/2})$
A priori L^2	$\mathcal{O}(h^2)$	$\mathcal{O}(h^{5/2})$
A posteriori L^2	$\mathcal{O}(h^2)$	$\mathcal{O}(h^{5/2})$

than the nodal interpolation. The key to this result is the representation (2.7) in which we can rewrite the boundary integral as

$$\int_{\Gamma_D} (u_D - u_{D,h})\, \partial_n \phi \, \mathrm{d}s = \int_{\Gamma_D} (u_D - u_{D,h})\, (\partial_n \phi - \psi_h) \, \mathrm{d}s$$

for all $\psi_h \in \mathcal{S}^1(\Gamma_D)$, see the proof of Theorem 7.1.

These results can be easily extended in several directions.

Remark 2.1. The computation of $\Pi_D u_D$ involves the solution of a linear system of equations of the size of the number of nodes on Γ_D. Other definitions of $u_{D,h}$ based on piecewise L^2 projections are possible and reduce the amount of work for the computation of the projection. One obtains the same bounds as for $u_{D,h} = \Pi_D u_D$. For example, let F be a connected face of $\partial\Omega$, i.e., the intersection of $\partial\Omega$ with a $d-1$ dimensional plane in $\mathbb{R}^d$, and let P be a connected component of $F \cap \Gamma_D$. Then one can define $u_{D,h}|_P \in \mathcal{S}^1(\Gamma_D)|_P$ by $u_{D,h}|_{\partial P \setminus \partial \Gamma_D} = I_D u_D$ and

$$\int_P (u_{D,h} - u_D) v_h \, \mathrm{d}s = 0 \quad \text{for all } v_h \in \mathcal{S}^1(\Gamma_D) \text{ with } v_h|_{\partial P \setminus \partial \Gamma_D} = 0.$$

In this case $u_{D,h}$ is defined by local L^2 projections on (maximally) affine parts of Γ_D. The error $u_{D,h} - u_D$ is then $L^2(\Gamma_D)$ orthogonal to functions in $\mathcal{S}^1(\Gamma_D)$ that vanish on the intersections of maximally affine subsets of Γ_D. This orthogonality is sufficient for the proofs of our estimates (cf. proof of Theorem 7.1).

Remark 2.2. The estimates in L^2 and H^1 can be extended to estimates in H^s, $0 \le s \le 1$, by standard interpolation techniques.

Remark 2.3. The case $\Omega \neq \Omega_h = \cup\mathcal{T}$ is excluded from our analysis but can be analyzed with similar techniques: Let $\tilde{u}_h \in H^1(\Omega)$ be an extension of u_h and let $\tilde{e} = u - \tilde{u}_h$ and $\tilde{u}_{D,h} = \tilde{u}_h|_{\Gamma_D}$. We define $\widetilde{\text{Res}}$ by replacing e by $\tilde{e}$ in the definition of Res, $\tilde{w}$ by replacing $u_{D,h}$ by $\tilde{u}_{D,h}$ in the definition of w, and $\tilde{\eta}_D^{(\pm 1/2)}$ by replacing w and $u_{D,h}$ by $\tilde{w}$ and $\tilde{u}_{D,h}$, respectively, in the definition of $\eta_D^{(\pm 1/2)}$. It can then be shown as above that

$$\begin{aligned} \|\nabla\tilde{e}\|_{L^2(\Omega)} &\leq \|\widetilde{\text{Res}}\|_{-1} + \tilde{\eta}_D^{(1/2)}, \\ \|\tilde{e}\|_{L^2(\Omega)} &\leq c_1\big(\|\widetilde{\text{Res}}\|_{-2} + \tilde{\eta}_D^{(-1/2)}\big). \end{aligned}$$

We believe that under appropriate assumptions on Γ_D similar estimates to those shown in Table 1 can be proved for $\tilde{\eta}_D^{(\pm 1/2)}$ with similar techniques as provided in this paper. For estimates of $\|\widetilde{\text{Res}}\|_{-k}$, $k = 1, 2$, we refer the reader to [DR].

We conclude this introduction with an overview of the paper. Section 3 summarizes well-known facts that will be used in the sequel without proofs and introduces the relevant notation. The proofs of the H^1 and L^2 estimates rely essentially on extension theorems for functions and vector fields that we obtain in Sections 4 and 5, respectively. The precise statements of our results are then given in Sections 6 and 7. We conclude the paper with an explicit example showing that the choice of $u_{D,h} = \Pi_D u_D$ leads genuinely to higher order contributions in the estimates which is not the case for $u_{D,h} = I_D u_D$.

3 Preliminaries

In this section we introduce the notation used throughout the paper and we collect some auxiliary results.

Notation. We say that a constant c in a given inequality depends only on the geometric properties of the triangulation if it depends only on the space dimension d and the constant $c_2 > 0$ that relates the maximal radius of a ball $B(x, r) \subset K$ and the minimal radius of a ball $B(y, R) \supset K$ via

$$c_2 r \leq h_K = \text{diam}(K) \leq c_2^{-1} R \quad \text{for all } K \in \mathcal{T}.$$

For simplicity we write frequently $a \lesssim b$ for $a \leq c\, b$ where the constant c depends only on the geometric properties of the given triangulation $\mathcal{T}$.

The lowest order conforming finite element space of piecewise affine and continuous functions is given by

$$\mathcal{S}^1(\mathcal{T}) = \{v_h \in C(\bar{\Omega}) : \ v_h|_T \text{ is affine on all elements } T \in \mathcal{T}\}.$$

We define $\mathcal{N}$ to be the set of all nodes (or vertices) of $\mathcal{T}$ and we write $(\varphi_z : z \in \mathcal{N})$ for the nodal basis of $\mathcal{S}^1(\mathcal{T})$. The subspace of all functions that vanish on Γ_D is given by

$$\mathcal{S}^1_D(\mathcal{T}) = \{v_h \in \mathcal{S}^1(\mathcal{T}) : v_h|_{\Gamma_D} = 0\}. \tag{3.1}$$

The open patches $\omega_z = \{x \in \Omega : 0 < \varphi_z(x)\}$ form an open cover $\{\omega_z : z \in \mathcal{N}\}$ of Ω with finite overlap and diameter $h_z = \text{diam}(\omega_z)$. Let $\mathcal{E}$ denote the set of all edges ($d = 2$) or faces ($d = 3$) of elements in $\mathcal{T}$, i.e.,

$$\mathcal{E} = \Big\{ \text{conv}\{z_1, \dots, z_d\} : \exists T \in \mathcal{T} \text{ such that } T \cap \mathcal{N} = \{z_1, \dots, z_d\} \Big\}.$$

For simplicity, we refer to the edges of the triangles in two dimensions also as faces. We suppose that Γ_D is matched exactly by faces of the triangulation, that is, there exist subsets $\mathcal{E}_D,\ \mathcal{E}_N \subset \mathcal{E}$ such that

$$\Gamma_D = \bigcup_{E \in \mathcal{E}_D} E, \qquad \bar{\Gamma}_N = \bigcup_{E \in \mathcal{E}_N} E.$$

In particular, $\mathcal{T}$ induces regular triangulations of Γ_D and Γ_N. Then the set of interior faces $\mathcal{E}_\Omega$ is defined by $\mathcal{E}_\Omega = \mathcal{E} \setminus (\mathcal{E}_D \cup \mathcal{E}_N)$. The lowest order conforming finite element space on $\mathcal{E}_D$ is given by defined through

$$\mathcal{S}^1(\Gamma_D) = \{v_h \in C(\Gamma_D) :\ v_h|_E \text{ is affine for all } E \in \mathcal{E}_D\}.$$

We denote the mesh-size function which is piecewise constant on the elements of $\mathcal{T}$ by $h_\mathcal{T}$, i.e., $h_\mathcal{T}|_T = h_T = \text{diam}(T)$. Similarly, $h_\mathcal{E} \in L^\infty(\cup\mathcal{E})$ describes the size of the edges of the triangulation by $h_\mathcal{E}|_E = h_E = \text{diam}(E)$; here $\cup\mathcal{E} = \cup_{T\in\mathcal{T}} \partial T$ is the set of points on the faces of the elements.

In this paper, we analyze two particular approximations of the given boundary data u_D in $\mathcal{S}^1(\Gamma_D)$: The nodal interpolation $I_D u_D$, defined through $I_D u_D(z) = u_D(z)$ for all nodes z on Γ_D, and the L^2 projection $\Pi_D u_D$ which is the unique function in $\mathcal{S}^1(\Gamma_D)$ with

$$\int_{\Gamma_D} (u_D - \Pi_D u_D)\, v_h \, \mathrm{d}s = 0 \quad \text{for all } v_h \in \mathcal{S}^1(\Gamma_D).$$

We define Sobolev spaces on the Dirichlet boundary Γ_D as follows.

Definition 3.1. *For an edge or a face $E \in \mathcal{E}$ and a function $g \in C^1(E)$ we denote by $\partial_\mathcal{E} g$ the surface gradient along E (with respect to a proper Cartesian coordinate system along the flat $d-1$ dimensional manifold E). We then say that $v|_E \in H^1(E)$ if v has weak derivatives on E and if $\|v\|^2_{H^1(E)} = \|v\|^2_{L^2(E)} + \|\partial_\mathcal{E} v\|^2_{L^2(E)} < \infty$. Similarly, we denote by $\partial^2_\mathcal{E} g$ the edgewise second derivative of g along Γ_D if $g|_E \in H^2(E)$ for all $E \in \mathcal{E}_D$.*

Definition 3.2. *We define* $H^1(\Gamma_D) = \{v \in C(\Gamma_D) : \forall E \in \mathcal{E}_D, v|_E \in H^1(E)\}$ *with the norm*

$$\|v\|^2_{H^1(\Gamma_D)} = \sum_{E\in\mathcal{E}_D} \|v\|^2_{H^1(E)}.$$

Some of our assertions require the projection Π_D to be stable in $H^1(\Gamma_D)$.

Definition 3.3. *The operator* $\Pi_D : H^1(\Gamma_D) \to \mathcal{S}^1(\Gamma_D)$ *is said to be* H^1 *stable if*

$$\|\Pi_D v\|_{H^1(\Gamma_D)} \lesssim \|v\|_{H^1(\Gamma_D)} \quad \textit{for all } v \in H^1(\Gamma_D).$$

Stability results can be found in [CT, BPS, Ca2]. A particular version of a red-green-blue refinement strategy on surfaces (such as Γ_D) allows for local refinements and guarantees that

$$\|\partial_{\mathcal{E}} \Pi_D v\|_{L^2(\Gamma_D)} \le C\|\partial_{\mathcal{E}} v\|_{L^2(\Gamma_D)} \quad \text{for all } v \in H^1(\Gamma_D).$$

Here the constant C does not depend on the mesh-size or the number of refinement levels, but depends on the shape of the elements [Ca3]. Thus, the assumption that Π_D be H^1 stable on Γ_D is indeed satisfied for a large class of meshes used in practise in two and three dimensional problems. We finally recall the following estimate for stable projections.

Lemma 3.4 ([CV], Lemma 3.3)**.** *Assume that* Π_D *is* H^1 *stable. Then*

$$\|h_{\mathcal{E}}^{-1}(v - \Pi_D v)\|_{L^2(\Gamma_D)} \lesssim \|\partial_{\mathcal{E}} v\|_{L^2(\Gamma_D)} \quad \textit{for all } v \in H^1(\Gamma_D).$$

We frequently use Poincaré-type inequalities in the estimates. In particular, if D is a Lipschitz domain and $\bar{u}$ is the mean value of u on D, then

$$\int_D |u - \bar{u}|^2 \,\mathrm{d}x \le c(D)\,\mathrm{diam}^2(D) \int_D |Du|^2 \,\mathrm{d}x.$$

Moreover, a scaling argument proves the following version for functions defined on the open patches ω_z. If $u \in H^1(\omega_z)$ with $u = 0$ on at least one face $E \in \mathcal{E}$ with $E \subset \partial\omega_z$, then

$$\int_{\omega_z} |u|^2 \,\mathrm{d}x \le c(\mathcal{T}) h_z^2 \int_{\omega_z} |Du|^2 \,\mathrm{d}x. \tag{3.2}$$

The next estimate of Poincaré-type is used in the proof of the extension theorems below.

Proposition 3.5. *Let $K \in \mathcal{T}$ be a triangle or tetrahedron of diameter h_K with nodes $p_1, \dots, p_{d+1} \in \mathbb{R}^d$. Then*

$$\int_K |\phi|^2 \,\mathrm{d}x \lesssim h_K^4 \int_K |D^2\phi|^2 \,\mathrm{d}x, \tag{3.3}$$

$$\int_K |\nabla\phi|^2 \,\mathrm{d}x \lesssim h_K^2 \int_K |D^2\phi|^2 \,\mathrm{d}x \tag{3.4}$$

for all $\phi \in H^2(K)$ with $\phi(p_j) = 0$ for $j = 1, \dots, d+1$.

Proof. Assume first that $K = \hat{K}$ is the standard simplex in $\mathbb{R}^d$ with $1 \lesssim h_K \lesssim 1$, and that (3.3) does not hold. Then there exists a sequence $\phi_k \in H^2(\hat{K})$ with $\phi_k(p_j) = 0$ for $j = 1, \dots, d+1$ and $\|\phi_k\|_{L^2(\hat{K})} = 1$ such that

$$k \int_{\hat{K}} |D^2\phi_k|^2 \,\mathrm{d}x \le \int_{\hat{K}} |\phi_k|^2 \,\mathrm{d}x.$$

It follows that the sequence ϕ_k is uniformly bounded in H^2 and that $\phi_k \rightharpoonup \phi$ (weakly) in H^2. The compact Sobolev embedding theorem implies that $\phi_k \to \phi$ in H^1 with $\|\phi\|_{L^2(\hat{K})} = 1$ and $\phi(p_j) = 0$ for $j = 1, \dots, d+1$. Moreover, we obtain from the weak lower semicontinuity of the norm that

$$\int_{\hat{K}} |D^2\phi|^2 \,\mathrm{d}x \le \liminf_{k\to\infty} \int_{\hat{K}} |D^2\phi_k|^2 \,\mathrm{d}x \le \limsup_{k\to\infty} \frac{1}{k} = 0,$$

and thus ϕ is an affine function. Since $\phi(p_j) = 0$, we deduce that $\phi \equiv 0$ and this contradicts $\|\phi\|_{L^2(\hat{K})} = 1$. A scaling argument completes the proof of (3.3). The proof of (3.4) is analogous. □

We finish the preliminaries by quoting some estimates for the residuals. For instance, if $f \in H^1(\Omega)$, then [CV, CB]

$$\begin{aligned}\|\operatorname{Res}\|_{-1} \lesssim\ & \|h_{\mathcal{T}}^2 \nabla f\|_{L^2(\Omega)} \\ & + \min_{p_h \in \mathcal{S}^1(\mathcal{T})^d} \big\{ \|\nabla u_h - p_h\|_{L^2(\Omega)} + \|h_{\mathcal{E}}^{1/2}(g - p_h \cdot n)\|_{L^2(\Gamma_N)} \big\},\end{aligned}$$

and

$$\begin{aligned}\|\operatorname{Res}\|_{-1} \lesssim\ & \|h_{\mathcal{T}}^2 \nabla f\|_{L^2(\Omega)} + \Big(\sum_{E \in \mathcal{E}_\Omega} h_E \|[\nabla u_h \cdot n_E]\|_{L^2(E)}^2 \Big)^{1/2} \\ & + \Big(\sum_{E \in \mathcal{E}_N} h_E \|g - \nabla u_h \cdot n\|_{L^2(E)}^2 \Big)^{1/2}.\end{aligned}$$

Local problem solving techniques, equilibrium estimators and other implicit a posteriori error estimation techniques [AO, BS] can also be applied.

Moreover, if the L^2 projection onto $\mathcal{S}^1(\mathcal{T})$ is H^1 stable, then we have the following bound for $\|\operatorname{Res}\|_{-2}$ (see [CV])

$$\|\operatorname{Res}\|_{-2} \lesssim \inf_{f_h \in \mathcal{S}^1_D(\mathcal{T})} \|h_{\mathcal{T}}^2(f - f_h)\|_{L^2(\Omega)} + \Big(\sum_{E \in \mathcal{E}_\Omega} h_E^3 \|[\nabla u_h \cdot n_E]\|^2_{L^2(E)}\Big)^{1/2} + \Big(\sum_{E \in \mathcal{E}_N} h_E^3 \|g - \nabla u_h \cdot n\|^2_{L^2(E)}\Big)^{1/2}.$$

4 The extension operator for functions

The H^1 estimates require good bounds on the quantities $\eta_D^{(1/2)}$ in (2.2). They rely on the construction of functions with given values on the Dirichlet boundary Γ_D which we describe in this section. Our first result is valid in any dimensions.

Proposition 4.1. *Let $K \in \mathcal{T}$ be a simplex in $\mathbb{R}^d$ and let $h_K = \operatorname{diam}(K)$. Assume that the nodes of K are given by $p_1, \ldots, p_{d+1} \in \mathbb{R}^d$ and that the faces of K are labeled $F_1, \ldots, F_{d+1}$. Suppose that $g \in C(\partial K)$ with $g|_{F_j} \in H^1(F_j)$ for $j = 1, \ldots, d+1$, and define w to be the harmonic extension of g to K. Then there exists a constant c_3 that only depends on the geometric properties of K such that*

$$\|\nabla w\|^2_{L^2(K)} \le c_3 \big\{h_K^{-1} \|g\|^2_{L^2(\partial K)} + h_K \|\partial_{\mathcal{E}} g\|^2_{L^2(\partial K)}\big\}.$$

Proof. Since the harmonic extension of g minimizes the Dirichlet integral, it suffices to construct a function $v \in H^1(K)$ with $v|_{\partial K} = g$ and

$$\|\nabla v\|^2_{L^2(K)} \le c_3 \big\{h_K^{-1} \|g\|^2_{L^2(\partial K)} + h_K \|\partial_{\mathcal{E}} g\|^2_{L^2(\partial K)}\big\}. \tag{4.1}$$

In order to construct v, let x_K denote the barycenter of K and define the simplices $K_j = \operatorname{conv}\{x_K, F_j\}$ for $j = 1, \ldots, d+1$. The idea is to interpolate the boundary data linearly along rays connecting the boundary points and x_K, see Figure 1. For each $x \in K \setminus \{x_K\}$ there exist unique $\lambda_x \in (0, 1]$ and $\xi_x \in \partial K$ such that $x = (1 - \lambda_x) x_K + \lambda_x \xi_x$. Define v on K by

$$v(x) = \begin{cases} \lambda_x g(\xi_x) & \text{if } x \in K, x \ne x_K, \\ 0 & \text{if } x = x_K. \end{cases}$$

It remains to prove (4.1). It suffices to show that

$$\|\nabla v\|^2_{L^2(K_j)} \le \frac{c_3}{d+1} \big\{h_K^{-1} \|g\|^2_{L^2(F_j)} + h_K \|\partial_{\mathcal{E}} g\|^2_{L^2(F_j)}\big\} \quad \text{for } j = 1, \ldots, d+1.$$

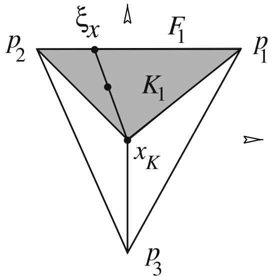

Fig. 1. Construction of the extension of the boundary data into the simplex in two dimensions

We may assume that $j = 1$, that $F_1 = \mathrm{conv}\{p_1, \dots, p_d\} \subset \{x \in \mathbb{R}^n : x_n = \rho\}$ with $\rho > 0$ and that $x_K = 0$. In the following we write $\hat{x}$ for the first $n-1$ coordinates of x, i.e., $\hat{x} = (x_1, \dots, x_{n-1})$. Then

$$v(x) = \begin{cases} \frac{x_n}{\rho} g\big(\frac{\rho}{x_n}\,\hat{x}, \rho\big) & \text{if } x \in K_1, x \neq x_K, \\ 0 & \text{if } x = x_K. \end{cases}$$

Thus for $i = 1, \dots, n-1$

$$\begin{aligned} \int_{K_1} \Big|\frac{\partial}{\partial x_i}\, v(x)\Big|^2 \mathrm{d}x &= \int_{K_1} \Big|\frac{\partial g}{\partial x_i}\Big(\frac{\rho}{x_n}\,\hat{x}, \rho\Big)\Big|^2 \mathrm{d}x \\ &= \int_0^\rho \int_{F_1} \Big|\frac{\partial g}{\partial x_i}(\hat{x}, \rho)\Big)\Big|^2 \Big(\frac{x_n}{\rho}\Big)^{d-1} \mathrm{d}\hat{x}\ \mathrm{d}x_n \le \rho \Big\| \frac{\partial g}{\partial x_i} \Big\|^2_{L^2(F_1)}. \end{aligned}$$

Finally,

$$\begin{aligned} \int_{K_1} \Big|\frac{\partial}{\partial x_n}\, v(x)\Big|^2 \mathrm{d}x &= \int_{K_1} \Big|\frac{1}{\rho}\, g\Big(\frac{\rho}{x_n}\,\hat{x}, \rho\Big) - \sum_{i=1}^{n-1} \frac{\partial g}{\partial x_i}\Big(\frac{\rho}{x_n}\,\hat{x}, \rho\Big)\, \frac{x_i}{x_n}\Big|^2 \mathrm{d}x \\ &\le \frac{1}{\rho}\, \|g\|^2_{L^2(F_1)} + \int_{K_1} \frac{|\hat{x}|^2}{x_n^2} \Big|\sum_{i=1}^{n-1} \frac{\partial g}{\partial x_i}\Big(\frac{\rho}{x_n}\,\hat{x}, \rho\Big)\Big|^2 \mathrm{d}x \\ &\le \frac{1}{\rho}\, \|g\|^2_{L^2(F_1)} + \frac{\mathrm{diam}^2(F_1)}{\rho} \sum_{i=1}^{n-1} \Big\| \frac{\partial g}{\partial x_i} \Big\|^2_{L^2(F_1)}. \end{aligned}$$

This implies the assertion since the regularity of the triangulation implies the existence of a constant $c_4 > 0$ such that

$$c_4 h_K \le \mathrm{diam}(F_j) \le c_4^{-1} h_K, \qquad c_4 h_K \le \rho \le c_4^{-1} h_K.$$

This concludes the proof of the proposition. □

The main result in this section is the following extension result that is crucial in the proofs of the H^1 estimates. For simplicity of the exposition, we state this result only for $d = 2$ and $d = 3$

Theorem 4.2. *Assume $d = 2$ or $d = 3$, and that $u_D \in C(\Gamma_D)$ with $u_D|_E \in H^2(E)$ for all faces $E \in \mathcal{E}_D$. Let $u_{D,h} = I_D u_D$. Then there exists a $w \in H^1(\Omega)$ such that $w|_{\Gamma_D} = u_D - u_{D,h}$, $\operatorname{supp} w \subseteq \{T \in \mathcal{T} : T \cap \Gamma_D \neq \emptyset\}$, and*

$$\|\nabla w\|_{L^2(\Omega)} \lesssim \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)}. \tag{4.2}$$

Proof. The idea of the proof is to successively extend functions from lower dimensional objects (faces) to higher dimensional ones (elements) by harmonic extension. In order to accomplish this, we define first the function w on all edges of the triangulation and then on all simplices.

Recall that we denote by $\partial_{\mathcal{E}}$ the (tangential) derivative of a function on the $d-1$ dimensional faces of the tetrahedra. Moreover, we use $\partial_{\mathcal{L}}$ for the derivative of a function along a one-dimensional edge (the line segment formed by the intersection of two faces) of a tetrahedron in three dimensions.

Step 1: Definition of w on the faces of the triangulation $\mathcal{T}$. Let $g = u_D - u_{D,h} \in C(\Gamma_D)$ and note that $g(z) = 0$ for all $z \in \mathcal{N} \cap \Gamma_D$. For each $E \in \mathcal{E}$ we define a function $w_E \in C(E)$ as follows.

(a) If $E \cap \Gamma_D = \{z_1, \ldots, z_\ell\}$ for $1 \leq \ell \leq d$ and $z_1, \ldots, z_\ell \in \mathcal{N}$ or $E \cap \Gamma_D = \emptyset$, then we set $w_E = 0$.
(b) If $E \subseteq \Gamma_D$, then we set $w_E = g|_E$.
(c) Suppose that $d = 3$ and that either (c_1) $E \cap \Gamma_D = \cup_{j=1}^J \operatorname{conv}\{a_j, b_j\}$ with $J \in \{1, 2, 3\}$, $a_j, b_j \in \mathcal{N}$, and $a_j \neq b_j$, or (c_2) $E \cap \Gamma_D = \operatorname{conv}\{a_1, b_1\} \cup \{z\}$ with $a_1, b_1 \in \mathcal{N}$, and $z \in \mathcal{N} \setminus \{a_1, b_1\}$. Let z_E denote the barycenter of E and define $S_j = \operatorname{conv}\{a_j, b_j\}$ and $G_j = \operatorname{conv}\{z_E, a_j, b_j\}$. For $j = 1, \ldots, J$, let $w_E|_{G_j}$ be the harmonic extension of $g|_{S_j}$ such that $w_E|_{\partial G_j \setminus S_j} = 0$. Finally we set $w_E = 0$ on $E \setminus \cup_{j=1}^J G_j$.

Here case (c) describes the situation that in three dimensions some (or all) of the edges of a face of a tetrahedron may be contained in the Dirichlet boundary even though the face is not a subset of Γ_D.

Step 2: An auxiliary estimate in case (c). We have

$$h_E^{-1} \|w_E\|_{L^2(E)}^2 \lesssim h_E \|\partial_{\mathcal{E}} w_E\|_{L^2(E)}^2 \lesssim \sum_{j=1}^J \left(h_E \|g\|_{H^1(F_j)}^2 + h_E^3 \|g\|_{H^2(F_j)}^2\right).$$

Note that by construction $w_E \in H^1(E) \cap C(E)$. Moreover, we obtain from Proposition 4.1 (with $d = 2$ applied to the triangles G_j whose union is the face E) and the one-dimensional Poincaré inequality for w_E on the line segment S_j that

$$\|\partial_{\mathcal{E}} w_E\|^2_{L^2(G_j)} \lesssim h_E \|\partial_{\mathcal{L}} w_E\|^2_{L^2(\partial G_j)} = h_E \|\partial_{\mathcal{L}} g\|^2_{L^2(S_j)}. \tag{4.3}$$

To estimate the derivative of g along S_j choose a face $F_j \in \mathcal{E}_D$ with $S_j \subseteq \partial F_j$, $j = 1, \ldots, J$. It follows by scaling from the trace inequality in $W^{1,1}(\hat{K})$,

$$\int_{\partial \hat{K}} |v| \, \mathrm{d}s \leq C(\hat{K}) \int_{\hat{K}} \big(|v| + |Dv|\big) \, \mathrm{d}x,$$

applied to $v = \phi^2$ that

$$\|\phi\|^2_{L^2(\partial K)} \lesssim h_K^{-1} \|\phi\|^2_{L^2(K)} + \|\phi\|_{L^2(K)} \|\nabla \phi\|_{L^2(K)} \quad \text{for all } \phi \in H^1(K).$$

This estimate applied to $K = F_j$ and the Poincaré inequality in Proposition 3.5 imply

$$h_E \|\partial_{\mathcal{L}} g\|^2_{L^2(S_j)} \lesssim \|g\|^2_{H^1(F_j)} + h_E^2 \|g\|^2_{H^2(F_j)}. \tag{4.4}$$

Since w_E vanishes on two of the three sides of the triangles G_j we may use Poincaré's inequality and we obtain

$$h_E^{-1} \|w_E\|^2_{L^2(E)} = h_E^{-1} \sum_{j=1}^{J} \|w_E\|^2_{L^2(G_j)} \lesssim h_E \sum_{j=1}^{J} \|\partial_{\mathcal{E}} w_E\|^2_{L^2(G_j)}.$$

The combination this inequality with (4.3) and (4.4) implies the assertion of Step 2.

Step 3: Proof of the theorem. We extend the function w_E (defined so far for all $E \in \mathcal{E}$) to a function w_T on $T \in \mathcal{T}$ in the following way. For all $T \in \mathcal{T}$ let w_T be the harmonic function with $w_T = w_E$ on ∂T. From the construction of w_E we have $w_T \not\equiv 0$ only if $T \cap \Gamma_D \neq \emptyset$. By Proposition 4.1 we deduce

$$\begin{aligned} \|\nabla w_T\|^2_{L^2(T)} &\lesssim h_T^{-1} \|w_T\|^2_{L^2(\partial T)} + h_T \|\partial_{\mathcal{E}} w_T\|^2_{L^2(\partial T)} \\ &\lesssim \sum_{E \in \mathcal{E}, E \subseteq \partial T} \big(h_E^{-1} \|w_E\|^2_{L^2(E)} + h_E \|\partial_{\mathcal{E}} w_E\|^2_{L^2(E)}\big). \end{aligned}$$

Recall that w_E is different from zero only if $E \in \mathcal{E}_D$ or (this applies only to the three-dimensional situation) if E is the face of a tetrahedron and at least one of the edges of this face is contained in Γ_D. In the former case, $w_E = g|_E$, and in the latter case w_E and its derivatives have been estimated above. Hence

$$\begin{aligned} \sum_{T \in \mathcal{T}} \|\nabla w_T\|^2_{L^2(T)} \lesssim \sum_{E \in \mathcal{E}, E \subseteq \Gamma_D} &\big(h_E^{-1} \|g\|^2_{L^2(E)} \\ &+ h_E \|\partial_{\mathcal{E}} g\|^2_{L^2(E)} + h_E^3 \|\partial^2_{\mathcal{E}} g\|^2_{L^2(E)}\big). \end{aligned}$$

Let χ_T be the characteristic function of the element T. Then,

$$w = \sum_{T\in\mathcal{T}} \chi_T w_T \in H^1(\Omega) \quad \text{with} \quad w|_{\Gamma_D} = g = u_D - u_{D,h}.$$

Proposition 3.5 shows, for all $E \in \mathcal{E}_D$,

$$h_E^{-1}\int_E |g|^2 \mathrm{d}s + h_E \int_E |\partial_{\mathcal{E}} g|^2 \mathrm{d}s \lesssim h_E^3 \int_E |\partial_{\mathcal{E}}^2 g|^2 \mathrm{d}s = h_E^3 \int_E |\partial_{\mathcal{E}}^2 u_D|^2 \mathrm{d}s.$$

These estimates allow us to rewrite the foregoing estimate as

$$\|\nabla w\|_{L^2(\Omega)}^2 \lesssim \sum_{E\in\mathcal{E}_D} h_E^3 \|\partial_{\mathcal{E}}^2 u_D\|_{L^2(E)}^2 .$$

This finishes the proof of the theorem. □

We conclude this section with the construction of a suitable extension of the difference $\Pi_D u_D - I_D u_D$ on Γ_D onto Ω.

Proposition 4.3. *Assume that* Π_D *is* H^1 *stable and that* $u_D \in C(\Gamma_D)$ *with* $u_D|_E \in H^2(E)$ *for all* $E \in \mathcal{E}_D$. *Then there exists a* $v_h \in \mathcal{S}^1(\mathcal{T})$ *such that* $v_h|_{\Gamma_D} = \Pi_D u_D - I_D u_D$, $\operatorname{supp} v_h \subseteq \{T \in \mathcal{T} : T \cap \Gamma_D \neq \emptyset\}$, *and*

$$\|\nabla v_h\|_{L^2(\Omega)} \lesssim \|h_{\mathcal{E}}\|_{L^\infty(\Gamma_D)}^{1/2} \|h_{\mathcal{E}} \partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)}.$$

Proof. Let $g_h = \Pi_D u_D - I_D u_D$. We define $v_h = \sum_{z\in\mathcal{N}} v_z \varphi_z$ where $v_z = g_h(z)$ if $z \in \mathcal{N} \cap \Gamma_D$ and $v_z = 0$ otherwise. Then $\operatorname{supp} v_h \subseteq \cup\{\omega_z : z \in \mathcal{N} \cap \Gamma_D\}$, and the quantities $\|v_h\|_{L^2(\omega_z)}$ and $h_z^{1/2}\|g_h\|_{L^2(\partial\omega_z\cap\Gamma_D)}$ are equivalent norms for $g_h|_{\partial\omega_z\cap\Gamma_D}$. We conclude from inverse estimates that

$$\|\nabla v_h\|_{L^2(\omega_z)} \lesssim h_z^{-1}\|v_h\|_{L^2(\omega_z)} \lesssim h_z^{-1/2}\|\Pi_D u_D - I_D u_D\|_{L^2(\partial\omega_z\cap\Gamma_D)}$$

for all $z \in \mathcal{N} \cap \Gamma_D$. We now take the sum for all $z \in \mathcal{N} \cap \Gamma_D$. Since the patches ω_z have finite overlap and since $h_z \lesssim h_E \lesssim h_z$ for $E \in \mathcal{E}$ and $z \in E \cap \mathcal{N}$ we obtain that

$$\|\nabla v_h\|_{L^2(\Omega)} \lesssim \|h_{\mathcal{E}}\|_{L^\infty(\Gamma_D)}^{1/2} \|h_{\mathcal{E}}^{-1}(\Pi_D u_D - I_D u_D)\|_{L^2(\Gamma_D)}.$$

It follows from the triangle inequality and Lemma 3.4 in view of $\Pi_D I_D u_D = I_D u_D$ that

$$\begin{aligned}\|\nabla v_h\|_{L^2(\Omega)} &\lesssim \|h_{\mathcal{E}}\|_{L^\infty(\Gamma_D)}^{1/2} \|h_{\mathcal{E}}^{-1}\Pi_D(u_D - I_D u_D)\|_{L^2(\Gamma_D)} \\ &\lesssim \|h_{\mathcal{E}}\|_{L^\infty(\Gamma_D)}^{1/2} \big(\|\partial_{\mathcal{E}}(u_D - I_D u_D)\|_{L^2(\Gamma_D)} \\ &\quad + \|h_{\mathcal{E}}^{-1}(u_D - I_D u_D)\|_{L^2(\Gamma_D)}\big).\end{aligned}$$

The assertion of the lemma follows now with standard interpolation inequalities. □

5 The extension operator for vector fields

In the case of the L^2 estimates, one needs good bounds on the quantities $\eta_D^{(-1/2)}$ defined in (2.7). The idea here is to write $\partial_n\phi = p\cdot n$ with $p = \nabla\phi$, and to use integration by parts to rewrite the boundary integral as a volume integral. This leads to the question of how to construct vector fields p_h that are suitably close to p; the answer is given in Theorem 5.1 below.

Definition 5.1. *For each $E\in\mathcal{E}_D$ let $n|_E, t_E^1,\dots,t_E^{d-1}\in\mathbb{R}^d$ be an orthonormal basis of $\mathbb{R}^d$. For $p\in H^1(\Omega)^d$ and $E\in\mathcal{E}_D$ let $\gamma_{t_E}(p)\in L^2(E)^d$ denote the tangential component of $p|_E$, i.e., $p|_E = \gamma_{t_E}(p) + (n|_E\cdot p|_E)\,n|_E$.*

Theorem 5.1. *Let $p\in H^1(\Omega)^d$ satisfy $\gamma_{t_E}(p) = 0$ for all $E\in\mathcal{E}_D$ and $p\cdot n = 0$ on Γ_N. Then there exists $p_h\in\mathcal{S}^1(\mathcal{T})^d$ with $\gamma_{t_E}(p_h) = 0$ for all $E\in\mathcal{E}_D$, $p_h\cdot n = 0$ on Γ_N, and*

$$\|h_{\mathcal{T}}^{-1}(p-p_h)\|_{L^2(\Omega)} + \|\nabla(p-p_h)\|_{L^2(\Omega)} \lesssim \|\nabla p\|_{L^2(\Omega)}. \tag{5.1}$$

Proof. For $z\in\mathcal{N}\setminus\partial\Omega$ set $p_z = |\omega_z|^{-1}\int_{\omega_z} p\,\mathrm{d}x$. We need a more sophisticated construction for nodes on $\partial\Omega$. For $z\in\mathcal{N}\cap\partial\Omega$ define $\widetilde{p}_z = |\omega_z|^{-1}\int_{\omega_z} p\,\mathrm{d}x$. Moreover, let $v_1^z,\dots,v_{k_z}^z$, $1\le k_z\le d$, be an orthonormal basis of

$$V_z = \operatorname{span}\{n|_E : z\in E\in\mathcal{E}_N\}\cup\operatorname{span}\{t_E^\ell : z\in E\in\mathcal{E}_D, \ell = 1,\dots,d-1\},$$

and let $(v_1^z,\dots,v_{k_z}^z, s_1^z,\dots,s_{d-k_z}^z)$ be an orthonormal basis of $\mathbb{R}^d$. Set

$$p_z = \sum_{j=1}^{d-k_z}(\widetilde{p}_z\cdot s_j^z)\,s_j^z$$

and define $p_h = \sum_{z\in\mathcal{N}} p_z\varphi_z$. By construction, $\gamma_{t_E}(p_h) = 0$ for all $E\in\mathcal{E}_D$ and $p_h\cdot n = 0$ on Γ_N. To see this, consider for example a face $E\in\mathcal{E}_D$. The spaces V_z contain the tangential directions for all nodes $z\in E$ and thus p_z is normal to E for all nodes $z\in E$. Consequently, the tangential component of p_h vanishes on all $E\in\mathcal{E}_D$. The argument for $\mathcal{E}_N$ is analogous. Since $\sum_{z\in\mathcal{N}}\varphi_z = 1$ and $h_z\lesssim h_T\lesssim h_z$ if $T\in\mathcal{T}$ and $z\in\mathcal{N}\cap T$, we have

(5.2)

$$\begin{aligned}\|h_{\mathcal{T}}^{-1}(p-p_h)\|_{L^2(\Omega)}^2 &= \sum_{z\in\mathcal{N}}\int_{\omega_z} h_{\mathcal{T}}^{-2}\varphi_z(p-p_z)\cdot(p-p_h)\,\mathrm{d}x\\ &\lesssim \Big(\sum_{z\in\mathcal{N}} h_z^{-2}\|p-p_z\|_{L^2(\omega_z)}^2\Big)^{1/2}\|h_{\mathcal{T}}^{-1}(p-p_h)\|_{L^2(\Omega)}.\end{aligned}$$

Moreover $(\varphi_z : z \in \mathcal{N})$ is a locally finite partition of unity and therefore

$$\|\nabla(p - p_h)\|_{L^2(\Omega)} \lesssim \Big(\sum_{z\in\mathcal{N}} \|\nabla\big(\varphi_z(p - p_z)\big)\|^2_{L^2(\omega_z)}\Big)^{1/2}. \tag{5.3}$$

If we expand the gradient using the product rule we obtain a term similar to (5.2) and a term proportional to $\|\nabla p\|_{L^2(\Omega)}$ since p_z is constant on the patch ω_z. It thus suffices to show that $\|p - p_z\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\|_{L^2(\omega_z)}$. Suppose first that $z \in \mathcal{N} \setminus \partial\Omega$. Since p_z is the mean value of p_h on ω_z we obtain with Poincaré's inequality that

$$\|p - p_z\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\|_{L^2(\omega_z)}. \tag{5.4}$$

It therefore remains to estimate this local norm for $z \in \mathcal{N} \cap \partial\Omega$. By construction, $p_z \cdot v_j^z = 0$ for $j = 1, \dots, k_z$, and this implies in view of the orthogonality of the vectors v_j^z and s_j^z that

$$\|p - p_z\|^2_{L^2(\omega_z)} = \sum_{j=1}^{k_z} \|p \cdot v_j^z\|^2_{L^2(\omega_z)} + \sum_{j=1}^{d-k_z} \|(p - p_z) \cdot s_j^z\|^2_{L^2(\omega_z)}. \tag{5.5}$$

The second term can be estimated by Poincaré's inequality since

$$\begin{aligned} \|(p - p_z) \cdot s_j^z\|_{L^2(\omega_z)} &= \|(p - \widetilde{p}_z) \cdot s_j^z\|_{L^2(\omega_z)} \\ &\lesssim h_z\|\nabla p\|_{L^2(\omega_z)} \quad \text{for } j = 1, \dots, d - k_z. \end{aligned} \tag{5.6}$$

In order to justify the application of the Poincaré inequality (3.2) in the first term, consider first $z \in E \in \mathcal{E}_N$. Then $E \subset \partial\omega_z$ and $\big(p \cdot n|_E\big)|_E = 0$. Hence

$$\|p \cdot n|_E\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\, n|_E\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\|_{L^2(\omega_z)}. \tag{5.7}$$

Similarly, for $z \in E \in \mathcal{E}_D$ we have that $\big(p \cdot t_E^\ell\big)|_E = 0$ for $\ell = 1, \dots, d - 1$ and consequently

$$\|p \cdot t_E^\ell\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\, t_E^\ell\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\|_{L^2(\omega_z)}. \tag{5.8}$$

Finally note that for all $z \in \partial\mathcal{N} \cap \Omega$ the vectors $v_j^z \in V_z$ are linear combinations of the vectors $n\big|_E$ and t_E^ℓ, and this allows us to find coefficients $\alpha_{z,E}^j$, $\beta_{z,E}^{j,\ell}$ with $|\alpha_{z,E}^j|$, $|\beta_{z,E}^{j,\ell}| \le C(\Omega)$ such that

$$v_j^z = \sum_{z\in E\in\mathcal{E}_N} \alpha_{z,E}^j\, n|_E + \sum_{z\in E\in\mathcal{E}_D} \sum_{\ell=1}^{d-1} \beta_{z,E}^{j,\ell}\, t_E^\ell, \quad j = 1, \dots, k_z.$$

By (5.7)–(5.8) we have

$$\|p \cdot v_j^z\|_{L^2(\omega_z)} \lesssim h_z\|\nabla p\|_{L^2(\omega_z)}.$$

By (5.5)–(5.8) one obtains (5.4) for $z \in \mathcal{N} \cap \partial\Omega$ as well. Since $\|\nabla\varphi_z\|_{L^\infty(\omega_z)} \lesssim h_z^{-1}$ for all $z \in \mathcal{N}$, the assertion of the theorem follows from (5.2)–(5.3). □

Lemma 5.2. *Let $p_h \in \mathcal{S}^1(\mathcal{T})^d$ be such that $\gamma_{t_E}(p_h) = 0$ for all $E \in \mathcal{E}_D$. Then $(p_h \cdot n)|_{\Gamma_D} \in \mathcal{S}^1(\Gamma_D)$.*

Proof. We have to show that $(p_h \cdot n)|_{\Gamma_D}$ is continuous across interfaces $E_1 \cap E_2$ of neighbouring faces $E_1, E_2 \in \mathcal{E}_D$ with $n|_{E_1} \neq n|_{E_2}$. Since $p_h|_{E_1\cap E_2}$ is perpendicular to $\operatorname{span}\{t^1_{E_1}, \dots, t^{d-1}_{E_1}, t^1_{E_2}, \dots, t^{d-1}_{E_2}\} = \mathbb{R}^d$ we have $p_h|_{E_1\cap E_2} = 0$ and hence $p_h \cdot n$ is continuous across $E_1 \cap E_2$. □

6 A priori and a posteriori estimates in H^1

In this section we state and prove the asymptotic estimates for the a priori and a posteriori estimates in H^1.

Theorem 6.1 (A priori estimate in H^1)**.** *Under the foregoing assumptions,*

$$\|\nabla e\|_{L^2(\Omega)} \lesssim \inf_{w_h \in \mathcal{S}^1(\mathcal{T}),\, w_h|_{\Gamma_D} = I_D u_D} \|\nabla(u - w_h)\|_{L^2(\Omega)} + \begin{cases} 0 & \text{if } u_{D,h} = I_D u_D, \\ \|h_\mathcal{E}\|^{1/2}_{L^\infty(\Gamma_D)} \|h_\mathcal{E} \partial^2_\mathcal{E} u_D\|_{L^2(\Gamma_D)} & \text{if } u_{D,h} = \Pi_D u_D. \end{cases}$$

Remark 6.2. If $u \in H^2(\Omega)$, then standard interpolation estimates imply that

$$\inf_{w_h \in \mathcal{S}^1(\mathcal{T}),\, w_h|_{\Gamma_D} = I_D u_D} \|\nabla(u - w_h)\|_{L^2(\Omega)} \lesssim \|h_\mathcal{T} D^2 u\|_{L^2(\Omega)}.$$

Proof. The Galerkin orthogonality

$$(\nabla e; \nabla v_h) = 0 \quad \text{for all } v_h \in \mathcal{S}^1_D(\mathcal{T}) \tag{6.1}$$

and Hölder's inequality yield

$$\|\nabla e\|^2_{L^2(\Omega)} = (\nabla e, \nabla(e - v_h)) \le \|\nabla e\|_{L^2(\Omega)} \|\nabla(e - v_h)\|_{L^2(\Omega)}$$

for all $v_h \in \mathcal{S}^1_D(\mathcal{T})$. If $u_{D,h} = I_D u_D$, then the assertion follows immediately by choosing $v_h = w_h - u_h$ where $w_h \in \mathcal{S}^1(\mathcal{T})$ satisfies $w_h|_{\Gamma_D} = I_D u_D$. If $u_{D,h} = \Pi_D u_D$, then let $v_h = w_h - u_h + y_h$, where $w_h,\ y_h \in \mathcal{S}^1(\mathcal{T})$ satisfy $y_h|_{\Gamma_D} = \Pi_D u_D - I_D u_D$ and $w_h|_{\Gamma_D} = I_D u_D$. It follows that

$$\|\nabla(e - v_h)\|_{L^2(\Omega)} \lesssim \|\nabla(u - w_h)\|_{L^2(\Omega)} + \|\nabla y_h\|_{L^2(\Omega)},$$

and the proof is an immediate consequence of Proposition 4.3. □

The a posteriori estimate relies on a good estimate of the contribution $\eta^{(1/2)}_D$ which is based on the extension results in Section 4. More precisely, we have the following theorem.

Theorem 6.2 (A posteriori estimate in H^1). *Under the foregoing assumptions,*

$$\|\nabla e\|_{L^2(\Omega)} \lesssim \|\operatorname{Res}\|_{-1} + \begin{cases} \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)} & \text{if } u_{D,h} = I_D u_D, \\ \|h_{\mathcal{E}}\|_{L^\infty(\Gamma_D)}^{1/2}\|h_{\mathcal{E}}\partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)} & \text{if } u_{D,h} = \Pi_D u_D. \end{cases}$$

Proof. In view of the representation (2.1) it suffices to estimate the terms $\eta_D^{(1/2)}$. Suppose first that $u_{D,h} = I_D u_D$. Since w minimizes the Dirichlet integral subject to the given Dirichlet conditions, we deduce from Theorem 4.2 that

$$\|\nabla w\|_{L^2(\Omega)} = \min_{v|_{\Gamma_D}=u_D-u_{D,h}} \|\nabla v\|_{L^2(\Omega)} \lesssim \|h_{\mathcal{E}}^{3/2}\partial_{\mathcal{E}} u_D\|_{L^2(\Gamma_D)}.$$

This proves the assertion for the nodal interpolation. Assume next that $u_{D,h} = \Pi_D u_D$ and that Π_D is H^1 stable. Note that for all $y_h \in \mathcal{S}^1(\mathcal{T})$ with $y_h|_{\Gamma_D} = \Pi_D u_D - I_D u_D$

$$\begin{aligned} \|\nabla w\|_{L^2(\Omega)} &= \min_{v|_{\Gamma_D}=u_D-u_{D,h}} \|\nabla v\|_{L^2(\Omega)} \\ &\le \min_{v|_{\Gamma_D}=u_D-u_{D,h}} \|\nabla (v-y_h)\|_{L^2(\Omega)} + \|\nabla y_h\|_{L^2(\Omega)} \\ &= \min_{y|_{\Gamma_D}=u_D-I_D u_D} \|\nabla y\|_{L^2(\Omega)} + \|\nabla y_h\|_{L^2(\Omega)}. \end{aligned}$$

The statement of the theorem follows now in view of the estimate for $\|\nabla y\|_{L^2(\Omega)}$ and $\|\nabla y_h\|_{L^2(\Omega)}$ with Theorem 4.2 and Proposition 4.3, respectively. □

7 A priori and a posteriori estimates in L^2

In this section we present the corresponding L^2 estimates.

Theorem 7.1 (A priori estimate in L^2). *Suppose that the dual problem* (2.4) *is H^2 regular and that Π_D is H^1 stable if $u_{D,h} = \Pi_D u_D$. Then*

$$\|e\|_{L^2(\Omega)} \lesssim \|h_{\mathcal{T}}\nabla e\|_{L^2(\Omega)} + \begin{cases} \|h_{\mathcal{E}}^2\partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)} & \text{if } u_{D,h} = I_D u_D, \\ \|h_{\mathcal{E}}\|_{L^\infty(\Gamma_D)}^{3/2}\|h_{\mathcal{E}}\partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)} & \text{if } u_{D,h} = \Pi_D u_D. \end{cases}$$

Proof. We first derive a representation of the error based on duality techniques. Let $z \in H^2(\Omega)$ satisfy (2.4) and (2.6). Then

$$\|e\|_{L^2(\Omega)}^2 = (e; -\Delta z) = (\nabla e; \nabla z) - \int_{\Gamma_D} e\,\partial_n z\,\mathrm{d}s. \tag{7.1}$$

Let $z_h \in \mathcal{S}^1_D(\mathcal{T})$ be the nodal interpolant of z so that the Galerkin orthogonality (6.1) implies

$$(\nabla e; \nabla z) = (\nabla e; \nabla(z - z_h)) \le \|h_{\mathcal{T}} \nabla e\|_{L^2(\Omega)} \|h_{\mathcal{T}}^{-1} \nabla(z - z_h)\|_{L^2(\Omega)}$$

$$\lesssim \|h_{\mathcal{T}} \nabla e\|_{L^2(\Omega)} \|z\|_{H^2(\Omega)}. \tag{7.2}$$

The second term on the right-hand side in (7.1) is bounded by

$$\int_{\Gamma_D} e\, \partial_n z \,\mathrm{d}s \le \eta_D^{(-1/2)} \|z\|_{H^2(\Omega)}. \tag{7.3}$$

It therefore suffices to estimate the terms $\eta_D^{(-1/2)}$. Suppose first that $u_{D,h} = I_D u_D$. By the continuity of the trace operator $H^1(\Omega) \to L^2(\Gamma_D)$ we have $\|\partial_n \phi\|_{L^2(\Gamma_D)} \le \|\nabla \phi\|_{L^2(\Gamma_D)} \lesssim \|\phi\|_{H^2(\Omega)}$. Therefore, Hölder's inequality yields for $\phi \in H^2_{DN}(\Omega)$,

$$\int_{\Gamma_D} (u_D - u_{D,h}) \partial_n \phi \,\mathrm{d}s \lesssim \|u_D - u_{D,h}\|_{L^2(\Gamma_D)} \|\phi\|_{H^2(\Omega)}.$$

and standard nodal interpolation estimates imply that

$$\eta_D^{(-1/2)} \lesssim \|h_{\mathcal{E}}^2 \partial_{\mathcal{E}}^2 u_D\|_{L^2(\Gamma_D)}.$$

It remains to prove the assertion for $u_{D,h} = \Pi_D u_D$. Fix $v \in H^1(\Omega)$ with $v|_{\Gamma_D} = u_D - u_{D,h}$ and $\phi \in H^2_{DN}(\Omega)$. Let $p = \nabla \phi$. Then $\gamma_{t_E}(p) = 0$ for all $E \in \mathcal{E}_D$ and $p \cdot n = 0$ on Γ_N and we may find a vector field $p_h \in \mathcal{S}^1(\mathcal{T})^d$ with the properties in Theorem 5.1. Since $u_D - u_{D,h}$ is $L^2(\Gamma_D)$ orthogonal to $\mathcal{S}^1(\Gamma_D)$ we have by Lemma 5.2, integration by parts, and Cauchy's inequality

$$\begin{aligned}
&\int_{\Gamma_D} (u_D - u_{D,h}) \partial_n \phi \,\mathrm{d}s \\
&\quad = \int_{\Gamma_D} (u_D - u_{D,h})(p - p_h) \cdot n \,\mathrm{d}s \\
&\quad = \int_{\partial\Omega} v(p - p_h) \cdot n \,\mathrm{d}s \\
&\quad = \int_{\Omega} \nabla v \cdot (p - p_h)\, \mathrm{d}x + \int_{\Omega} v \operatorname{div}(p - p_h)\, \mathrm{d}x \\
&\quad \le \|h_{\mathcal{T}} \nabla v\|_{L^2(\Omega)} \, \|h_{\mathcal{T}}^{-1}(p - p_h)\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \, \|\operatorname{div}(p - p_h)\|_{L^2(\Omega)} \\
&\quad \le \big(\|h_{\mathcal{T}} \nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\big)^{1/2} \big(\|h_{\mathcal{T}}^{-1}(p - p_h)\|^2_{L^2(\Omega)} \\
&\qquad + \|\operatorname{div}(p - p_h)\|^2_{L^2(\Omega)}\big)^{1/2}.
\end{aligned}$$

The choice $v = w - v_h$ with w of Theorem 4.2 and v_h of Proposition 4.3 shows

$$\|h_{\mathcal{T}} \nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \lesssim \|h_{\mathcal{E}}\|^3_{L^\infty(\Gamma_D)} \|h_{\mathcal{E}} \partial_{\mathcal{E}}^2 u_D\|^2_{L^2(\Gamma_D)}.$$

The combination of the previous estimates with Theorem 5.1 and $\|\nabla p\|_{L^2(\Omega)} \leq \|\phi\|_{H^2(\Omega)}$ show that

$$\eta_D^{(-1/2)} \lesssim \|h_\mathcal{E}\|^{3/2}_{L^\infty(\Gamma_D)} \|h_\mathcal{E} \partial^2_\mathcal{E} u_D\|_{L^2(\Gamma_D)}.$$

This concludes the proof of the theorem. □

The inequality $\|\partial_n \phi\|_{L^2(\Gamma_D)} \lesssim \|\phi\|_{H^2(\Omega)}$ appears suboptimal in the foregoing proof for $u_{D,h} = I_D u_D$. However, the estimate for $\eta_D^{(-1/2)}$ in the proof of Theorem 7.1 can in general not be improved, see Section 8.

The next theorem describes the corresponding results for a posteriori error estimates.

Theorem 7.2 (A posteriori estimate in L^2). *Suppose that the dual problem* (2.4) *is H^2 regular. Then*

$$\|e\|_{L^2(\Omega)} \lesssim \|\operatorname{Res}\|_{-2} + \begin{cases} \|h^2_\mathcal{E} \partial^2_\mathcal{E} u_D\|_{L^2(\Gamma_D)} & \text{if} \, u_{D,h} = I_D u_D, \\ \|h_\mathcal{E}\|^{3/2}_{L^\infty(\Gamma_D)} \|h_\mathcal{E} \partial^2_\mathcal{E} u_D\|_{L^2(\Gamma_D)} & \text{if} \, u_{D,h} = \Pi_D u_D. \end{cases}$$

Proof. With (7.1), (7.3), and (2.6)–(2.7),

$$\begin{aligned} \|e\|^2_{L^2(\Omega)} &\leq \left(\|\operatorname{Res}\|_{-2} + \eta_D^{(-1/2)}\right) \|z\|_{H^2(\Omega)} \\ &\leq c_1 \left(\|\operatorname{Res}\|_{-2} + \eta_D^{(-1/2)}\right) \|e\|_{L^2(\Omega)}. \end{aligned}$$

The assertion follows now as in the proof of Theorem 7.1 □

8 Model Example

In this section we discuss an example which demonstrates the different scaling for the different choices for the approximation of the Dirichlet data. In particular, the boundary contribution

$$\|e\|^2_{L^2(\Omega)} - \operatorname{Res}(z) = -\int_{\Gamma_D} (u_D - u_{D,h}) \partial_n z \, ds$$

in (2.5) is not of higher order for the discrete boundary data $u_{D,h} = I_D u_D$. Moreover, the L^2 error $\|e\|_{2,\Omega}$ is significantly reduced if one replaces the discrete Dirichlet data $u_{D,h} = I_D u_D$ by $u_{D,h} = \Pi_D u_D$ while the H^1 errors are comparable.

For the precise argument consider $\Omega = (0,1)^2$ and $\Gamma_D = \partial\Omega$. Let $h = 1/n$ for $n \in \mathbb{N}$. We fix the uniform triangulation $\mathcal{T}$ with nodes $\mathcal{N} = \{(jh, kh) : j, k = 0, 1, \dots, n\}$ and sides parallel to the x-axis, the y-axis and the direction $(1, 1)$. In this situation, the quadratic function $u(x, y) = u_D(x, y) = x(1-x) + y(1-y)$ is the solution of $-\Delta u = 4$ subject to its own boundary data. The following theorem describes the asymptotic scaling of the boundary contribution.

Theorem 8.1. *Suppose that $\mathcal{T}$ and u are as above and that $u_{D,h} = I_D u_D$. Then*

$$\eta_D^{(-1/2)} \geq \frac{\sqrt{5}}{3\sqrt{22}c_1}\left(h^2 + \mathcal{O}(h^{5/2})\right).$$

In order to prove this result, let $I_h u \in \mathcal{S}^1(\mathcal{T})$ denote the nodal interpolant of u. We write e_h^I and e_h^Π for the finite element errors for the choices $u_{D,h} = I_D u_D$ and $u_{D,h} = \Pi_D u_D$, respectively. Finally, let z_h^I and z_h^Π denote the solutions in $H_D^1(\Omega)$ of the corresponding dual problems

$$-\Delta z_h^I = e_h^I \text{ in } \Omega, \qquad z_h^I = 0 \text{ on } \partial\Omega,$$
$$-\Delta z_h^\Pi = e_h^\Pi \text{ in } \Omega, \qquad z_h^\Pi = 0 \text{ on } \partial\Omega.$$

The following proposition compares the four contributions to the two L^2 residual relations

$$\|e_h^I\|_{L^2(\Omega)}^2 = \operatorname{Res}(z_h^I) - \int_{\Gamma_D} (1 - I_D)u_D\, \partial_n z_h^I \,\mathrm{d}s,$$
$$\|e_h^\Pi\|_{L^2(\Omega)}^2 = \operatorname{Res}(z_h^\Pi) - \int_{\Gamma_D} (1 - \Pi_D)u_D\, \partial_n z_h^\Pi \,\mathrm{d}s.$$

Proposition 8.2. *The approximation errors e_h^I and e_h^Π are related by $e_h^I = e_h^\Pi + h^2/6$ and satisfy the following estimates:*

$$-\int_{\Gamma_D} (1 - I_D)u_D\, \partial_n z_h^I \,\mathrm{d}s = h^4/18 + \mathcal{O}(h^{9/2}),$$
$$-\int_{\Gamma_D} (1 - \Pi_D)u_D\, \partial_n z_h^\Pi \,\mathrm{d}s = \mathcal{O}(h^{9/2}).$$

Moreover, $\|e_h^I\|_{L^2(\Omega)}^2 = 11h^4/90$ *and* $\|e_h^\Pi\|_{L^2(\Omega)}^2 = 7h^4/180$.

Remark. The L^2 error is reduced to 56% when the discrete boundary data are obtained by the L^2 projection instead of the nodal interpolation of u_D.

Proof. The second partial derivatives of u are constant and therefore a second order difference quotient is exact. The stiffness matrix for the uniform triangulation $\mathcal{T}$ is equivalent to the discrete problem in the related difference scheme. Hence, $e_h^I = (1 - I_h)u$.

Let s denote the arc-length parameter on an edge $E \subset \Gamma_D$ on which $e_h^I = (1 - I_D)u$ is given by $e_h^I(s) = s(h - s)$ with $0 \leq s \leq h$. The identities

$$\int_0^h s\,(e_h^I - h^2/6)\mathrm{d}s = 0 \quad \text{and} \quad \int_0^h (h - s)\,(e_h^I - h^2/6)\mathrm{d}s = 0$$

imply that $e_h^I - h^2/6 = (1 - I_D)u_D - h^2/6$ is $L^2(\Gamma_D)$ orthogonal to $\mathcal{S}^1(\Gamma_D)$ so that we have $\Pi_D u_D = I_D u_D + h^2/6$. Since $I_h u + h^2/6 \in \mathcal{S}^1(\mathcal{T})$ equals

$u_{D,h} = \Pi_D u_D$ on Γ_D and satisfies the difference equations there holds $e_h^\Pi = e_h^I - h^2/6$.

A short calculation shows that

$$\int_0^h \int_0^h e_h^I(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{11}{90} h^6 \text{ and } \int_0^h \int_0^h e_h^\Pi(x,y)^2 \, \mathrm{d}x \, \mathrm{d}y = \frac{7}{180} h^6.$$

This verifies the last two identities in the assertion of the proposition since e_h^I and e_h^Π are (h,h)-periodic.

The L^2 projection is stable for (quasi-) uniform meshes and therefore the estimates in the proof Theorem 7.1 show $\eta_D^{(-1/2)} = \mathcal{O}(h^{5/2})$. By assumption, the dual problem is H^2 regular and thus

$$\|\partial_n z_h^\Pi\|_{L^2(\Gamma_D)} \le \|z_h^\Pi\|_{H^2(\Omega)} \lesssim \|e_h^\Pi\|_{L^2(\Omega)}.$$

Hence

$$-\int_{\Gamma_D} (1-\Pi_D)u_D \, \partial_n z_h^\Pi \, \mathrm{d}s \lesssim \eta_D^{(-1/2)} \, \|e_h^\Pi\|_{L^2(\Omega)} = \mathcal{O}(h^{9/2}).$$

Let $\zeta \in H_D^1(\Omega) = H_0^1(\Omega)$ be the solution of $\Delta\zeta = 1$ in Ω. Then,

$$z_h^I = z_h^\Pi - h^2/6\,\zeta.$$

We obtain in view of $e_h^I = e_h^\Pi + h^2/6$ that

$$\begin{aligned} -\int_{\Gamma_D} (1-I_D)u_D \, \partial_n z_h^I \, \mathrm{d}s &= -\int_{\partial\Omega} (e_h^\Pi + h^2/6)\, \partial_n (z_h^\Pi - \frac{h^2}{6}\zeta)\, \mathrm{d}s \\ &= -\int_{\partial\Omega} e_h^\Pi \, \partial_n z_h^\Pi \, \mathrm{d}s + \frac{h^2}{6}\int_{\partial\Omega} e_h^\Pi \, \partial_n \zeta \, \mathrm{d}s \\ &\quad + \frac{h^4}{36}\int_{\partial\Omega} \partial_n \zeta \, \mathrm{d}s - \frac{h^2}{6}\int_{\partial\Omega} \partial_n z_h^\Pi \, \mathrm{d}s. \end{aligned}$$

The first term on the right-hand side is of order $h^{9/2}$ as shown previously. The second term is of the same order by the same arguments since $\|\zeta\|_{H^2(\Omega)} \lesssim 1$. Partial integration in the third term shows

$$\int_{\partial\Omega} \partial_n \zeta \, \mathrm{d}s = \int_\Omega \Delta\zeta \, \mathrm{d}x = 1.$$

Combined with a direct calculation of $\int_\Omega e_h^I \, \mathrm{d}x = h^2/3$, the same arguments prove for the last term

$$-\int_{\partial\Omega} \partial_n z_h^\Pi \, \mathrm{d}s = -\int_\Omega \Delta z_h^\Pi \, \mathrm{d}x = \int_\Omega e_h^\Pi \, \mathrm{d}x = h^2/6.$$

These estimates prove that

$$-\int_{\Gamma_D} (1-I_D)u_D \, \partial_n z_h^I \, \mathrm{d}s = h^4/18 + \mathcal{O}(h^{9/2}),$$

as asserted. □

Proof of Theorem 8.1. With u_D, f, $\mathcal{T}$, z_h^I, and e_h^I as in the proposition,

$$\|z_h^I\|_{H^2(\Omega)}\,\eta_D^{(-1/2)} \geq \int_{\Gamma_D} (1 - I_D)u_D \partial_n z_h^I \,\mathrm{d}s = h^4/18 + \mathcal{O}(h^{9/2}).$$

The estimate $\|z_h^I\|_{H^2(\Omega)} \leq c_1 \|e_h^I\|_{L^2(\Omega)}$ and the above identity for $\|e_h^I\|_{L^2(\Omega)}$ conclude the proof. □

The point in the example is that the approximation error $u_D - I_D u_D$ in the Dirichlet data does not change its sign. This is always the case on parts of the boundary where u_D is convex or concave. We may therefore expect that the boundary contribution

$$-\int_{\Gamma_D} e_h^I \partial_n z_h^I \mathrm{d}s \tag{8.1}$$

is not of higher order. In this sense, the model example describes a rather generic situation unless long-range cancellations lead to a global integral (8.1) of higher order.

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